Hokkaido Mathematical Journal Vol. 35 (2006) p. 427–436

Mixed basic subgroups

Takashi Okuyama

(Received October 19, 2004)

Abstract. Every torsion abelian group has basic subgroups and all its basic subgroups are isomorphic. We extend the concept of basic subgroups from torsion abelian groups to arbitrary abelian groups. The generalized basic subgroups are called *mixed basic subgroups*. An example of a mixed group is given in which not all mixed basic subgroups are isomorphic. This example also shows that a mixed basic subgroup of a splitting group need not be splitting.

Key words: mixed basic subgroup, purifiable subgroup, T-high subgroup, splitting group.

1. Introduction

We extend the concept of basic subgroups from p–groups to arbitrary abelian groups. *Mixed basic subgroups* of arbitrary abelian groups are defined as follows. Let G be an abelian group and T the maximal torsion subgroup of G. Then there exists a subgroup L of G such that

$$\begin{aligned} G/T(L) = L/T(L) \oplus T/T(L), \\ T(L) \text{ is a basic subgroup of } T. \end{aligned} \tag{1.1}$$

Such a subgroup L satisfies the following three conditions:
(1) $T(L)$ is a direct sum of cyclic groups;
(2) L is pure in G;
(3) G/L is torsion divisible.
Conversely, if a subgroup L of G satisfies the above three conditions, then L satisfies the equation (1.1). The following questions arise.

Problem 1 Are all mixed basic subgroups of an abelian group isomorphic ?

Problem 2 Which properties of a mixed group can be translated into corresponding properties of a mixed basic subgroup and thus reduced to the case that the torsion part is a direct sum of cyclic groups ?

1991 Mathematics Subject Classification : 20K21, 20K27.

This work is supported by Grant–in–Aid for Science Research (c)

In this note, we consider Problem 1.

In Section 3, we give the definition and a few properties of mixed basic subgroups. In Section 4, we give an example of a mixed group that shows that the answer to the Problem 1 is no and that not all mixed basic subgroups of splitting groups are splitting.

All groups considered in this note are arbitrary abelian groups unless stated otherwise. The terminologies and notations not expressly introduced follow the usage of [1]. Throughout this note, $\mathbf{Z}$ denotes the ring of integers, $\mathbf{N}$ the set of all positive integers, $\mathbf{P}$ the set of all prime integers, p always denotes a prime, T the maximal torsion subgroup, and G_p the p–component of an abelian group G.

2. Notation and basics

We recall definitions and properties mentioned in [2]. We frequently use them in this note.

From the definition [2, Definition 1.1] of p–almost–dense subgroups and its characterization [2, Proposition 1.3, Proposition 1.4], we can define almost–dense subgroups as follows.

Definition 2.1 A subgroup A of a group G is said to be *almost–dense* in G if, for all integers $n \geqq 0$ and for every $p \in \mathbf{P}$,

$$p^n G[p] \subseteq A + p^{n+1} G.$$

Recall definition of purifiable subgroups in a group G.

Definition 2.2 Let G be a group. A subgroup A of G is said to be *purifiable* in G if, among the pure subgroups of G containing A, there exists a minimal one. Such a minimal pure subgroup is called a *pure hull* of A.

Proposition 2.3 [2, Theorem 1.8, Theorem 1.11] *Let G be a group and A a subgroup of G. Let H be a pure subgroup of G containing A. Then H is a pure hull of A in G if and only if the following three conditions are satisfied*:

(1) *A is almost–dense in H*;

(2) *H/A is torsion*;

(3) [*for every $p \in \mathbf{P}$*,] *there exists a nonnegative integer m_p such that*

$$p^{m_p} H[p] \subseteq A.$$

From Proposition 2.3, for purifiable torsion–free subgroups, we immediately obtain the following.

Corollary 2.4 *Let G be a group and A a subgroup of G. Suppose that A is purifiable in G. Let H be a pure hull of A in G. If A is torsion–free, then H_p is bounded for all $p \in \mathbf{P}$.*

The following is a relationship between purifiability and p–purifiability.

Proposition 2.5 [2, Theorem 1.12] *Let G be a group. A subgroup A of G is purifiable in G if and only if, for every $p \in \mathbf{P}$, A is p–purifiable in G.*

Definition 2.6 Let G be a group and A a subgroup of G. For every nonnegative integer n, we define the *nth p–overhang* of A in G to be the vector space

$$V_{p,\,n}(G,\,A) = \frac{(A + p^{n+1}G) \cap p^nG[p]}{(A \cap p^nG)[p] + p^{n+1}G[p]}.$$

Moreover, the set

$$O_A^G(p) = \{t \mid V_{p,\,t}(G,\,A) \neq 0\}. \tag{2.7}$$

is called the *p–overhang* set of A in G.

