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Local well-posedness and smoothing effects of strong solutions for nonlinear Schrödinger equations with potentials and magnetic fields

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Abstract. In this paper, we study the existence and the regularity of local strong solutions for the Cauchy problem of nonlinear Schrödinger equations with time-dependent potentials and magnetic fields. We consider these equations when the nonlinear term is the critical and/or power type which is, for example, equal to $\lambda|u|^{p-1}u$ with some $1 \leq p < \infty$, $\lambda \in \mathbf{C}$. We prove local well-posedness of strong solutions under the additional assumption $1 \leq p < \infty$ for space dimension $n = 4$, $1 \leq p \leq 1 + 4/(n-4)$ for $n \geq 5$, and local smoothing effects of it under the additional assumption $1 \leq p \leq 1 + 2/(n-4)$ when $n \geq 5$ without any restrictions on n.

Key words: nonlinear Schrödinger equations with time-dependent potentials and magnetic fields, well-posedeness of the Cauchy problem with the subcritical/critical power, local smoothing effects.

1. Introduction

We study local well-posedness and smoothing effects of the following nonlinear Schrödinger equation with magnetic fields:

$$i\partial_t u = \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,x))^2 u + V(t,x)u + F(u), \qquad (t,x) \in \mathbf{R}\times\mathbf{R}^n, \tag{1.1}$$

$$u(0,x) = \phi(x), \quad x \in \mathbf{R}^n, \tag{1.2}$$

where u is a complex valued unknown function on $\mathbf{R}\times\mathbf{R}^n$, the initial data ϕ is a complex valued given function on $\mathbf{R}^n$, the components of the vector potential A_j $(j = 1, \ldots, n)$ are real valued given functions on $\mathbf{R}\times\mathbf{R}^n$, the linear scalar potential V is a real valued given function on $\mathbf{R}\times\mathbf{R}^n$, and the nonlinear function F is a complex valued given function on $\mathbf{C}$. We can

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find this type equation, for example, in the Maxwell Schrödinger equations, which are the classical approximation to the quantum field equations for an electrodynamical nonrelativistic many body system (see, e.g., Shimomura [19] and Tsutsumi [26, 27]).

We will construct the strong solutions by using the contraction methods. For (1.1), the corresponding time-dependent linear Schrödinger equation is as follows:

$$i\partial_t u = H(t)u, \tag{LS}$$

where

$$\begin{aligned} H(t) &= \frac{1}{2}\sum_{j=1}^{n}(-i\partial_j - A_j(t,\cdot))^2 + V(t,\cdot) \\ &= -\frac{1}{2}\Delta + iA(t,\cdot)\cdot\nabla + \frac{i}{2}\nabla\cdot A(t,\cdot) + \frac{1}{2}|A(t,\cdot)|^2 + V(t,\cdot) \end{aligned} \tag{1.3}$$

is time-dependent Schrödinger operator acting in $L^2(\mathbf{R}^n)$. In [31], Yajima constructed the fundamental solution generated by this Hamiltonian as an extension of Fujiwara's results [4, 5]. We will solve the integral equation corresponding to (1.1)–(1.2) by using some properties of the propagator to this Hamiltonian. In what follows, we set

$$\tilde{V}(t) = \frac{i}{2}\nabla\cdot A(t) + \frac{1}{2}|A(t)|^2 + V(t).$$

The Cauchy problem for nonlinear Schrödinger equations with subcritical and/or critical power nonlinearities and linear potentials or magnetic fields has been investigated by many authors. (see also [3, 1, 6, 7, 10, 11, 12, 16, 24, 25] and references therein). Especially, de Bouard [3] studied on (1.1)–(1.2) with A and V independent of t (cf. Remark 1.7), the first author [16] studied well-posedness of weak solution to (1.1)–(1.2) with A and V depending on t. For the proof of well-posedness for nonlinear Schrödinger equations, we usually employ the Strichartz estimate, which is an estimate on a space-time integral of solutions to the linear problem. For the free Schrödinger group, this was proved by Strichartz [23] (see also [7, 11, 29]). It is well-known that this estimate also holds for $A = 0$ and $V \neq 0$ with some conditions (cf. [8, 11, 13]). In this paper, we use the Strichartz estimate with $A \neq 0$ and $V \neq 0$ which is obtained by Yajima [31] (Lemma 2.5). We also use so-called the endpoint Strichartz estimates obtained by Keel and

Tao [14].

On the other hand, solutions of the Schrödinger type equations have smoothing effects, that is, the solution is smoother than the initial data for almost all time t. For the free Schrödinger group, Sjölin [20] has proved the following inequality to exhibit this property

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n} |\varphi(t,x)(1-\Delta)^{1/4}e^{it\Delta}f|^2 dx\,dt \le C\|f\|_{L^2}^2, \qquad f \in L^2,\ \varphi \in C_0^\infty(\mathbf{R}^{n+1})$$

(cf. [2, 28]). Yajima [30] has proved it for the equation (LS), which we will quote as Lemma 4.1 below. Recently, Yajima and Zhang [32, 33] have proved this property for (LS) and well-posedness for (1.1)–(1.2), when $A = 0$ and V is superquadratic at infinity. When $1 \le n \le 7$, the first author [15] and Sjölin [21] showed this property for the strong solutions to (1.1)–(1.2) with $A = V = 0$. We will prove the smoothing effects of the strong solutions to (1.1)–(1.2) with scalar potentials and magnetic fields for all space dimensions, time-locally. This property for the weak solutions to (1.1)–(1.2) with potentials and magnetic fields was studied in the previous paper [16] (cf. [22] for the case $A = V = 0$).

