Hokkaido Mathematical Journal Vol. 36 (2007) p. 121–127

On the universal Burnside module

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(Received September 2, 2005)

Abstract. Let G be a group. In the case where G is finite, Oliver-Petrie defined a Burnside module $\Omega(G, \mathcal{F})$ consisting of all equivalent classes of $\mathcal{F}$-complex. The purpose of this paper is to define the universal Burnside module $U(G, \mathcal{F})$. If G is finite, we have $U(G, \mathcal{F}) \cong \Omega(G, \mathcal{F})$.

Key words: G-CW-complex, $\mathcal{F}$-complex, Universal Burnside module.

1. Introduction

This paper is an outgrowth of the author's thesis [5], in which he showed that a Burnside module has Burnside congruences. The Burnside module is introduced by R. Oliver-T. Petrie [7] for a finite group. This notion is a generalization of a Burnside ring. It is well known that the Burnside ring is defined for a finite group. On the other hand, T. tom Dieck defined the Burnside ring for a compact Lie group [2]. In this paper, we study the next problem:

Problem What group is the Burnside module defineable for ?

Throughout this paper let G be a group (not necessarily finite) and $S(G)$ denote the set of all subgroups of G. Suppose that Π is a partially ordered set and G acts on it preserving the partially order. We regard $S(G)$ as a G-set via the action $(g, H) \mapsto gHg^{-1} (g \in G$ and $H \in S(G))$ and as a partially ordered set via

$$H \leq K \quad \text{if and only if} \quad H \supseteq K \quad (H, K \in S(G)).$$

For any $\alpha \in \Pi$ we set

$$\Pi_\alpha = \{\beta \in \Pi \mid \beta \geq \alpha\} \quad \text{and} \quad G_\alpha = \{g \in G \mid g\alpha = \alpha\}.$$

Let $\rho\colon \Pi \to S(G)$ be an order preserving map. A pair (Π, ρ) is called a

2000 Mathematics Subject Classification : 57S15, 57S25.

G-poset if it is satisfying the following condition: for any $\alpha \in \Pi$,

$$\rho(\alpha) \lhd G_\alpha \quad \text{and} \quad \rho\colon \Pi_\alpha \to S(G)_{\rho(\alpha)} \quad \text{is injective.}$$

Note that $S(G)_{\rho(\alpha)} = S(\rho(\alpha)) \subset S(G_\alpha)$ and $G_\alpha \subset G_{\rho(\alpha)} = N_G(\rho(\alpha))$, the normalizer of $\rho(\alpha)$ in G. As example of a G-poset, consider $(S(G), \mathrm{id})$.

Definition 1.1 Let (Π, ρ) be a G-poset. A G-space X is called a *(Π, ρ)-space* if it is equipped with a specified subspaces $\{X_\alpha \mid \alpha \in \Pi\}$, satisfying the following three conditions:
(i) $gX_\alpha = X_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,
(ii) $X_\alpha \subsetneqq X_\beta$ if $\alpha \leqq \beta$ in Π, and
(iii) for any $H \in S(G)$,

$$X^H = \coprod_{\rho(\alpha)=H} X_\alpha.$$

If X is a G-CW-complex and X_α's are subcomplexes we call $(X, \{X_\alpha \mid \alpha \in \Pi\})$ also (Π, ρ)-complex. Let X and Y be (Π, ρ)-complexes. A map $f\colon X \to Y$ is a *(Π, ρ)-map* if it is a cellular G-map such that $f(X_\alpha) \subset Y_\alpha$ for any $\alpha \in \Pi$. A (Π, ρ)-map is simply written to be a Π-map. Let (Π, ρ) be the G-poset. By a family $\mathcal{F} \subset \Pi$ is meant any G-invariant subset. A *$\mathcal{F}$-complex* X is a (Π, ρ)-complex such that, for any $\alpha \in \Pi$, $X_\alpha = \cup\{X_\beta \mid \beta \in \mathcal{F}, \beta \leq \alpha\}$. Let $\alpha \in \mathcal{F}$. The G-set $G/\rho(\alpha)$ has a unique structure as (Π, ρ)-complex obtained by setting

$$(G/\rho(\alpha))_\beta = \{g\rho(\alpha) \mid g\alpha \leq \beta\} \text{ for any } \beta \in \Pi.$$

Note that $G/\rho(\alpha)$ is a $G\alpha$-complex and an $\mathcal{F}$-complex for any family $\mathcal{F}$ containing α. A quotient set $\mathcal{F}/G$ consists of all orbits of $\mathcal{F}$ under G. Let $\mathcal{A} \subset \mathcal{F}$ be a complete set of representatives for $\mathcal{F}/G$. A (Π, ρ)-homotopy equivalence class of a $\mathcal{F}$-complex X is denoted by $[X]_h$.

One obtains the Grothendieck group of all (Π, ρ)-homotopy equivalence classes of $\mathcal{F}$-complexes modulo $\langle [Y]_h - [X]_h - [C_f]_h \rangle$, where f runs over the set

$$\{f\colon X \to Y \text{ is a } \Pi\text{-map} \mid X \text{ and } Y \text{ are } (\Pi, \rho)\text{-complexes}\}.$$

The set is denoted by $U(G, \mathcal{F})$ which is also called the *universal Burnside module*.