We immediately obtain the following properties.

Proposition 2.8 *Let G be a group and A a subgroup of G. If $G_p = 0$, then $O_A^G(p) = \emptyset$.*

Proposition 2.9 [2, Proposition 2.2] *Let G be a group and A a subgroup of G. For p–pure subgroup K of G containing A,*

$$O_A^G(p) = O_A^K(p).$$

The following is useful when we consider purifiable torsion–free subgroups. A proof was written in [5, Corollary 2.15].

Corollary 2.10 *Let G be a group and A a subgroup of G. Suppose that A is purifiable torsion–free in G. Let H be a pure hull of A in G. Then the following are equivalent*:

(1) $O_A^G(p) = \emptyset$ *for all* $p \in \mathbf{P}$;

(2) *H is torsion–free.*

We conclude this section with definition of N–high subgroups.

Definition 2.11 Let G be a group and let A and N be subgroups of G. If A is maximal with respect to the property of being disjoint from N, then A is called an N*–high* subgroup of G.

3. Mixed basic subgroups

It is well–known that there exist basic subgroups for all torsion groups. In this section we extend the concept of basic subgroups from torsion groups to arbitrary abelian groups.

Proposition 3.1 *Let G be a group and A a T–high subgroup of G. Then there exists a subgroup L containing A such that*

$$G/T(L) = L/T(L) \oplus T/T(L) \tag{3.2}$$

and $T(L)$ is a basic subgroup of T.

Proof. Let B be a basic subgroup of T. Since T/B is divisible, T/B is an absolute direct summand of G/B. Since $(A+B)/B \cap T/B = 0$, there exists a subgroup L of G containing A such that

$$G/B = L/B \oplus T/B.$$

Since L/B is torsion–free, $B = T(L)$. □

The subgroup L in Proposition 3.1 satisfying (3.2) satisfies the following three conditions:
(1) $T(L)$ is a direct sum of cyclic groups;
(2) L is pure in G;
(3) G/L is torsion divisible.
It is well–known that, if G is torsion, then a subgroup L of G satisfying the above three conditions is a basic subgroup of G. In general, we show that, for any group G, the conditions (1)–(3) imply (3.2).

Theorem 3.3 *Let L be a subgroup of a group G. Then L satisfies the following three condition*:
(1) *$T(L)$ is a direct sum of cyclic groups*;
(2) *L is pure in G*;
(3) *G/L is torsion divisible,*
if and only if

$$G/T(L) = L/T(L) \oplus T/T(L)$$

and $T(L)$ is a basic subgroup of T.

Proof. It suffices to show necessity. First we prove that $G = T + L$. Let $g \in G$. By the conditions (3) and (2), $mg \in L \cap mG = mL$ for some integer m. Then we have $mg = mx$ for some $x \in L$. Since $g - x \in T$, $g \in T + L$. Hence $G = T + L$ and so $G/T(L) = L/T(L) \oplus T/T(L)$.

By the condition (3), $T/T(L)$ is divisible. Hence, by the condition (1), $T(L)$ is a basic subgroup of T. □

Proposition 3.1 and Theorem 3.3 establish the existence of mixed basic subgroups of arbitrary abelian groups.

Definition 3.4 A subgroup L of a group G is said to be a *mixed basic subgroup* of G if L satisfies the following three conditions:
(1) $T(L)$ is a direct sum of cyclic groups;
(2) L is pure in G;
(3) G/L is torsion divisible.

By Proposition 3.1, Theorem 3.3, and [1, Lemma 35.1], we immediately have the following corollary.

Corollary 3.5 *Let G be a group. If G_p is unbounded for some $p \in \mathbf{P}$, then we obtain an infinite properly decreasing chain*

$$L_1 \supset L_2 \supset \cdots \supset L_n \supset \ldots$$

where every subgroup L_i is a mixed basic subgroup of G for $i \geqq 1$ such that $T(L_i)$ are all isomorphic.

Proposition 3.1 and Theorem 3.3 combined lead to the following useful property.

Corollary 3.6 *Let G be a group. Let A be any subgroup such that $A \cap T = 0$ and B any basic subgroup of T. Then there exists a mixed basic subgroup L containing A such that $B = T(L)$.*

4. An example

By definition of purifiable subgroups, the following is immediate.

Proposition 4.1 *Let G be a group and A a torsion–free subgroup of G. If A is purifiable in some mixed basic subgroup of G, then A is purifiable in G.*

We now present a splitting mixed group G containing a purifiable torsion–free subgroup A and a mixed basic subgroup L such that $A \subseteq L$ but A is not purifiable in L. This example also shows that not all mixed basic subgroups of splitting groups are splitting and not all mixed basic subgroups are isomorphic.