We make the following assumptions on the vector potential and the scalar potential, which are introduced by Yajima [30, 31].

Assumption (A) For $j = 1, \dots, n$, A_j is a continuous function of $(t,x) \in \mathbf{R}\times\mathbf{R}^n$ and a C^∞ class function of x for each t. $\partial_x^\alpha A_j$ is a C^1 class function of $(t,x) \in \mathbf{R}\times\mathbf{R}^n$ for any multi-index α. A satisfies for $|\alpha| \ge 1$,

$$|\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha \langle x\rangle^{-1-\varepsilon},$$
$$|\partial_x^\alpha A(t,x)| + |\partial_t\partial_x^\alpha A(t,x)| \le C_\alpha$$

with some $\varepsilon > 0$, where $A(t,x) = (A_1(t,x), \dots, A_n(t,x))$, $B_{jk}(t,x) = \partial_j A_k(t,x) - \partial_k A_j(t,x)$.

Assumption (V) V is a continuous function of $(t,x) \in \mathbf{R}\times\mathbf{R}^n$ and a C^∞ class function of x for each t. $\partial_x^\alpha V$ is a continuous function of $(t,x) \in \mathbf{R}\times\mathbf{R}^n$ for any multi-index α. V satisfies

$$|\partial_x^\alpha V(t,x)| \le C_\alpha$$

for $|\alpha| \ge 2$.

We also assume the following assumptions on the nonlinear function F. (cf. [10, 11])

Assumption (F1) $F \in C^1(\mathbf{C}, \mathbf{C})$ in the real sense with $F(0) = 0$.

Let $F \in C^1(\mathbf{C}, \mathbf{C})$. For $z \in \mathbf{C}$, we define the linear operator $F'(z)$ on $\mathbf{C}$ by

$$F'(z)\omega = \partial_z F(z)\omega + \partial_{\bar{z}} F(z)\bar{\omega}, \qquad \text{for } \omega \in \mathbf{C},$$

where $\partial_z = (1/2)(\partial_\xi - i\partial_\eta)$ and $\partial_{\bar{z}} = (1/2)(\partial_\xi + i\partial_\eta)$ with $z = \xi + i\eta$, ξ, $\eta \in \mathbf{R}$.

Assumption (F2) There exists $M > 0$ such that for $|z| > 1$,

$$|F'(z)| \equiv \max\{|\partial_z F(z)|, |\partial_{\bar{z}} F(z)|\} \leq M|z|^{p-1}$$

with some $1 \leq p < \infty$.

We introduce the following function spaces. We set for $k = 0, 1, \dots,$

$$\Sigma(k) = \{f \in L^2 \colon \|f\|_{\Sigma(k)} < \infty\},$$
$$\|f\|_{\Sigma(k)} = \sum_{\substack{|\alpha+\beta| \leq k \\ |\alpha|, |\beta| \geq 0}} \|x^\alpha \partial_x^\beta f\|_2,$$

and let $\Sigma(-k)$ be the dual space of $\Sigma(k)$. Then $\Sigma(k)$ is a Banach space with the norm $\|\cdot\|_{\Sigma(k)}$.

Definition We call the components (q, r) *an admissible pair* if they satisfy

$$\frac{2}{r} = n\left(\frac{1}{2} - \frac{1}{q}\right), \tag{1.4}$$

and

$$\begin{cases} 2 \leq q \leq \infty & \text{if} \quad n = 1, \\ 2 \leq q < \infty & \text{if} \quad n = 2, \\ 2 \leq q \leq 2n/(n-2) & \text{if} \quad n \geq 3. \end{cases}$$

Let

$$\mathcal{X}_T = \bigcap_{(q,r)} L^r(I_T, L^q), \qquad \bar{\mathcal{X}}_T = \mathcal{X}_T \cap C(I_T, L^2), \tag{1.5}$$

where (q, r) is every admissible pair and $I_T = [0, T]$.

Remark 1.1 For this definition, we take the results of Keel and Tao [14] into consideration.

We state the main results of this paper.

Theorem 1 *Suppose that* Assumptions (A), (V) *and* (F1) *are satisfied. In addition, suppose that* Assumption (F2) *with* $1 \le p < \infty$ *if* $n = 4$, *with* $1 \le p \le 1 + 4/(n-4)$ *if* $n \ge 5$ *is satisfied. Let* $\phi \in \Sigma(2)$, *in addition with* $\|\phi\|_{\Sigma(2)}$ *sufficiently small if* $n \ge 5$ *and* $p = 1 + 4/(n-4)$. *Then there exists* $T > 0$ *depending only on* $\|\phi\|_{\Sigma(2)}$ *such that* (1.1)–(1.2) *has a unique solution* u *in* $C(I_T, \Sigma(2))$. *Furthermore* $\partial_t u \in \bar{\mathcal{X}}_T$, *in particular* $u \in C(I_T, \Sigma(2)) \cap C^1(I_T, L^2)$. *Moreover this solution depends on the initial datum continuously. Namely if* $\phi_m \to \phi$ *as* $m \to \infty$ *in* $\Sigma(2)$, *then the coresponding solution* $u_m \in C(I_{T_0}, \Sigma(2))$ *to the datum* ϕ_m *converges to* u *in* $C(I_{T_0}, \Sigma(2))$, *where* $T_0 > 0$ *depends only on* $\|\phi\|_{\Sigma(2)}$.