The theory of Burnside module would be better developed in the category of finite, base pointed G-CW-complexes. Let X and Y be base pointed, finite G-CW-complexes. For $[X]_h$, $[Y]_h \in U(G, \mathcal{F})$, we set

$$[X]_h + [Y]_h = [X \vee Y]_h.$$

Our main theorem is the following.

Theorem 1.2 *The group $U(G, \mathcal{F})$ is an abelian group generated by $\{[G/\rho(\alpha)^+]_h \mid \alpha \in \mathcal{A}\}$, where $(G/\rho(\alpha))^+$ is the disjoint union of $G/\rho(\alpha)$ and a base point. If G is a compact Lie group, then $U(G, \mathcal{F})$ is a free abelian group with a basis $\{[G/\rho(\alpha)^+]_h \mid \alpha \in \mathcal{A}\}$.*

If G is a finite group, we have $U(G, \mathcal{F}) \cong \Omega(G, \mathcal{F})$, where $\Omega(G, \mathcal{F})$ is the Burnside module by Oliver-Petrie [7]. For a base pointed, finite G-CW-complex X there exists integers a_α, such that

$$[X]_c = \sum_{\alpha \in \mathcal{A}} a_\alpha [(G/\rho(\alpha))^+]_c,$$

where the sign $[\quad]_c$ is an equivalence class, which will be explained in §2.

In the rest of this paper we describe the detail of the above result, although we will argue using the category of finite, base pointed G-CW-complexes not that of finite G-CW-complexes. In Section 2, we review fundamentals of Burnside module. Finally, in Section 3, we prove Theorem 1.2.

2. Basic properties of Burnside modules

We shall review fundamental properties on the Burnside module. See our general reference R. Oliver-T. Petrie [7] for details. In this section, we assume that G is a finite group. A base pointed version of a (Π, ρ)-complex replaces Definition 1.1 (iii) with the following condition:

$$\text{for any } H \in S(G), \quad X^H = \bigvee_{\rho(\alpha)=H} X_\alpha \quad \text{(the wedge sum of } X_\alpha\text{'s)}.$$

Here each subcomplex X_α ($\alpha \in \Pi$) has a base point q. Let X and Y are finite, base pointed (Π, ρ)-complexes. We call them equivalent if $\chi(X_\alpha) = \chi(Y_\alpha)$ for all $\alpha \in \Pi$ (χ is the Euler characteristic). An equivalence class of X is denoted by $[X]_c$. Wedge sum together with the definition $(X \vee Y)_\alpha = X_\alpha \vee Y_\alpha$ defines an additon of equivalence classes. As a result one obtains an additve abelian group

$$\Omega(G,\,\Pi) = \{[X]_c \mid X \text{ is a } (\Pi,\,\rho)\text{-complex}\},$$

which is called a *Burnside module associated to* Π. This notion is a generalization of the Burnside ring $\Omega(G)$. Obviously

$$\Omega(G) = \Omega(G,\, S(G)).$$

Let α be any element of Π and X a $(\Pi,\,\rho)$-complex. Construct a new space X' by attaching α-cells $G/\rho(\alpha) \times D^i$'s to X. Each attachment map

$$\varphi\colon G/\rho(\alpha) \times S^{i-1} \to X$$

is defined such that $\varphi(g\rho(\alpha) \times S^{i-1}) \subset X_{g\alpha}$. The space X' is equipped with a $(\Pi,\,\rho)$-complex structure:

$$(X')_\beta = X_\beta \cup \left(\bigcup \{g\rho(\alpha) \times D^i \mid g\alpha \le \beta,\, g \in G\}\right) \text{ for } \beta \in \Pi.$$

Any $(\Pi,\,\rho)$-complex is constructed from one point by attaching α-cells for $\alpha \in \Pi$. Clearly, a $(\Pi,\,\rho)$-complex is an $\mathcal{F}$-complex if and only if it is constructed by attaching α-cells for $\alpha \in \mathcal{F}$. We describe generators for the Burnside module $\Omega(G,\,\mathcal{F})$. A set $A \subset \mathcal{F}$ is the same as mentioned in §1. Then

Proposition 2.1 ([7, Proposition 1.5])

$$\Omega(G,\,\mathcal{F}) \cong \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}.$$

Any finite $(\Pi,\,\rho)$*-complex* X *is equivalent in* $\Omega(G,\,\mathcal{F})$ *to a sum of the form* $\sum_{\alpha\in\mathcal{A}} a_\alpha[(G/\rho(\alpha))^+]_c$*, and the map* $[X]_c \to \{a_\alpha\}_{\alpha\in\mathcal{A}}$ *defines the isomorphism.*

3. Proof of Theorem 1.2

We recall a definition. The set $U(G,\,\mathcal{F})$ is called the *universal Burnside module* if it is a Grothendieck group of all $(\Pi,\,\rho)$-homotopy equivalence classes of $\mathcal{F}$-complexes modulo $\langle [Y]_h - [X]_h - [C_f]_h \rangle$, where f runs over the set

$$\{f\colon X \to Y \text{ is a } \Pi\text{-map} \mid X \text{ and } Y \text{ are } (\Pi,\,\rho)\text{-complexes}\}.$$