Example 4.2 Let $G = A \oplus B$ where $A = \mathbf{Z}[p^{-1}] = \{\frac{m}{p^n} \mid m \in \mathbf{Z},\, n \in \mathbf{N}\}$ and $B = \bigoplus_{n=1}^{\infty} \langle x_n \rangle$ with $o(x_n) = p^n$. Let $a = 1$ and $a_n = \frac{1}{p^n}$. Define

$$L = \langle a_n + x_n \mid n \geqq 1 \rangle.$$

Then we have the following properties. First, by Definition 3.4, we obtain:

Property 4.3 G is a mixed basic subgroup of G and $T(G) = B$.

By the definition of purifiability and Proposition 2.3, we have:

Property 4.4 $\langle a \rangle$ is purifiable in G and A is a pure hull of $\langle a \rangle$ in G.

We will show that L is a mixed basic subgroup of G by verifying the conditions of Definition 3.4.

Property 4.5 For every $n \geqq 1$, let $y_n = x_n - px_{n+1}$. Then the following hold.

(1) $y_n \in L$ for all $n \geqq 1$.

(2) Let $B' = \langle y_n \mid n \geqq 1 \rangle$. Then

$$B' = \bigoplus_{n=1}^{\infty} \langle y_n \rangle$$

is pure in B and the maximal torsion subgroup of L.

(3) $x_n \notin B'$ for all $n \geqq 1$.

(4) B/B' is divisible. Hence B' is a proper basic subgroup of B.

Proof. (1) By definition, $a_n - pa_{n+1} = 0$. Hence

$$\begin{aligned} y_n &= x_n - px_{n+1} \\ &= (a_n + x_n) - p(a_{n+1} + x_{n+1}) - (a_n - pa_{n+1}) \\ &= (a_n + x_n) - p(a_{n+1} + x_{n+1}) \end{aligned}$$

and so $y_n \in L$.

Except for $B' = T(L)$, all assertions are clear by [1, Lemma 35.1]. So we prove $B' = T(L)$.

We show $L[p] = B'[p]$. Let $x \in L[p]$. Then

$$x = \sum_{i=1}^{k} \beta_i(a_i + x_i) \quad \text{and} \quad px = 0 \tag{4.6}$$

where β_i is an integer for $1 \leqq i \leqq k$. Since $G = A \oplus B$, we have

$$\sum_{i=1}^{k} \beta_i a_i = 0, \quad x = \sum_{i=1}^{k} \beta_i x_i. \tag{4.7}$$

Since $px = 0$, by (4.7), $\sum_{i=1}^{k} \beta_i p x_i = 0$. Since $\{x_i\}$ is p–independent in B, we have $\beta_i = p^{i-1}\beta'_i$ for some integer β'_i. By (4.7), $\sum_{i=1}^{k} \beta'_i p^{i-1} a_i = 0$. Note that, by definition, $p^{i-1}a_i = a_1$. Therefore

$$\sum_{i=1}^{k} \beta'_i = 0. \tag{4.8}$$

and

$$\begin{aligned}
x &= \sum_{i=1}^{k-1} \beta'_i p^{i-1} x_i \\
&= \beta'_1(x_1 - px_2) + (\beta'_1 + \beta'_2)px_2 + \sum_{i=3}^{k} \beta'_i p^{i-1} x_i \\
&= \beta'_1(x_1 - px_2) + (\beta'_1 + \beta'_2)p(x_2 - px_3) \\
&\quad + (\beta'_1 + \beta'_2 + \beta'_3)p^2 x_3 + \sum_{i=4}^{k} \beta_i p^{i-1} x_i \\
&= \dots \\
&= \beta'_1(x_1 - px_2) + (\beta'_1 + \beta'_2)p(x_2 - px_3) \\
&\quad + (\beta'_1 + \beta'_2 + \beta'_3)p^2(x_3 - px_4) + \dots \\
&\quad + (\beta'_1 + \beta'_2 + \cdots + \beta'_{k-1})p^{k-2}(x_{k-1} - px_k) \\
&\quad + \left(\sum_{i=1}^{k} \beta'_i\right) p^{k-1} x_k
\end{aligned}$$

$$= \sum_{i=1}^{k-1}\left(\sum_{j=1}^{i}\beta_j'\right)p^{i-1}y_i + \left(\sum_{i=1}^{k}\beta_i'\right)p^{k-1}x_k$$
$$\overset{(4.8)}{=} \sum_{i=1}^{k-1}\left(\sum_{j=1}^{i}\beta_j'\right)p^{i-1}y_i \in B'[p].$$

Hence we have

$$L[p] = B'[p]. \tag{4.9}$$

Note that $B' \subseteq T(L) \subseteq B$. By Property 4.5(2), B' is pure in B and in $T(L)$. Further, by (4.9), $B' = T(L)$ is the maximal torsion subgroup of L. □

Property 4.10 L is a proper mixed basic subgroup of G.

