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The singular Embry quartic moment problem

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Abstract. Given a collection of complex numbers $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i+j \le 2n,\ |i-j| \le n)$ with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, we consider the moment problem for γ in the case of $n = 2$, which is refered to Embry quartic moment problem. In this note we give a solution for the singular case.

Key words: truncated complex moment problem, representing measure, quartic moment problem, flat extension.

1. Introduction and Preliminaries

In [8, Proposition 2.8], it was shown that Bram-Halmos' characterization for subnormality of a cyclic operator induces moment matrices $M(n)$ which were studied in [3] and [4]. As a parallel study, in [8, Proposition 2.8] they obtained matrices $E(n)$ corresponding to the Embry's characterization of such operator.

For $n \in \mathbb{N}$, let $m = m(n) := ([n/2]+1)([(n+1)/2]+1)$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the algebra of $m \times m$ complex matrices), we denote the successive rows and columns according to the following ordering:

$$\underbrace{\mathit{1}}_{(1)},\ \underbrace{Z}_{(1)},\ \underbrace{Z^2,\ \bar{Z}Z}_{(2)},\ \underbrace{Z^3,\ \bar{Z}Z^2}_{(2)},\ \underbrace{Z^4,\ \bar{Z}Z^3,\ \bar{Z}^2Z^2}_{(3)},\ \ldots \tag{1.1}$$

For a collection of complex numbers

$$\gamma \equiv \{\gamma_{ij}\} \quad (0 \le i+j \le 2n,\ |i-j| \le n) \quad \text{with } \gamma_{00} > 0 \text{ and } \gamma_{ji} = \bar{\gamma}_{ij}, \tag{1.2}$$

we define the moment matrix $E(n) \equiv E(n)(\gamma)$ in $\mathcal{M}_m(\mathbb{C})$ as follows:

$$E(n)_{(k,l)(i,j)} := \gamma_{l+i,\,j+k}.$$

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For example, if $n=2$, i.e.,

$$\gamma: \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{12}, \gamma_{21}, \gamma_{13}, \gamma_{22}, \gamma_{31},$$

then we obtain the moment matrix

$$E(2) = \begin{array}{c} \begin{array}{cccc} \mathit{1} & Z & Z^2 & \bar{Z}Z \end{array} \\ \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{array}.$$

We consider a collection of complex numbers γ as in (1.2). *The (Embry) truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu(z) \quad (0 \le i+j \le 2n,\ |i-j| \le n);$$

μ is called a *representing measure* for γ.

The Embry quadratic moment problem, that is for $n=1$, was solved completely (see [9]). In this paper, we solve the singular quartic moment problem of $E(2)$ according to its ranks. According to [8, Proposition 3.10], we must characterize the double flat extension $E(4)$.

Some of the calculations in this article were obtained throughout computer experiments using the software tool *Mathematica* [10].

2. Solution of The Singular Case

Assume that $E(2)$ is positive and let $r := \operatorname{rank} E(2)$. Then obviously $1 \le r \le 4$. The singular case is of $\det E(2) = 0$, i.e., $r = 1$, 2 and 3.

2.1. The case of $r = 1$

By a direct computation, we have the following proposition.

Proposition 2.1 *Assume that $E(2) \ge 0$ and $r = 1$. Then there exists the unique flat extension $E(3)$ of $E(2)$. Therefore γ admits the unique 1-atomic representing measure $\mu = \gamma_{00}\delta_{\gamma_{01}/\gamma_{00}}$.*

Example 2.2 Let us consider a matrix

$$E(2) = \begin{pmatrix} 1 & 1+i & 2i & 2 \\ 1-i & 2 & 2+2i & 2-2i \\ -2i & 2-2i & 4 & -4i \\ 2 & 2+2i & 4i & 4 \end{pmatrix}.$$

$E(2)$ is positive with $\operatorname{rank} E(2) = 1$. The representing measure is δ_{1+i}.

