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Motivic interpretation of Milnor K-groups attached to Jacobian varieties

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Abstract. In the paper [Som90], Somekawa conjectures that his Milnor K-group $K(k, G_1, \ldots, G_r)$ attached to semi-abelian varieties $G_1, \ldots, G_r$ over a field k is isomorphic to $\mathrm{Ext}^r_{\mathcal{M}_k}(\mathbb{Z}, G_1[-1] \otimes \cdots \otimes G_r[-1])$ where $\mathcal{M}_k$ is a certain category of motives over k. The purpose of this note is to prove this conjecture, when the varieties G_i are Jacobians of smooth curves over a perfect field and we take $\mathcal{M}_k$ as Voevodsky's category of motives $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Key words: motivic cohomology, Milnor K-groups

1. Introduction

To unify the Moore exact sequence and the Bloch exact sequence, K. Kato defined the generalized Milnor K-groups attached to finite family of semi-abelian varieties over a base field k in [Som90]. (See also [Kah92]). Given semi-abelian varieties $G_1, \ldots, G_r$ over k, one defines $K(k, \{G_i\}_{i=1}^r) = F/R$, where F is the group

$$\bigoplus_{E/k:\text{finite}} G_1(E) \otimes \cdots \otimes G_r(E)$$

and R is a subgroup generated by various elements corresponding to the projection formula relation and Weil reciprocity relation; for the precise definition, see Section 2. This group is a generalization of the Milnor K-group as the following example shows.

Example 1.1 (cf. [Som90, 1.4]) In the notation above, if $G_1 = G_2 = \cdots = G_r = \mathbf{G}_{\mathrm{m}}$, the following equality holds.

$$K\big(k, \{G_i\}_{i=1}^r\big) = K_r^M(k).$$

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Further generalizations were proposed and studied by W. Raskind and M. Spiess [RS00] and R. Akhtar [MilKt], [ZerCy] and [TorMi]. In [Som90], Somekawa conjectures that the Somekawa K-groups should be motivic cohomology groups attached to semi-abelian varieties. More precisely

Conjecture 1.2 (Somekawa conjecture) *Let $G_1, \dots, G_r$ be semi-abelian varieties over k, then we have the canonical isomorphism*

$$K\big(k, \{G\}_{i=1}^r\big) \xrightarrow{\sim} \mathrm{Ext}^r_{\mathcal{M}_k}\left(\mathbb{Z}, \bigotimes_{i=1}^r G_i[-1]\right)$$

where $\mathcal{M}_k$ is a certain category of motives over k and $G_i[-1]$ means 1-motif (cf. [Del74]).

In this paper we will examine this conjecture, if we take $\mathcal{M}_k$ as Voevodsky's category of motives $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Main Theorem 1.3 (Somekawa conjecture for Jacobian varieties) *Let $(C_1, a_1), \dots, (C_n, a_n)$ be pointed projective smooth geometrically connected curves over perfect field k. Then we have the isomorphism*

$$K\big(k, \{J\}_{i=1}^n\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-}(k)}\left(\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec}\, k), \mathbb{Z}\Big(\bigwedge_{i=1}^n (C_i, a_i)\Big)[n]\right)$$

where J_i is the Jacobian of C_i and $\mathbb{Z}(\bigwedge_{i=1}^n (C_i, a_i)) := C^*(\bigotimes_{i=1}^n \mathbb{Z}_{\mathrm{tr}}(C_i, a_i)) \cdot [-n]$.

2. Proof

First, we will briefly review the definition of mixed K-groups from [ZerCy] and [RS00].

2.1 Let k be a field, and X a smooth quasi-projective varieties over k. We use the notation $\mathrm{CH}_0(X)$ for the group of zero-cycles on X modulo rational equivalence. If G is a group scheme defined over k and A is k-algebra, we use the notation $G(A)$ for the group of A-rational points, i.e., the set of morphisms $\mathrm{Spec}\, A \to G$ compatible with the structure map.

2.2 Suppose k is a field and G is a semi-abelian variety defined over k, that is, there is an exact sequence of group schemes (viewed as sheaves in the flat topology) over k:

$$0 \to T \to G \to A \to 0$$

where T is a torus and A is an abelian variety.