Theorem 2 *Suppose that* Assumptions (A), (V) *and* (F1) *are satisfied. In addition, suppose that* Assumption (F2) *with* $1 \le p < \infty$ *if* $n = 4$, *with* $1 \le p \le 1+2/(n-4)$ *if* $n \ge 5$ *is satisfied. Let* u *be the solution of* (1.1)–(1.2) *obtained in* Theorem 1. *Then for* $\mu > 1/2$,

$$\int_{I_T} \|\langle x\rangle^{-\mu-5/2} \langle D_x\rangle^{5/2} u(t)\|_2^2 \, dt < \infty,$$

where $\langle D_x\rangle = (I - \Delta)^{1/2}$.

Remark 1.2 When $n \ge 5$, we prove local well-posedness of (1.1)–(1.2) under the assumption $1 \le p \le 1+4/(n-4)$. On the other hand, we obtain local smoothing effects only in the case of $1 \le p \le 1 + 2/(n-4)$, because for the solution u of (1.1)–(1.2) we have only the fact that the nonlinear term $F(u)$, the time derivative $\partial_t F(u)$, $|x|^2 F(u)$ and $x_j \partial_k F(u)$, j, $k = 1, \ldots, n$, belong to $L^1(I_T; L^2(\mathbf{R}^n))$ by the Sobolev embedding theorem (see Lemma 4.2).

Remark 1.3 For our solution u, we have no information on $\partial_x^\alpha u$, $|\alpha| = 2$, except $\Delta u \in L^\infty(I_T; L^2)$ (cf. [1, 11, 12]).

Remark 1.4 If $A \equiv 0$ then for $n \ge 5$ and $F(u) = |u|^{p-1}u$ with $p = 1 + 4/(n-4)$, we don't need the assumption on the size of the initial datum. (cf. [1]).

Remark 1.5 Following the argument below, we can construct $\Sigma(k)$-solution with positive integer k. In fact, for, e.g., $F(u) = M_1|u|^{[(k+1)/2]-1}u + M_2|u|^{4/(n-2k)}u$ with $n > 2k$ and $[(k+1)/2] - 1 < 4/(n-2k)$, we may use the contraction arguement in the following Banach space:

$$\begin{aligned} X = \{u \in C(I_T, H^k) \mid x^\alpha \partial_t^l \partial_x^\beta u \in L^r(I_T; L^q),\ |\alpha| + 2l + |\beta| = k, \\ |\alpha| = 0, 1, \ldots, k,\ l = 0, 1, \ldots, [k/2], \\ |\beta| = 0, \ldots, k - 2[k/2]\} \end{aligned}$$

where (q, r) is the admissible pair with $1/q = (k-1)/k + 1/(k^2p - k^2 + 2k)$. We note that $\Sigma(k) \subset L^{kp-k+2}$ and $k - 2[k/2] = 0$ or 1. In this paper, we treat the case of $k = 2$, especially. Because when we regard (1.1) as L^2-valued ordinary differential equation, $\Sigma(2)$ is included in $D(H(t))$, the space corresponding to the strong solution to (1.1).

Remark 1.6 In fact the fundamental solution of (LS) can be constructed for the scalar potentials with singularities under the suitable conditions (see Theorem 7 in Yajima [31]). Thus using this property, we can show the local well-posedness of (1.1)–(1.2) for these scalar potentials with singularities. But since we do not have the local smoothing property of the propagator of (LS) for singular potentials even when $A = F = 0$, we need to assume the continuity for the scalar potentials to prove the local smoothing effects of (1.1)–(1.2).

Remark 1.7 When A and V are independent of t, that is, $H(t) = H$, it is rather easy to prove Theorem 1 because ∂_t is commutable with H. If V satisfies the some growth conditions, which are weaker than Assumption (V), then H defined on C_0^∞ is essentially self-adjoint in $L^2(\mathbf{R}^n)$ (see [9] and, e.g., pp. 199 in [18]). Thus we can prove theorems by using $e^{-it\bar{H}}$ instead of $U(t, 0)$, where $\bar{H}$ is the self-adjoint realization of H. (cf. [3]).