2.2. The case of $r = 2$

Assume that rank $E(2) = 2$. Then

$$Z^2 = \alpha \mathit{1} + \beta Z \quad \text{and} \quad \bar{Z}Z = \alpha' \mathit{1} + \beta' Z, \tag{2.1}$$

for some complex numbers α, β, α', β'. By a direct computation, we have

$$\alpha = -\frac{\gamma_{01}\gamma_{12} - \gamma_{02}\gamma_{11}}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}, \quad \beta = \frac{\gamma_{00}\gamma_{12} - \gamma_{10}\gamma_{02}}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}},$$

$$\alpha' = -\frac{\gamma_{01}\gamma_{21} - \gamma_{11}^2}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}, \quad \beta' = \frac{\gamma_{00}\gamma_{21} - \gamma_{10}\gamma_{11}}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}.$$

Proposition 2.3 *Assume that $E(2) \geq 0$ and $r = 2$. If*

$$\bar{\alpha}\gamma_{12} + \bar{\beta}\gamma_{22} = \alpha'\gamma_{21} + \beta'\gamma_{22}, \tag{2.2}$$

then there exists a unique flat extension $E(3)$ of $E(2)$. Therefore, γ admits unique 2-atomic representing measure $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1}$, the two atoms z_0, z_1 are the roots of

$$z^2 - (\alpha + \beta z) = 0, \tag{2.3}$$

and the densities are

$$\rho_0 = \frac{\gamma_{01} - \gamma_{00}z_1}{z_0 - z_1} \quad \textit{and} \quad \rho_1 = \frac{z_0\gamma_{00} - \gamma_{01}}{z_0 - z_1}.$$

Proof. By (2.1), we have

$$Z^3 = \alpha Z + \beta Z^2 \quad \text{and} \quad \bar{Z}Z^2 = \alpha' Z + \beta' Z^2. \tag{2.4}$$

Let us take $\gamma_{32} := \bar{\alpha}\gamma_{12} + \bar{\beta}\gamma_{22}$ (or $= \alpha'\gamma_{21} + \beta'\gamma_{22}$). Then

$$\begin{aligned} \alpha\gamma_{31} + \beta\gamma_{32} &= \alpha(\alpha'\gamma_{20} + \beta'\gamma_{21}) + \beta(\alpha'\gamma_{21} + \beta'\gamma_{22}) \\ &= \alpha'(\alpha\gamma_{20} + \beta\gamma_{21}) + \beta'(\alpha\gamma_{21} + \beta\gamma_{22}) \\ &= \alpha'\gamma_{22} + \beta'\gamma_{23}. \end{aligned}$$

Since $E(2)$ admits a flat extension $E(3)$ if and only if

$$\alpha\gamma_{31} + \beta\gamma_{32} = \alpha'\gamma_{22} + \beta'\gamma_{23}, \tag{2.5}$$

$E(2)$ admits a flat extension $E(3)$. The remaining parts follow from [8, Theorem 3.9]. $\square$

Notice that the flat extension $E(3)$ of $E(2)$ can be written as

$$A = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} \\ \gamma_{30} & \gamma_{31} & \gamma_{32} & \gamma_{41} & \gamma_{33} & \gamma_{42} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{32} & \gamma_{24} & \gamma_{33} \end{pmatrix}$$

with

$$\begin{aligned} &\gamma_{03} = \alpha\gamma_{01} + \beta\gamma_{02}, \ \gamma_{23} = \alpha\gamma_{21} + \beta\gamma_{22}, \ \gamma_{14} = \alpha\gamma_{12} + \beta\gamma_{13}, \\ &\gamma_{33} = \alpha\gamma_{31} + \beta\gamma_{32}, \quad \text{and} \quad \gamma_{24} = \alpha\gamma_{22} + \beta\gamma_{23}, \end{aligned} \tag{2.6}$$

which can be used in the following example.

Example 2.4 Let us consider a positive matrix

$$E(2) = \begin{pmatrix} 1 & i & -2 & 2 \\ -i & 2 & 0 & 0 \\ -2 & 0 & 8 & -8 \\ 2 & 0 & -8 & 8 \end{pmatrix}$$

with $E(2) = 2$. By a simple computation we have $\alpha = -4$, $\beta = -2i$, $\alpha' = 4$, and $\beta' = 2i$, so that (2.2) may hold. By (2.7), we have $\gamma_{03} = 0$, $\gamma_{23} = -16i$, $\gamma_{14} = 16i$, $\gamma_{33} = 64$, $\gamma_{24} = -64$. Thus, the flat extension of $E(2)$ is

$$A = \begin{pmatrix} 1 & i & -2 & 2 & 0 & 0 \\ -i & 2 & 0 & 0 & -8 & 8 \\ -2 & 0 & 8 & -8 & -16i & 16i \\ 2 & 0 & -8 & 8 & 16i & -16i \\ 0 & -8 & 16i & -16i & 64 & -64 \\ 0 & 8 & -16i & 16i & -64 & 64 \end{pmatrix}.$$