2.3 In the notation above, let K/k be an algebraic function field and v a place of K/k. Let L/K_v be a finite unramified Galois extension such that $T \times_k F \xrightarrow{\sim} \mathbf{G}_{\mathrm{m}}{}^n$ for the residue field F of L and some n; let w be the unique extension of v of L. We obtain the following commutative diagram of exact sequences defining a map $r_w = (r_w^1, \ldots, r_w^n)$;

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T(\mathcal{O}_w) \arrow[r] \arrow[d] & G(\mathcal{O}_w) \arrow[r] \arrow[d] & A(\mathcal{O}_w) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & T(L) \arrow[r] \arrow[d, "\mathrm{ord}_w"'] & G(L) \arrow[r] \arrow[d, "r_w=(r_w^1,\ldots,r_w^n)"] & A(L) \arrow[r] & 0 \\
 & \mathbb{Z}^n \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Z}^n \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

2.4 In the notation above, we are going to construct a map

$$\partial_v : G(K_v) \otimes K_v^\times \to G(k(v)).$$

Fix $g \in G(K_v)$ and $h \in K_v^\times$. For each $i = 1, \ldots, n$, we define $h_i \in T(L)$ to be the n-th tuple having h in the i-th coordinate and 1 elsewhere. Then set

$$\varepsilon(g,h) = \left((-1)^{\mathrm{ord}_w(h) r_w^1(g)}, \ldots, (-1)^{\mathrm{ord}_w(h) r_w^n(g)}\right) \in T(\mathcal{O}_w) \subset G(\mathcal{O}_w)$$

and

$$\tilde{\partial_v}(g,h) = \varepsilon(g,h) g^{\mathrm{ord}_w(h)} \prod_{i=1}^{n} h_i^{-r_w^i(g)} \in G(\mathcal{O}_w).$$

We define the "extended tame symbol" $\partial_v(g,h)$ to be the image of $\tilde{\partial}_v(g,h)$ under the canonical map $G(\mathcal{O}_w) \to G(F)$; Then $\partial_v(g,h)$ is invariant under the action of $\mathrm{Gal}(F/k(v))$, so that it belongs to $G(k(v))$. This definition of ∂_v is independent of the choice of L and of the isomorphism from the torus to ${\mathbf{G}_{\mathrm{m}}}^{\oplus n}$.

2.5 Let $r \geq 0$ and $s \geq 0$ be integers; let $X_1, \ldots, X_r$ be smooth quasiprojective varieties defined over k and $G_1, \ldots, G_r$ a finite (possibly empty) family of semi-abelian varieties defined over k. We define Mixed K-groups $K(k, \{\mathcal{CH}_0(X_i)\}_{i=1}^r; \{G_j\}_{j=1}^s)$ as follows. If $r = 0$ and $s = 0$, we write $K(k, \emptyset)$ for our groups and set $K(k, \emptyset) = \mathbb{Z}$. For $r = 1$, we define

$$K\big(k, \{\mathcal{CH}_0(X_i)\}_{i=1}^r; \{G_j\}_{j=1}^s\big) = F/R$$

where

$$F = \bigoplus_{E/k:\text{finite}} \bigotimes_{i=1}^{r} \mathrm{CH}_0((X_i)_E) \otimes \bigotimes_{j=1}^{s} G_i(E)$$

and $R \subset F$ is the subgroup generated by the relations **R1**-**R2** below. To simplify the notations, set $H_i(E) = \mathrm{CH}_0((X_i)_E)$ for $i = 1, \ldots, r$ and $H_j(E) = G_{j-r}(E)$ for $j = r+1, \ldots, r+s$.

R1 For any finite extensions $k \hookrightarrow E_1 \overset{\psi}{\hookrightarrow} E_2$, let $h_{i_0} \in H_{i_0}(E_2)$ and $h_i \in H_i(E_1)$ for $i \neq i_0$, the relation

$$\big(\psi^*(h_1) \otimes \cdots \otimes h_{i_0} \otimes \cdots \otimes \psi^*(h_r)\big)_{E_2} \\ - \big(h_1 \otimes \cdots \otimes \psi_*(h_{i_0}) \otimes \cdots \otimes h_r\big)_{E_1}$$

where ψ^* or ψ_* means the pullback or pushforward map for the Chow group structure on H_i (if $1 \leqq i \leqq r$) or the group scheme structure on H_i (if $s \leqq i \leqq r+s$).