Notations Let $L^q(\mathbf{R}^n) = \left\{\psi \colon \|\psi\|_q = \left(\int_{\mathbf{R}^n} |\psi(x)|^q\, dx\right)^{1/q} < \infty\right\}$ for $1 \le q < \infty$, and let $L^\infty(\mathbf{R}^n) = \left\{\psi \colon \|\psi\|_\infty = \operatorname{ess\,sup}_{x \in \mathbf{R}^n} |\psi(x)| < \infty\right\}$. Let the Sobolev space $H^k(\mathbf{R}^n) = \left\{\psi \colon \|\psi\|_{H^k} = \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_2 < \infty\right\}$, for positive integer k. For simplicity, we denote the space $L^q(\mathbf{R}^n)$ by L^q and the space $H^k(\mathbf{R}^n)$ by H^k, respectively. For a Banach space X and an interval $I_T = [0, T]$, let $C(I_T, X)$ be the set of X-valued strongly continuous functions on I_T, and let $L^q(I_T, X)$ be the set of X-valued L^q-functions on I_T. We put $L_T^r(X) = L^r(I_T, X)$ with the norm

$$\|f\|_{L^r_T(X)} = \left(\int_{I_T} \|f(t,\cdot)\|_X^r \, dt\right)^{1/r}, \qquad 1 \le r < \infty,$$
$$\|f\|_{L^\infty_T(X)} = \operatorname{ess\,sup}_{t\in I_T} \|f(t,\cdot)\|_X.$$

We denote the set of rapidly decreasing functions on $\mathbf{R}^n$ by $\mathcal{S}(\mathbf{R}^n)$. We denote various constants by C, M and so forth. They may differ from line to line, when it does not cause any confusion.

We use the following symbols:

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_k = \frac{\partial}{\partial x_k} \quad \text{for } k = 1, \dots, n,$$
$$\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$
$$\text{for any multi-index } \alpha = (\alpha_1, \dots, \alpha_n),$$
$$\nabla = (\partial_1, \dots, \partial_n), \quad \Delta = \partial_1^2 + \cdots + \partial_n^2,$$
$$\langle x \rangle = (1 + |x|^2)^{1/2},$$
$$a \vee b = \max\{a, b\}.$$

Outline of this paper is as follows. In Section 2, we introduce some results of (LS) obtained in Yajima [31]. In Section 3, we prove Theorem 1, that is, the local well-posedness of the strong solutions to (1.1)–(1.2) by the contraction argument in the suitable function spaces. In Section 4, we prove Theorem 2, that is, the local smoothing effects of the strong solutions to (1.1)–(1.2) by using the smoothing property of (LS) obtained in Yajima [30].

2. Preliminaries

We introduce some results for the linear equation (LS) in Yajima [31].

Lemma 2.1 (Yajima [31]) *Assume* Assumptions (A) *and* (V). *Then there exists a unique propagator* $\{U(t,s)\}_{t,\,s\in\mathbf{R}}$ *for* (LS) *satisfying the following properties*:

1. *For any $t \neq s$, $U(t,s)$ maps $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^n)$ continuously and extends a unitary operator in $L^2(\mathbf{R}^n)$ which satisfies $U(t,r)U(r,s) = U(t,s)$.*
2. *For $\psi \in \Sigma(2)$, $U(\cdot,\cdot)\psi \in C(\mathbf{R}^2, \Sigma(2)) \cap C^1(\mathbf{R}^2, L^2)$, and the following equations hold*:

$$i\partial_t U(t,s)\psi = H(t)U(t,s)\psi,$$
$$i\partial_s U(t,s)\psi = -U(t,s)H(s)\psi.$$

Furthermore there exists $\tilde{T} > 0$ such that for $0 < |t-s| < \tilde{T}$, $U(t,s)$ can be represented as in the form of oscillatory integral

$$(U(t,s)f)(x) = (2\pi i(t-s))^{-n/2} \int_{\mathbf{R}^n} e^{iS(t,s,x,y)} b(t,s,x,y) f(y)\, dy.$$

Then $\{U(t,s)\colon |t-s| < \tilde{T},\ t,\, s \in \mathbf{R}\}$ is strongly continuous in $L^2(\mathbf{R}^n)$. Here $S(t,s,x,y)$ and $b(t,s,x,y)$ are uniquely determined functions satisfy some properties.

The following lemma is the L^p-L^q estimate for $U(t,s)$.

Lemma 2.2 (Yajima [31]) *Let $\tilde{T}$ be the same constant as in* Lemma 2.1*. For any $2 \le q \le \infty$ and for $0 < |t-s| < \tilde{T}$, $t \neq s$, there exists a constant $C = C(n,q)$ such that*

$$\|U(t,s)f\|_q \le C|t-s|^{-n(1/2-1/q)} \|f\|_{q'},$$

where q' is the dual of q.

We define linear operators U and G as follows:

$$(U\phi)(t) = U(t,0)\phi, \quad t \in \mathbf{R},$$

$$(Gf)(t) = \int_0^t U(t,s) f(s)\, ds, \quad t \in \mathbf{R}.$$

These operators have the following properties (see Yajima [30], [31]). Let I be a compact sub-interval of $[0,\tilde{T}]$.

Lemma 2.3 *U is a bounded operator from L^2 into $C(I,L^2) \cap AC(I,\Sigma(-2))$ satisfying*

$$i\partial_t U\phi = H(t) U\phi$$

for $\phi \in L^2$ and a.e. $t \in I$ in $\Sigma(-2)$, where $AC(I,\Sigma(-2))$ is the class of $\Sigma(-2)$-valued absolutely continuous functions.