According to Proposition 2.3, we obtain the representing measure

$$\mu = \left(\frac{1}{5}\sqrt{5} + \frac{1}{2}\right)\delta_{(\sqrt{5}-1)i} + \left(-\frac{1}{5}\sqrt{5} + \frac{1}{2}\right)\delta_{-(\sqrt{5}+1)i}.$$

2.3. The case of $r = 3$

For a positive $n \times n$ matrix A, let us denote by $[A]_k$ the compression of A to the first k rows and columns. We denote by M_{ij} the determinant of the cofactor of $E(2)$ with respect to (i, j) and $\Delta_d = \det([E(2)]_d)$, for $d = 1, 2, 3$, and 4.

We now assume that rank $E(2) = 3$. Then there exist a_0, a_1, a_2 in $\mathbb{C}$ such that

$$\bar{Z}Z = a_0 \mathit{1} + a_1 Z + a_2 Z^2. \tag{2.7}$$

In fact,

$$a_0 = \frac{M_{41}}{\Delta_3},\ a_1 = -\frac{M_{42}}{\Delta_3},\ a_2 = \frac{M_{43}}{\Delta_3}.$$

To establish a flat extension $E(3)$, we should choose suitable γ_{23}. By (2.7) we have

$$\bar{Z}Z^2 = a_0 Z + a_1 Z^2 + a_2 Z^3. \tag{2.8}$$

Let us take

$$\gamma_{23} := a_0\gamma_{12} + a_1\gamma_{13} + a_2\gamma_{14}. \tag{2.9}$$

Since $\{\mathit{1}, Z, Z^2, Z^3\}$ is linearly dependent, we have

$$Z^3 = b_0 \mathit{1} + b_1 Z + b_2 Z^2, \quad \text{for some} \quad b_i \in \mathbb{C}. \tag{2.10}$$

Then

$$b_0 = \frac{1}{\Delta_3} \cdot \begin{vmatrix} \gamma_{03} & \gamma_{01} & \gamma_{02} \\ \gamma_{13} & \gamma_{11} & \gamma_{12} \\ \gamma_{23} & \gamma_{21} & \gamma_{22} \end{vmatrix},$$

$$b_1 = \frac{1}{\Delta_3} \cdot \begin{vmatrix} \gamma_{00} & \gamma_{03} & \gamma_{02} \\ \gamma_{10} & \gamma_{13} & \gamma_{12} \\ \gamma_{20} & \gamma_{23} & \gamma_{22} \end{vmatrix},$$

and

$$b_2 = \frac{1}{\Delta_3} \cdot \begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{03} \\ \gamma_{10} & \gamma_{11} & \gamma_{13} \\ \gamma_{20} & \gamma_{21} & \gamma_{23} \end{vmatrix}.$$

Define

$$\gamma_{14} := b_0\gamma_{11} + b_1\gamma_{12} + b_2\gamma_{13}. \tag{2.11}$$

Note that γ_{23} is determined by γ_{14}. The following lemma is useful to establish Algorithm.

Lemma 2.5 *If $\Delta_3 \neq |M_{34}|$, then we can take the unique $\gamma_{23} \in \mathbb{C}$ satisfying* (2.9) *and* (2.11).

Proof. Let us consider γ_{23} in (2.9). Then we have

$$\begin{aligned} \gamma_{23} &= a_0\gamma_{12} + a_1\gamma_{13} + a_2\gamma_{14} \\ &= a_0\gamma_{12} + a_1\gamma_{13} + a_2(b_0\gamma_{11} + b_1\gamma_{12} + b_2\gamma_{13}), \end{aligned}$$

and thus

$$\Delta_3^2\gamma_{23} = M_{43}M_{34}\gamma_{23} + \Omega(= |M_{34}|^2\gamma_{23} + \Omega),$$

where

$$\begin{aligned} \Omega = {} & \Delta_3^2(a_0\gamma_{12} + a_1\gamma_{13}) \\ & + M_{43}\gamma_{11}\left(- \begin{vmatrix} \gamma_{01} & \gamma_{02} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} \gamma_{13} + \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} \gamma_{03} \right) \\ & + M_{43}\gamma_{12}\left(\begin{vmatrix} \gamma_{00} & \gamma_{02} \\ \gamma_{20} & \gamma_{22} \end{vmatrix} \gamma_{13} - \begin{vmatrix} \gamma_{10} & \gamma_{12} \\ \gamma_{20} & \gamma_{22} \end{vmatrix} \gamma_{03} \right) \\ & + M_{43}\gamma_{13}\left(- \begin{vmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{20} & \gamma_{21} \end{vmatrix} \gamma_{13} + \begin{vmatrix} \gamma_{10} & \gamma_{11} \\ \gamma_{20} & \gamma_{21} \end{vmatrix} \gamma_{03} \right). \end{aligned}$$