R2 For every algebraic function field K/k and all choices $f_i \in \mathrm{CH}_0((X_i)_K)$ for $i = 1, \ldots, r$ and $g_j \in G_j(K)$ for $j = 1, \ldots, s$, $h \in K^\times$ such that for each place v of K/k, there exists $i(v)$ such that $g_i \in G_i(\mathcal{O}_v)$ for all $i \neq i(v)$, the relation for $s > 0$:

$$\sum_{v:\text{place of } K/k} \big(s_v(f_1)\otimes\cdots\otimes s_v(f_r)\otimes g_1(v) \\ \otimes\cdots\otimes \partial_v(g_{i(v)},h)\otimes\cdots\otimes g_r(v)\big)_{k(v)/k}$$

Here $\mathcal{O}_v$ is the valuation ring of v, $s_v : \mathrm{CH}_0((X_i)_K) \to \mathrm{CH}_0((X_i)_{k(v)})$ is the specialization map for Chow groups (cf. [Ful84, 20.3]) and $g_i(v) \in G_i(k(v))$ $(i \neq i(v))$ denotes the reduction of $g_i \in G(\mathcal{O}_v)$ modulo m_v.

If $s = 0$, the element

$$\sum_{v:\text{place of } K/k} \mathrm{ord}_v(h)\big(s_v(f_1)\otimes\cdots\otimes s_v(f_r)\big)_{k(v)/k}.$$

The class in F/R of an element $a_1\otimes\cdots\otimes a_r \in G_1(E)\otimes\ldots G_r(E)$ will be denoted $\{a_1,\ldots,a_r\}_{E/k}$. If $r = 0$, we simply write F/R by $K(k,\{G_i\}_{i=1}^s)$ above.

Remark 2.6

(1) By the relation **R1**, if ψ is a k-isomorphism $E_1 \xrightarrow{\sim} E_2$, then we have the equality

$$\{g_1,\ldots,g_r\}_{E_1/k} = \{\psi^*(g_1),\ldots,\psi^*(g_r)\}_{E_2/k}$$

This shows that symbols form a set.

(2) If $\sigma : Y \to \operatorname{Spec} k$ is a projective variety, we will define

$$A_0(Y) := \ker\big(\sigma_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(\operatorname{Spec} k) \xrightarrow{\sim} \mathbb{Z}\big)$$

and note that if Y contains a k-rational point, then σ_* induces the direct summand decomposition

$$\mathrm{CH}_0(Y) \xrightarrow{\sim} \mathbb{Z} \oplus A_0(Y)$$

(3) Suppose that $X_1,\ldots,X_q$ are smooth quasiprojective varieties and $Y_1,\ldots,Y_r$ smooth projective varieties over k. By replacing $\mathcal{CH}_0$ with A_0 in the appropriate instances, we can define groups $K(k,\{\mathcal{CH}_0(X_i)\}_{i=1}^q, \{A_0(Y_j)\}_{j=1}^r, \{G_k\}_{k=1}^s)$ as was done previously.

(4) The Chow groups $\mathrm{CH}(\mathcal{M})$ of a Chow motive $\mathcal{M} = (X,p,m)$ are defined as $p_*\,\mathrm{CH}_{*+m}(X)$. One can also define specialization map for Chow

groups of motives. (cf. [RS00, 2.3]). Hence for Chow motives $\mathcal{M}_1, \dots, \mathcal{M}_r$, we can define the $K(k, \{\mathcal{CH}_0(\mathcal{M}_i)\}_{i=1}^r)$ in exactly the same way as above.

2.7 Now we recall fundamental isomorphisms from [RS00] and [ZerCy].

(1) (cf. [RS00, 2.2]). For projective smooth varieties $X_1, \dots, X_n$ over a field k, we have isomorphisms

$$\begin{aligned} \mathrm{CH}_0(X_1 \times \cdots \times X_n) &\xrightarrow{\sim} K\big(k, \{\mathcal{CH}_0(X_i)\}_{i=1}^n\big) \\ &\xrightarrow{\sim} K\big(k, \{\mathcal{CH}_0(h(X_i))\}_{i=1}^n\big) \end{aligned}$$

where $h(X_i)$ means the Chow motive associated to X_i.