Lemma 2.4 *If $f \in L^1(I,L^2)$, then $Gf \in C(I,L^2) \cap AC(I,\Sigma(-2))$ and it follows that*

$$i\partial_t Gf = H(t) Gf + if$$

for a.e. $t \in I$ in $\Sigma(-2)$.

The following Strichartz estimates are obtained in Yajima [31]. Let $I_T = [0, T] \subset [0, \tilde{T}]$.

Lemma 2.5 *Assume that the components (q_i, r_i) are arbitrary admissible pairs, where $i = \emptyset$, 1, 2, and let (q_i', r_i') be dual of (q_i, r_i), namely $1/q_i + 1/q_i' = 1$ and $1/r_i + 1/r_i' = 1$. Then U is a bounded operator from L^2 into $L^r_T(L^q)$, and G is a bounded operator from $L^{r_2'}_T(L^{q_2'})$ into $L^{r_1}_T(L^{q_1})$, with the bounds independent of T. Namely, there exist C, $C' > 0$ independent of T such that*

$$\begin{aligned} &\|U\phi\|_{L^r_T(L^q)} \le C\|\phi\|_2, \\ &\|Gf\|_{L^{r_1}_T(L^{q_1})} \le C'\|f\|_{L^{r_2'}_T(L^{q_2'})}. \end{aligned}$$

Furthermore, $U\phi \in C(I_T, L^2)$ and $Gf \in C(I_T, L^2)$ for any $\phi \in L^2$ and $f \in L^{r_2'}_T(L^{q_2'})$.

Remark 2.1 We can obtain the endpoint estimates of $U(t, s)$ by the same argument as in [14]. We remark that these estimates hold locally in time.

Remark 2.2 Under Assumptions (A) and (V), it is easily seen that there exists $C > 0$, depending on $\tilde{T}$, such that

$$\begin{aligned} &|A(t, x)| \le C\langle x\rangle, \\ &|V(t, x)| \le C\langle x\rangle^2, \\ &|\partial_x V(t, x)| \le C\langle x\rangle \end{aligned}$$

for any $t \in I$ and $x \in \mathbf{R}^n$. Then we have

$$|\partial_x^\alpha \tilde{V}(t, x)| \le C\langle x\rangle$$

for any $t \in I$, $x \in \mathbf{R}^n$ and $|\alpha| \ge 1$.

Remark 2.3 Under Assumptions (F1) and (F2), it is easily seen that F can be decomposed in the form

$$\begin{aligned} &F = F_1 + F_2, \quad F_1,\ F_2 \in C^1(\mathbf{C}, \mathbf{C}), \quad F_1(0) = F_2(0) = 0, \\ &|F_1(z)| \le M_1|z|, \quad |F_1'(z)| \le M_1, \\ &|F_2(z)| \le M_2|z|^p, \quad |F_2'(z)| \le M_2|z|^{p-1} \end{aligned}$$

for $z \in \mathbf{C}$ (see Kato [11]).

3. Proof of Theorem 1

We only prove the case $n \geq 4$. The lower dimension cases are much simple and we omit them. We note that $1 \leq p < \infty$ if $n = 4$, $1 \leq p \leq 1 + 4/(n-4)$ if $n \geq 5$.

Let $I_T = [0, T]$ for $0 < T \leq \tilde{T}$, where $\tilde{T}$ is introduced in Lemma 2.2. And set $l = (n/4)(1 - 1/p)$ so that $0 \leq l \leq 1$, $\gamma = 2/l$ and $\rho = 4p/(p+1)$. We introduce the following function spaces and their norms.

$$\begin{aligned}
&X_T = L_T^\infty(L^2) \cap L_T^\gamma(L^\rho),\\
&\|u\|_{X_T} = \|u\|_{L_T^\infty(L^2)} \vee \|u\|_{L_T^\gamma(L^\rho)},\\
&X_T' = L_T^1(L^2) + L_T^{\gamma'}(L^{\rho'}),\\
&\|v\|_{X_T'} = \inf\bigl\{\|v_1\|_{L_T^1(L^2)} + \|v_2\|_{L_T^{\gamma'}(L^{\rho'})} \colon v = v_1 + v_2\bigr\},\\
&\bar{X_T} = C(I_T, L^2) \cap L_T^\gamma(L^\rho),
\end{aligned}$$

where ρ' and γ' are the dual of ρ and γ, respectively, namely $\rho' = 4p/(3p-1)$ and $\gamma' = 2/(2-l)$. We note that X_T and X_T' are Banach spaces.

Remark 3.1 The pairs $(2, \infty)$ and (ρ, γ) are admissible.

We define the function space Z_T as follows:

$$\begin{aligned}
&Z_T = \{u \colon \|u\|_{Z_T} < \infty\},\\
&\bar{Z_T} = \{u \in Z_T \colon u \in C(I_T, \Sigma(2)),\ \partial_t u \in C(I_T, L^2)\},
\end{aligned}$$

where

$$\begin{aligned}
\|u\|_{Z_T} = \|u\|_{L_T^\infty(L^2)} \vee \|\Delta u\|_{L_T^\infty(L^2)} \vee \left(\sum_{j,\,k=1}^n \|x_j \partial_k u\|_{X_T}\right)\\
\vee \||x|^2 u\|_{X_T} \vee \|\partial_t u\|_{X_T}.
\end{aligned}$$

Then Z_T is a Banach space.