Since Ω does not have γ_{23} term and $\Delta_3 \neq |M_{34}|$, we may choose the unique datum γ_{23}. □

Lemma 2.6 *$E(2)$ has a flat extension of $E(3)$ if and only if we may take γ_{03} satisfying*

$$b_0\gamma_{30} + b_1\gamma_{31} + b_2\gamma_{32} = a_0\gamma_{22} + a_1\gamma_{23} + a_2\gamma_{24}, \tag{2.12}$$

such that γ_{23} satisfies (2.9) *and* (2.11).

Proof. Compare the columns of the flat extension $E(3)$ of $E(2)$. $\square$

Algorithm 2.7
(I) Determine γ_{23} by Lemma 2.5;
(II) Calculate b_0, b_1 and b_2;
(III) Define γ_{14}, γ_{24} as

$$\gamma_{14} := b_0\gamma_{11} + b_1\gamma_{12} + b_2\gamma_{13} \tag{2.13a}$$

$$\gamma_{24} := b_0\gamma_{21} + b_1\gamma_{22} + b_2\gamma_{23}; \tag{2.13b}$$

(IV) Solve the equation (2.12) with respect to γ_{03}. If it has a solution, then go to the next step;
(V) Define γ_{33} as

$$\gamma_{33} := b_0\gamma_{30} + b_1\gamma_{31} + b_2\gamma_{32}, \text{ or } := a_0\gamma_{22} + a_1\gamma_{23} + a_2\gamma_{24}. \tag{2.14}$$

(VI) Obtain a flat extension $E(3)$ of $E(2)$.

Example 2.8 Let us consider a positive matrix

$$E(2) = \begin{pmatrix} 1 & 0 & i & 2 \\ 0 & 2 & 0 & 0 \\ -i & 0 & 4 & -2i \\ 2 & 0 & 2i & 4 \end{pmatrix}.$$

with $\operatorname{rank} E(2) = 3$. Note that $\bar{Z}Z = 2\mathit{1}$. By Lemma 2.5, we may take $\gamma_{23} := 0$. Then we can obtain $b_0 = (4/3)\gamma_{03}$, $b_1 = i$, $b_2 = (1/3)i\gamma_{03}$. Substituting them to (2.12), we have $|\gamma_{03}| = 3/\sqrt{2}$. From (2.13a), (2.13b) and (2.14) we have $\gamma_{14} = 2\gamma_{03}$, $\gamma_{24} = 4i$, $\gamma_{33} = 8$. Thus we can obtain the flat extension $E(3)$ of $E(2)$ as follows

$$F = \begin{pmatrix} 1 & 0 & i & 2 & \gamma_{03} & 0 \\ 0 & 2 & 0 & 0 & 2i & 4 \\ -i & 0 & 4 & -2i & 0 & 0 \\ 2 & 0 & 2i & 4 & 2\gamma_{03} & 0 \\ \gamma_{30} & -2i & 0 & 2\gamma_{30} & 8 & -4i \\ 0 & 4 & 0 & 0 & 4i & 8 \end{pmatrix}.$$

We can easily check that $\det([F]_4) = \det([F]_5) = \det F = 0$. (To be continued in Example 2.12.)

Next we consider the double flat extension. Since

$$\begin{aligned} Z^4 &= b_0 Z + b_1 Z^2 + b_2 Z^3, \qquad (2.15) \\ \bar{Z} Z^3 &= a_0 Z^2 + a_1 Z^3 + a_2 Z^4, \\ \bar{Z}^2 Z^2 &= a_0 \bar{Z} Z + a_1 \bar{Z} Z^2 + a_2 \bar{Z} Z^3, \end{aligned}$$

we may define

$$\gamma_{34} := b_0 \gamma_{31} + b_1 \gamma_{32} + b_2 \gamma_{33}, \qquad (2.16a)$$

$$\gamma_{35} := b_0 \gamma_{32} + b_1 \gamma_{33} + b_2 \gamma_{34}. \qquad (2.16b)$$

Hence by comparing columns of $E(4)$ we have the following lemma.