Lemma 3.1 *For any $\mathcal{F}$-complex X, we have*

$$[\Sigma X]_h = -[X]_h \quad \text{in } U(G,\, \mathcal{F}).$$

Proof. Let X be a $\mathcal{F}$-complex and CX the cone of X. Let $\iota\colon X \to CX$ is an inclusion map. By definition, we obtain

$$[CX]_h = [X]_h + [C_\iota]_h.$$

Since CX is contractible, it follows that $[CX]_h = 0$ in $U(G,\, \mathcal{F})$. Note that C_ι is homeomorphic to ΣX, and proves the lemma. □

Let X be a base pointed topological space. The symbol Σ^i means a reduced i-th suspension operator, that is, $\Sigma^0 X = X$, $\Sigma^i X = \Sigma\big(\Sigma^{i-1} X\big)$.

Corollary 3.2 *One has that*

$$\left[\Sigma^{k+1} X\right]_h = -\left[\Sigma^k X\right]_h \quad \text{in } U(G,\, \mathcal{F}).$$

Proof. Clear by definition and Lemma 3.1. □

Proof of Theorem 1.2. We proceed by induction on the dimension of $\mathcal{F}$-complex. Since any $\mathcal{F}$-complex X' is constructed by attaching α-cells for $\alpha \in \mathcal{F}$ to the lower dimensional complex X. That is,

$$X' = X \bigcup_{\varphi} (G/\rho(\alpha) \times D^i),$$

where $\varphi\colon G/\rho(\alpha) \times S^{i-1} \to X$ is an attachment map such that $\varphi(g\rho(\alpha) \times S^{i-1}) \subset X_{g\alpha}$. If $\iota\colon X \to X'$ is an inclusion map, C_ι is $(\Pi,\, \rho)$-homotopy equivalent to $\Sigma^i (G/\rho(\alpha))^+$.
By definition,

$$[X']_h = [X]_h + [C_\iota]_h.$$

Hence we have

$$\begin{aligned} [X']_h &= [X]_h + [\Sigma^i (G/\rho(\alpha))^+]_h \\ &= [X]_h + (-1)^i [(G/\rho(\alpha))^+]_h. \end{aligned}$$

By assumption of induction, $[X]_h$ is generated by the $[(G/\rho(\alpha))^+]_h$ for $\alpha \in \mathcal{A}$. Assume that G is a compact Lie group. We shall show that the $[(G/\rho(\beta))^+]_h$'s are linearly independent. Suppose $\sum_{\beta \in \mathcal{A}} n_\beta [(G/\rho(\beta))^+]_h =$

0 in $U(G, \mathcal{F})$. Let α be minimal with respect to inclusion such that $n_\alpha \neq 0$. Then we have

$$0 = \bar{\chi}_{\alpha,\rho(\alpha)}(0) = \sum_{\beta \in \mathcal{A}} n_\beta \bar{\chi}(((G/\rho(\beta))^+_\alpha)^{\rho(\alpha)}) = n_\alpha \bar{\chi}(G_\alpha/\rho(\alpha)),$$

where $\bar{\chi}_{\alpha,K}([X]_h) = \bar{\chi}(X^K_\alpha)$ for any subgroup K of G and $\bar{\chi}(X) = \chi(X) - 1$ for any space X. Let $W\rho(\alpha)$ be a group $N\rho(\alpha)/\rho(\alpha)$, the Weyl group of $\rho(\alpha)$. When $W\rho(\alpha)$ is a finite group, it follows that $G_\alpha/\rho(\alpha)$ is finite. Then we have that n_α is zero, this is a contradiction. We consider the case $W\rho(\alpha)$ is not finite. By definition,

$$(G/\rho(\alpha))^{\rho(\alpha)}_\alpha = (G/\rho(\alpha))_\alpha = G_\alpha/\rho(\alpha).$$

Since $(G/\rho(\alpha))^{\rho(\alpha)}$ carries a free $W\rho(\alpha)$-action and hence a free S^1 -action, then

$$\chi((G/\rho(\alpha))^{\rho(\alpha)}) = 0.$$

Thus $\chi((G/\rho(\alpha))^{\rho(\alpha)}_\alpha) = 0$, and our theorem is proved. □

Proposition 3.3 *For a finite group G, we have $U(G, \mathcal{F}) \cong \Omega(G, \mathcal{F})$.*

Proof. A map $\psi\colon U(G, \mathcal{F}) \to \Omega(G, \mathcal{F})\colon [(G/\rho(\alpha))^+]_h \mapsto [(G/\rho(\alpha))^+]_c$ is a surjective homomorphism by definition. Both $\{[(G/\rho(\alpha))^+]_h \mid \alpha \in \mathcal{A}\}$ and $\{[(G/\rho(\alpha))^+]_c \mid \alpha \in \mathcal{A}\}$ are linearly independent. Thus ψ is injective. □

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