Remark 3.2 Since $1 \leq p < \infty$ if $n = 4$ and $1 \leq p \leq 1 + 4/(n-4)$ if $n \geq 5$, it follows from the Sobolev embedding theorem that

$$\Sigma(2) \hookrightarrow H^2 \hookrightarrow H^{2l} \hookrightarrow L^{2p}. \tag{3.1}$$

Recall that $l = (n/4)(1 - 1/p)$

Lemma 3.1 *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$, $f \in \mathcal{S}(\mathbf{R}^{n+1})$, and let $v = U\phi - iGf$. Then*

$$\begin{aligned}\partial_k^2 v = U(\partial_k^2\phi) - iG[(\partial_k^2\tilde{V})v + 2(\partial_k\tilde{V})\partial_k v\\ + i\partial_k^2 A\cdot\nabla v + 2i\partial_k A\cdot\partial_k\nabla v + \partial_k^2 f],\end{aligned} \tag{3.2}$$

$$\begin{aligned}x_j\partial_k v = U(x_j\partial_k\phi) - iG[x_j(\partial_k\tilde{V})v + ix_j\partial_k A\cdot\nabla v\\ + (\partial_j - iA_j)\partial_k v + x_j\partial_k f],\end{aligned} \tag{3.3}$$

$$x_k{}^2 v = U(x_k{}^2\phi) - iG[2x_k(\partial_k - iA_k)v + v + x_k{}^2 f]. \tag{3.4}$$

Proof. We differentiate the equation $v = U\phi - iGf$. Then we have

$$i\partial_t v = H(t)v + f$$

and hence,

$$\begin{aligned}i\partial_t(x_k^2 v) &= x_k^2 H(t)v + x_k^2 f\\ &= H(t)(x_k^2 v) - [H(t), x_k^2]v + x_k^2 f\\ &= H(t)(x_k^2 v) + 2x_k(\partial_k - iA_k)v + v + x_k^2 f.\end{aligned}$$

Noting that $(x_k^2 v)(0) = x_k^2\phi$, we have (3.4). In a similar way, we can prove (3.2) and (3.3). $\square$

Lemma 3.2 *If $T > 0$ is sufficiently small, U is a bounded operator from $\Sigma(2)$ into Z_T, with the bound independent of T. Namely, if $T > 0$ is sufficiently small, there exists $C_1 > 0$ independent of T such that*

$$\|U\phi\|_{Z_T} \le C_1\|\phi\|_{\Sigma(2)}. \tag{3.5}$$

Furthermore, $U\phi \in \bar{Z}_T$ for any $\phi \in \Sigma(2)$.

Proof. We first assume $\phi \in \mathcal{S}(\mathbf{R}^n)$. By Lemma 2.5, we have

$$\|U\phi\|_{L_T^\infty(L^2)} \le c\|\phi\|_2. \tag{3.6}$$

By the application of Lemma 2.5 to the equalities in Lemma 3.1, we have the following estimates

$$\begin{aligned}&\|\Delta U\phi\|_{X_T}\\ &\quad\le \Bigg\| U(\Delta\phi) - iG\Bigg[(\Delta\tilde{V})U\phi + 2(\nabla\tilde{V})\cdot(\nabla U\phi)\\ &\qquad + 2i\sum_{k=1}^n(\partial_k A)\cdot(\nabla\partial_k U\phi) + i(\Delta A)\cdot(\nabla U\phi)\Bigg]\Bigg\|_{X_T}\end{aligned}$$