Lemma 2.9 *$E(2)$ has a double flat extension $E(4)$ if and only if γ_{03} satisfies* (2.12) *in* Lemma 2.6 *and*

$$b_0 \gamma_{41} + b_1 \gamma_{42} + b_2 \gamma_{43} = a_0 \gamma_{33} + a_1 \gamma_{34} + a_2 \gamma_{35}. \qquad (2.17)$$

Algorithm 2.10 (continued)

(VII) Define γ_{34} as (2.16a) and then γ_{35} as (2.16b);

(VIII) Solve the equation (2.17) with respect to γ_{03}, i.e., γ_{03} satisfies both (2.12) and (2.17). If we may take a required γ_{03}, then go to the next step;

(IX) Define γ_{04}, γ_{15}, γ_{25}, γ_{44} and γ_{26} as

$$\gamma_{04} := b_0 \gamma_{01} + b_1 \gamma_{02} + b_2 \gamma_{03},$$

$$\gamma_{15} := b_0 \gamma_{12} + b_1 \gamma_{13} + b_2 \gamma_{14},$$

$$\gamma_{25} := b_0 \gamma_{22} + b_1 \gamma_{23} + b_2 \gamma_{24},$$

$$\gamma_{44} := b_0 \gamma_{41} + b_1 \gamma_{42} + b_2 \gamma_{43},$$

$$\gamma_{26} := b_0 \gamma_{23} + b_1 \gamma_{24} + b_2 \gamma_{25};$$

(X) Obtain a double flat extension $E(4)$ of $E(2)$.

From Lemma 2.6 and Lemma 2.9, we have the following theorem.

Theorem 2.11 *Assume that $E(2)$ is positive and $r = 3$. Then γ admits a 3-atomic representing measure if and only if we may take γ_{03} satisfying* (2.12) *in* Lemma 2.6 *and* (2.17) *in* Lemma 2.9.

Example 2.12 (Example 2.8 revisited) By (2.16a) and (2.16b), we have $\gamma_{34} = 0$, $\gamma_{35} = 8i$. Then the equation (2.17) is $2|\gamma_{03}|^2 = 9$, and so γ_{03} with

$|\gamma_{03}| = 3/\sqrt{2}$ satisfies both (2.12) and (2.17). Now we define $\gamma_{04} = -1 + (i/3\gamma_{03})\gamma_{03}$, $\gamma_{15} = 2\gamma_{04}$, $\gamma_{25} = 4\gamma_{03}$, $\gamma_{44} = 16$, $\gamma_{26} = 2$. Thus the double flat extension of $E(2)$ is

$$\begin{pmatrix} E(2) & B \\ B^* & C \end{pmatrix},$$

where

$$B = \begin{pmatrix} \gamma_{03} & 0 & -1 + i/3\gamma_{03}\gamma_{03} & 2i & 4 \\ 2i & 4 & 2\gamma_{03} & 0 & 0 \\ 0 & 0 & 4i & 8 & -4i \\ 2\gamma_{03} & 0 & 1 & 4i & 8 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 8 & -4i & 0 & 0 & 4\gamma_{30} \\ 4i & 8 & 4\gamma_{03} & 0 & 0 \\ 0 & 4\gamma_{30} & 16 & -8i & 2 \\ 0 & 0 & 8i & 16 & -8i \\ 4\gamma_{03} & 0 & 2 & 8i & 16 \end{pmatrix}.$$

Since $|\gamma_{03}|^2 = 9/2$, if we choose $\gamma_{03} = (3/2)(1-i)$, then the three atoms are

$$z_0 = \frac{3+\sqrt{7}}{4} + \frac{3-\sqrt{7}}{4}i,\ z_1 = \frac{3-\sqrt{7}}{4} + \frac{3+\sqrt{7}}{4}i,\ \text{and } z_2 = -1-i.$$

From the Vandermonde equation

$$\begin{pmatrix} 1 & 1 & 1 \\ z_0 & z_1 & z_2 \\ z_0^2 & z_1^2 & z_2^2 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{02} \end{pmatrix},$$

we obtain the densities $\rho_0 = 2/7$, $\rho_1 = 2/7$, $\rho_2 = 3/7$. Hence the 3-atomic representing measure is

$$\mu = \frac{2}{7}\delta_{(3+\sqrt{7})/4+\{(3-\sqrt{7})/4\}i} + \frac{2}{7}\delta_{(3-\sqrt{7})/4+\{(3+\sqrt{7})/4\}i} + \frac{3}{7}\delta_{-1-i}.$$

We close this section as a special case of $\bar{Z}Z = 1$.