(2) (cf. [ZerCy, 2.6]). For projective smooth varieties $X_1, \dots, X_r, \dots, X_{r+s}$, if $X_r = \operatorname{Spec} k$, then we have the canonical isomorphism

$$\begin{aligned} &K\big(k, \{\mathcal{CH}_0(X_i)\}_{i=1}^r; \{A_0(X_j)\}_{j=r+1}^{r+s}\big) \\ &\quad \xrightarrow{\sim} K\big(k, \{\mathcal{CH}_0(X_i)\}_{i=1}^{r-1}; \{A_0(X_j)\}_{j=r+1}^{r+s}\big). \end{aligned}$$

(3) (cf. [RS00, 2.4] and [ZerCy, 2.10], see also [Som90, 2.4]). For smooth projective geometrically connected curves $C_1, \dots, C_d$ over k with Jacobian $J_1, \dots, J_d$ such that $C_i(k) \neq \emptyset$ for each i, we have the isomorphisms

$$K\big(k, \{\mathcal{CH}_0(h(C_i^+))\}_{i=1}^d\big) \xrightarrow{\sim} K\big(k, \{A_0(C_i)\}_{i=1}^d\big) \xrightarrow{\sim} K\big(k, \{J_i\}_{i=1}^d\big)$$

where $h(C_i^+)$ is the Chow motive $(C_i, [\Delta_{C_i}] - [\{\epsilon\} \times C_i] - [C_i \times \{\epsilon\}])$ with $\epsilon \in C_i(k)$.

(4) (cf. [ZerCy, 2.8].) For projective smooth geometrically connected curves $C_1, \dots, C_r, \dots, C_{r+s}$ over a field k with $C_i(k) \neq \emptyset$, the canonical projection map

$$C_1 \times \cdots \times C_{r-1} \times C_r \times C_{r+1} \times \cdots \times C_{r+s} \to C_1 \times C_{r-1} \times C_{r+1} \times \cdots \times C_{r+s}$$

induces the split exact sequences

$$0 \to K_A \to K_{\mathrm{CH}} \to K_{\mathbb{Z}} \to 0$$

where

$$K_A := K\big(k, \{\mathcal{CH}_0(C_i)\}_{i=1}^{r-1}; \{A_0(C_j)\}_{j=r}^{r+s}\big),$$

$$K_{\mathrm{CH}} := K\big(k, \{\mathcal{CH}_0(C_i)\}_{i=1}^{r}; \{A_0(C_j)\}_{j=r+1}^{r+s}\big) \quad \text{and}$$

$$K_{\mathbb{Z}} := K\big(k, \{\mathcal{CH}_0(C_i)\}_{i=1}^{r-1}; \{A_0(C_j)\}_{j=r+1}^{r+s}\big).$$

Here we utilize the isomorphism $K_{\mathbb{Z}} \xrightarrow{\sim} K(k, \{\mathcal{CH}_0(C_i)\}_{i=1}^{r-1}, \mathcal{CH}_0(\operatorname{Spec} k); \{A_0(C_j)\}_{j=r+1}^{r+s})$ in (2). This result is considered as a generalization of well-known split sequence

$$0 \to A_0(C_r) \to \mathrm{CH}_0(C_r) \to \mathbb{Z} \to 0.$$

Notations 2.8 We consider the category of (effective) Chow motives $\mathbf{Chow}^{\mathrm{eff}}(k)$ over field k. (See [Man68] or [Sch91]). For projective geometrically connected smooth curves $C_1, \dots, C_r, \dots, C_{r+s}$ over a field k, we put the effective Chow motive

$$h(C_i)_{i=1}^{r} \otimes h\big(C_j^+\big)_{j=r+1}^{r+s} := h(C_1) \otimes \cdots \otimes h(C_r) \otimes h\big(C_{r+1}^+\big) \otimes \cdots \otimes h\big(C_{r+s}^+\big).$$

We put the trivial Chow motive $\mathbb{Z}(0) := (\operatorname{Spec} k, \Delta_{\operatorname{Spec} k})$. As in 2.7 (4), the canonical decomposition

$$h(C_r) \xrightarrow{\sim} \mathbb{Z}(0) \oplus h\big(C_r^+\big)$$

induces the split sequence

$$0 \to H_A \to H_{\mathrm{CH}} \to H_{\mathbb{Z}} \to 0$$

where

$$H_A := \operatorname{Hom}_{\mathbf{Chow}^{\mathrm{eff}}(k)}\big(\mathbb{Z}(0), h(C_i)_{i=1}^{r-1} \otimes h(C_j^+)_{j=r}^{r+s}\big),$$

$$H_{\mathrm{CH}} := \operatorname{Hom}_{\mathbf{Chow}^{\mathrm{eff}}(k)}\big(\mathbb{Z}(0), h(C_i)_{i=1}^{r} \otimes h(C_j^+)_{j=r+1}^{r+s}\big) \quad \text{and}$$

$$H_{\mathbb{Z}} := \operatorname{Hom}_{\mathbf{Chow}^{\mathrm{eff}}(k)}\big(\mathbb{Z}(0), h(C_i)_{i=1}^{r-1}; h(C_j^+)_{j=r+1}^{r+s}\big).$$