$$
\begin{aligned}
&\le c\|\Delta\phi\|_2 + c\bigg\|(\Delta\tilde{V})U\phi + 2(\nabla\tilde{V})\cdot(\nabla U\phi) \\
&\quad + 2i\sum_{k=1}^{n}(\partial_k A)\cdot(\nabla\partial_k U\phi) + i(\Delta A)\cdot(\nabla U\phi)\bigg\|_{L^1_T(L^2)} \\
&\le c\|\Delta\phi\|_2 + cT\big(\|\langle x\rangle U\phi\|_{L^\infty_T(L^2)} + \|\langle x\rangle\nabla U\phi\|_{L^\infty_T(L^2)} \\
&\quad + \|\Delta U\phi\|_{L^\infty_T(L^2)} + \|\nabla U\phi\|_{L^\infty_T(L^2)}\big) \\
&\le c\|\Delta\phi\|_2 + cT\bigg(\|U\phi\|_{L^\infty_T(L^2)} + \||x|^2 U\phi\|_{L^\infty_T(L^2)} \\
&\quad + \sum_{j,\,k=1}^{n}\|x_j\partial_k U\phi\|_{L^\infty_T(L^2)} + \|\Delta U\phi\|_{L^\infty_T(L^2)}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
&\|x_j\partial_k U\phi\|_{X_T} \\
&\quad\le \|U(x_j\partial_k\phi) - iG[ix_j(\partial_k A)\cdot\nabla U\phi \\
&\quad\quad + (\partial_j - iA_j)\partial_k U\phi + x_j(\partial_k\tilde{V})U\phi]\|_{X_T} \\
&\quad\le c\|x_j\partial_k\phi\|_2 + cT\|ix_j(\partial_k A)\cdot\nabla U\phi \\
&\quad\quad + (\partial_j - iA_j)\partial_k U\phi + x_j(\partial_k\tilde{V})U\phi\|_{L^1_T(L^2)} \\
&\quad\le c\|x_j\partial_k\phi\|_2 + cT\big(\|x_j\nabla U\phi\|_{L^\infty_T(L^2)} + \|\Delta U\phi\|_{L^\infty_T(L^2)} \\
&\quad\quad + \|\langle x\rangle\partial_k U\phi\|_{L^\infty_T(L^2)} + \|x_j\langle x\rangle U\phi\|_{L^\infty_T(L^2)} \\
&\quad\le c\|x_j\partial_k\phi\|_2 + cT\bigg(\|U\phi\|_{L^\infty_T(L^2)} + \sum_{j,\,k=1}^{n}\|x_j\partial_k U\phi\|_{L^\infty_T(L^2)} \\
&\quad\quad + \||x|^2 U\phi\|_{L^\infty_T(L^2)} + \|\Delta U\phi\|_{L^\infty_T(L^2)}\bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\||x|^2 U\phi\|_{X_T} \\
&\quad\le \bigg\|U(|x|^2\phi) - iG\bigg[\sum_{k=1}^{n}\{2x_k(\partial_k - iA_k)U\phi\} + nU\phi\bigg]\bigg\|_{X_T} \\
&\quad\le c\||x|^2\phi\|_2 + c\bigg\|\sum_{k=1}^{n}\{2x_k(\partial_k - iA_k)U\phi\} + nU\phi\bigg\|_{L^1_T(L^2)} \\
&\quad\le c\||x|\phi\|_2 + cT\big(\|x\cdot\nabla U\phi\|_{L^\infty_T(L^2)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \||x|^2 U\phi\|_{L^\infty_T(L^2)} + \|U\phi\|_{L^\infty_T(L^2)}\big) \\
&\le c\||x|\phi\|_2 + cT\bigg(\sum_{j,\,k=1}^{n} \|x_j\partial_k U\phi\|_{L^\infty_T(L^2)} \\
&\quad + \||x|^2 U\phi\|_{L^\infty_T(L^2)} + \|U\phi\|_{L^\infty_T(L^2)}\bigg).
\end{aligned}
$$

We have used Remark 2.2. By Lemmas 2.3 and 2.5, we obtain

$$
\begin{aligned}
&\|\partial_t U\phi\|_{X_T} \\
&\quad \le \|H(t)U\phi\|_{X_T} \\
&\quad \le c\bigg(\|U\phi\|_{X_T} + \||x|^2 U\phi\|_{X_T} + \sum_{j,\,k=1}^{n} \|x_j\partial_k U\phi\|_{X_T} + \|\Delta U\phi\|_{X_T}\bigg).
\end{aligned}
$$

From the above estimates, it follows that

$$
\|U\phi\|_{Z_T} \le c\|\phi\|_{\Sigma(2)} + cT\|U\phi\|_{Z_T}.
$$

Therefore, if $T > 0$ is small enough, (3.5) holds for $\phi \in \mathcal{S}(\mathbf{R}^n)$. By the density argument, we see that if $T > 0$ is small enough, (3.5) holds for any $\phi \in \Sigma(2)$. Actually, $U\phi \in \bar{Z}_T$ for any $\phi \in \Sigma(2)$. This follows from Lemmas 2.3 and 2.5 immediately. □

Remark 3.3 According to the proof of Lemma 3.2, we see that $\Delta U\phi \in X_T$ for any $\phi \in \Sigma(2)$.

Recall that $(Gf)(t) = \int_0^t U(t,s)f(s)ds$, $t \in \mathbf{R}$.

Lemma 3.3 *Let $f \in L^\infty_T(L^2)$ and $\partial_t f$, $|x|^2 f$, $x_j\partial_k f \in X'_T$ for $j,\, k = 1, \ldots, n$. Assume that $T > 0$ is sufficiently small and that $f(0) \in L^2$ exists. Then $Gf \in Z_T$. Furthermore there exists $C_2 > 0$ independent of T such that*

$$
\begin{aligned}
\|Gf\|_{Z_T} \le C_2\bigg(&\|f\|_{L^\infty_T(L^2)} + \||x|^2 f\|_{X'_T} + \sum_{j,\,k=1}^{n} \|x_j\partial_k f\|_{X'_T} \\
&+ \|\partial_t f\|_{X'_T}\bigg). \qquad (3.7)
\end{aligned}
$$

In particular, if $f \in C(I_T, L^2)$, then $Gf \in \bar{Z}_T$.