Lemma 2.13 *The equation* $A|z|^2 + 2\,\mathrm{Re}(Cz) = B$, $(A > 0, C \in \mathbb{C}$, $B \in \mathbb{R})$ *has a solution if and only if* $AB + |C|^2 \geq 0$.

Proof. Observe that

$$A|z|^2 + 2\,\mathrm{Re}(Cz) = B \Longleftrightarrow \left|\bar{z} + \frac{C}{A}\right|^2 = \frac{AB + |C|^2}{A^2},$$

for $A > 0$, $C \in \mathbb{C}$, $B \in \mathbb{R}$. $\square$

Proposition 2.14 *Assume that $E(2) \geq 0$ and $r = 3$. If $\bar{Z}Z = \mathit{1}$, then there exists a 3-atomic representing measure for γ.*

Proof. Since $\bar{Z}Z = \mathit{1}$, we have $\gamma_{00} = \gamma_{11} = \gamma_{22}$, $\gamma_{10} = \gamma_{21}$ and $\gamma_{31} = \gamma_{20}$. So we put $a := \gamma_{00} = \gamma_{11} = \gamma_{22}$, $x := \gamma_{10} = \gamma_{21}$, and $u := \gamma_{31} = \gamma_{20}$. Then

$$E(2) = \begin{pmatrix} a & y & v & a \\ x & a & y & x \\ u & x & a & u \\ a & y & v & a \end{pmatrix} \quad \text{with} \quad y = \bar{x},\ v = \bar{u},\ a > 0.$$

In this case, $\Delta_3 = a^3 - 2ayx + vx^2 + uy^2 - uva > 0$. Moreover, the equalities (2.12) and (2.17) are same. By (2.9), we have $\gamma_{23} = y$. Set $z := \gamma_{30}$ again. Then the equation (2.12) is

$$A|z|^2 + 2\,\mathrm{Re}(Cz) = B, \tag{2.18}$$

where

$$\begin{aligned} A &= a^2 - yx, \\ B &= a^4 - 3a^2yx + 2avx^2 + 2auy^2 - 2uva^2 + v^2u^2 + y^2x^2 - 2yuvx, \\ C &= -2xua + u^2y + x^3. \end{aligned}$$

Since $AB + |C|^2 = \Delta_3^2 > 0$, by Lemma 2.13, the equation (2.18) has a solution. Thus γ admits a 3-atomic representing measure. $\square$

Example 2.15 Let us consider a matrix

$$E(2) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then obviously $E(2)$ is positive, $r = 3$, and $\bar{Z}Z = \mathit{1}$. Note $\gamma_{23} = 0$. If we take γ_{03} satisfying $|\gamma_{03}| = 1$, then it is the solution of (2.12) and (2.17). Hence $E(2)$ has double flat extension $E(4)$. Since $Z^3 = \gamma_{03}\mathit{1}$, the three

atoms are the roots of $z^3 = \gamma_{03}$. Thus we have that
(a) if $\gamma_{03} = i$, then the representing measure is

$$\mu = \frac{1}{3}(\delta_{1/2i-(1/2)\sqrt{3}} + \delta_{1/2i+(1/2)\sqrt{3}} + \delta_{-i});$$

(b) if $\gamma_{03} = 1$, then the representing measure is

$$\mu = \frac{1}{3}(\delta_1 + \delta_{-1/2+(1/2)i\sqrt{3}} + \delta_{-1/2-(1/2)i\sqrt{3}});$$

(c) if $\gamma_{03} = -i$, then the representing measure is

$$\mu = \frac{1}{3}(\delta_{-1/2i+(1/2)\sqrt{3}} + \delta_{-1/2i-(1/2)\sqrt{3}} + \delta_i);$$

(d) if $\gamma_{03} = -1$, then the representing measure is

$$\mu = \frac{1}{3}(\delta_{-1} + \delta_{1/2+(1/2)i\sqrt{3}} + \delta_{1/2-(1/2)i\sqrt{3}}).$$

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