Proof. By definition, $G = L + B$. By Property 4.5(2), $G/B' = L/B' \oplus B/B'$ and $B' = T(L)$. Further, by Property 4.5(4), $B' = T(L)$ is a proper basic subgroup of B. Hence, by Theorem 3.3, L is a proper mixed basic subgroup of G. □

Before showing a few properties of G, we need the following lemma.

Lemma 4.11 *Let G be a group and H a pure subgroup of G such that G/H is torsion. Then the following hold.*
(1) *$G = H + T$.*
(2) *If $G_p = H_p \oplus U_p$ for every $p \in \mathbf{P}$, then $G = H \oplus U$ where $U = \oplus_{p\in\mathbf{P}} U_p$.*
(3) *Suppose that A is purifiable in G. Let K be a pure hull of A in G. If G/A is torsion, then $G = K \oplus T'$ for some subgroup T' of T.*

Proof. (1) Let $g \in G$. Since G/H is torsion, $ng \in H \cap nG = nH$ for some $n \in \mathbf{Z}$. Then we have $ng = nh$ for some $h \in H$ and $g - h \in T$. Hence $G = H + T$.
(2) By hypothesis and (1), the assertion is clear.
(3) By Proposition 2.5 and Corollary 2.4, for every $p \in \mathbf{P}$, K_p is bounded and so K_p is a direct summand of G_p. If G/A is torsion, then, by (2), the assertion is confirmed. □

In [4, Theorem], we proved that all pure hulls of purifiable torsion–free subgroups are isomorphic. However, in the group G of Example 4.2, we can easily prove directly that all pure hulls of $\langle a\rangle$ in G are isomorphic.

Property 4.12 All pure hulls of $\langle a\rangle$ in G are isomorphic.

Proof. By Property 4.4, A is a pure hull of $\langle a \rangle$ in G. Let K be another pure hull of $\langle a \rangle$ in G. Note that $G/\langle a \rangle$ is torsion. Thus, by Lemma 4.11, $G = K \oplus T'$ for some subgroup T' of B. Since A is torsion–free, by Proposition 2.8, $O^A_{\langle a \rangle}(p) = \emptyset$. By Proposition 2.9, $O^K_{\langle a \rangle}(p) = \emptyset$. Further, by Corollary 2.10, K is torsion–free. Therefore $T' = T(G) = B$. Hence $A \cong G/B \cong K$. □

Property 4.13 L is not splitting.

Proof. Suppose that L is splitting. Then $L = F \oplus B'$ where F is a torsion–free subgroup of L. Since $G = L + B$, we have

$$G = L + B = F + B + B' = F \oplus B.$$

Hence $A \cong G/B \cong F$ and F is p–divisible. Since A is the maximal p–divisible subgroup of G, $F \subseteq A$ and hence $F = A$. This contradicts $x_1 \notin L$. Hence L is not splitting. □

Property 4.13 leads to the following general result.

Corollary 4.14 *There exist splitting groups with a mixed subgroup that is not splitting.*

Property 4.13 also leads to the following result.

Property 4.15 $L \ncong G$.

Property 4.15 leads to the following general result.

Corollary 4.16 *There exist groups that contain non–isomorphic mixed subgroups.*

Property 4.17 $\langle a \rangle$ is not purifiable in L.

Proof. Suppose that $\langle a \rangle$ is purifiable in L. Let M be a pure hull of $\langle a \rangle$ in L. By definition of G, $L/\langle a \rangle$ is torsion. Hence, by Lemma 4.11, $L = M \oplus U$ for some subgroup U of B'. Since A is torsion–free, by Proposition 2.8, $O^A_{\langle a \rangle}(p) = \emptyset$ for all $p \in \mathbf{P}$. By Proposition 2.9, $O^M_{\langle a \rangle}(p) = \emptyset$ for all $p \in \mathbf{P}$. Further, by Corollary 2.10, M is torsion–free and L is splitting. This contradicts Property 4.13. Hence $\langle a \rangle$ is not purifiable in L. □

Property 4.17 also follows from [3, Theorem 3.2], because $h^L_p(a) = \omega$.

Acknowledgment Part of this work was completed during the author's visit at the University of Hawaii. The author thanks the Department of

Mathematics at the University of Hawaii and Professor Adolf Mader for their hospitality.

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Department of Mathematics
Faculty of Engineering
Oita University
700, Dan–noharu, Oita-shi
Oita–ken, 870–1192, Japan
E-mail: okuyama@csis.oita-u.ac.jp