Corollary 2.9 *In the Notation 2.8, we have the isomorphism*

$$\operatorname{Hom}_{\mathbf{Chow}^{\mathrm{eff}}(k)}\big(\mathbb{Z}(0), h(C_i)_{i=1}^{r} \otimes h(C_j^+)_{j=r+1}^{r+s}\big)$$
$$\xrightarrow{\sim} K\big(k, \{\mathcal{CH}_0(C_i)\}_{i=1}^{r}; \{A_0(C_j)\}_{j=r+1}^{r+s}\big).$$

Proof. We prove the assertion by induction on s. For $s = 0$, we have the isomorphism

$$\mathrm{Hom}_{\mathbf{Chow}^{\mathrm{eff}}(k)}\left(\mathbb{Z}(0), h(C_i)_{i=1}^r\right) \xrightarrow{\sim} \mathrm{CH}_0(C_1 \times \cdots \times C_r).$$

Therefore the assertion follows from 2.7 (1). For the inductive step, let us notice the split exact sequences

$$0 \to H_A \to H_{\mathrm{CH}} \to H_{\mathbb{Z}} \to 0,$$
$$0 \to K_A \to K_{\mathrm{CH}} \to K_{\mathbb{Z}} \to 0$$

in 2.7 (4) and 2.8. If we have the isomorphisms $H_{\mathbb{Z}} \xrightarrow{\sim} K_{\mathbb{Z}}$ and $H_{\mathrm{CH}} \xrightarrow{\sim} K_{\mathrm{CH}}$ compatible with the short exact sequences above, then we also get the isomorphism $H_A \xrightarrow{\sim} K_A$. Hence we get the result. □

From now on, let k be a perfect field and (C_i, a_i) $(i = 1, \cdots, n)$ smooth projective geometrically connected curves.

2.10 By [Voe00, 2.1.4, 3.2.6], we have the fully faithful embeddings

$$\mathbf{Chow}^{\mathrm{eff}}(k) \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$$

which sends $h(X)$ to $C^*(\mathbb{Z}_{\mathrm{tr}}(X))$ for any smooth projective variety X over k. Since

$$h(C_i)_{i=1}^r \otimes h(C_j^+)_{j=r+1}^{r+s} = \mathrm{Coker}\left(\bigoplus_{k=r+1}^{r+s} h\left(\prod_{\substack{i=1\\ i\neq k}}^{r+s} C_i\right) \to h\left(\prod_{i=1}^{r+s} C_i\right)\right)$$

is a direct summand of $h(\prod_{i=1}^{r+s} C_i)$, it turns out that $h(C)_1^r \otimes h(C^+)_{r+1}^{r+s}$ goes to

$$\begin{aligned}
&\bigotimes_{i=1}^{r} C^*(\mathbb{Z}_{\mathrm{tr}}(C_i)) \otimes \mathbb{Z}\left(\bigwedge_{i=r+1}^{r+s} (C_i, a_i)\right)[-s] \\
&\quad = \mathrm{Coker}\left(\bigoplus_{k=r+1}^{r+s} C^*\left(\mathbb{Z}_{\mathrm{tr}}\left(\prod_{\substack{i=1\\ i\neq k}}^{r+s} C_i\right)\right) \to C^*\left(\mathbb{Z}_{\mathrm{tr}}\left(\prod_{i=1}^{r+s} C_i\right)\right)\right).
\end{aligned}$$

The following is an immediate consequence of 2.9 and 2.10.

Corollary 2.11 *In the notation above, we have the isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}\Big(\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec}\, k), \bigotimes_{i=1}^{r} C^*(\mathbb{Z}_{\mathrm{tr}}(C_i)) \otimes \mathbb{Z}\Big(\bigwedge_{i=r+1}^{r+s}(C_i, a_i)\Big)[-s]\Big)$$
$$\xrightarrow{\sim} K\big(k, \{\mathcal{CH}_0(C_i)\}_{i=1}^{r}; \{J_j\}_{j=r+1}^{r+s}\big)$$

where J_i is the Jacobian of C_i.

The main theorem is just the case for $r = 0$ in the Corollary 2.11 above.

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