Proof. First we assume that $f \in \mathcal{S}(\mathbf{R}^{n+1})$. By Lemma 2.5, we have

$$
\|Gf\|_{L^\infty_T(L^2)} \le c\|f\|_{L^1_T(L^2)}
$$

and that $Gf \in C(I_T, L^2)$. By the application of Lemma 2.5 to the equalities in Lemma 3.1, we have the following estimates

$$\begin{aligned}
&\|x_j \partial_k Gf\|_{X_T} \\
&\quad \le \big\| G\big[x_j(\partial_k \tilde{V})Gf + i x_j \partial_k A \cdot \nabla Gf + (\partial_j - iA_j)\partial_k Gf + x_j \partial_k f\big]\big\|_{X_T} \\
&\quad \le cT\Bigg(\|Gf\|_{L^\infty_T(L^2)} + \||x|^2 Gf\|_{L^\infty_T(L^2)} + \sum_{j,k=1}^{n} \|x_j \partial_k Gf\|_{L^\infty_T(L^2)} \\
&\qquad + \|\Delta Gf\|_{L^\infty_T(L^2)} \Bigg) + c\|x_j \partial_k f\|_{X'_T}
\end{aligned}$$

and

$$\begin{aligned}
&\||x|^2 Gf\|_{X_T} \\
&\quad \le \left\| G\left[\sum_{k=1}^{n} \{2x_k(\partial_k - iA_k)Gf\} + nGf + |x|^2 f \right] \right\|_{X_T} \\
&\quad \le cT\Bigg(\|Gf\|_{L^\infty_T(L^2)} + \sum_{j,k=1}^{n} \|x_j \partial_k Gf\|_{L^\infty_T(L^2)} + \||x|^2 Gf\|_{L^\infty_T(L^2)} \Bigg) \\
&\qquad + c\||x|^2 f\|_{X'_T}.
\end{aligned}$$

We have used Remark 2.2. Recall that $\tilde{V}(t,x) = (i/2)\nabla \cdot A(t,x) + (1/2)|A(t,x)|^2 + V(t,x)$, $(t,x) \in \mathbf{R} \times \mathbf{R}^n$. Since

$$\begin{aligned}
\partial_t Gf = G\partial_t f + Uf(0) + \frac{i}{2}(\Delta Gf - G\Delta f) \\
- G\{(A \cdot \nabla - i\tilde{V})f\} + (A \cdot \nabla - i\tilde{V})Gf,
\end{aligned} \tag{3.8}$$

and, from (3.2),

$$\begin{aligned}
&\Delta Gf - G\Delta f \\
&\quad = -iG\Bigg[(\Delta \tilde{V})Gf + i(\Delta A) \cdot \nabla Gf \\
&\qquad + 2(\nabla \tilde{V}) \cdot \nabla Gf + 2i \sum_{j,k=1}^{n} (\partial_j A_k)\partial_{jk} Gf \Bigg],
\end{aligned} \tag{3.9}$$

it follows from Lemma 2.5 that

$$\|\partial_t Gf\|_{X_T}$$

$$
\begin{aligned}
&\le c\biggl(\|\partial_t f\|_{X'_T} + \|f\|_{L^1_T(L^2)} + \|f(0)\|_2 + \||x|^2 f\|_{X'_T} \\
&\quad + \sum_{j,\,k=1}^{n} \|x_j \partial_k f\|_{X'_T} + \||x|^2 Gf\|_{X_T} + \sum_{j,\,k=1}^{n} \|x_j \partial_k Gf\|_{X_T} \biggr) \\
&\quad + cT\|\Delta Gf\|_{L^\infty_T(L^2)}.
\end{aligned}
$$

By (1.3), Lemmas 2.4 and 2.5, we have

$$
\begin{aligned}
&\|\Delta Gf\|_{L^\infty_T(L^2)} \\
&\quad \le \bigl\| 2\partial_t Gf - 2A \cdot \nabla Gf + 2i\tilde{V} Gf + 2if \bigr\|_{L^\infty_T(L^2)} \\
&\quad \le c\biggl(\sum_{j,\,k=1}^{n} \|x_j \partial_k Gf\|_{L^\infty_T(L^2)} + \||x|^2 Gf\|_{L^\infty_T(L^2)} \\
&\qquad + \|\partial_t f\|_{X'_T} + \|f\|_{L^1_T(L^2)} + \|f(0)\|_2 + \||x|^2 f\|_{X'_T} \\
&\qquad + \sum_{j,\,k=1}^{n} \|x_j \partial_k f\|_{X'_T} + \|f\|_{L^\infty_T(L^2)} \biggr) + cT\|\Delta Gf\|_{L^\infty_T(L^2)}
\end{aligned}
$$

and hence for $T > 0$ sufficiently small,

$$
\begin{aligned}
&\|\Delta Gf\|_{L^\infty_T(L^2)} \\
&\quad \le c\biggl(\sum_{j,\,k=1}^{n} \|x_j \partial_k Gf\|_{L^\infty_T(L^2)} + \||x|^2 Gf\|_{L^\infty_T(L^2)} + \|\partial_t f\|_{X'_T} \\
&\qquad + \|f\|_{L^1_T(L^2)} + \|f(0)\|_2 + \||x|^2 f\|_{X'_T} + \sum_{j,\,k=1}^{n} \|x_j \partial_k f\|_{X'_T} \\
&\qquad + \|f\|_{L^\infty_T(L^2)} \biggr).
\end{aligned}
$$

From the above estimates, we have

$$
\begin{aligned}
\|Gf\|_{Z_T} \le c\biggl(&\|f\|_{L^1_T(L^2)} + \|f(0)\|_2 + \|f\|_{L^\infty_T(L^2)} + \||x|^2 f\|_{X'_T} \\
&+ \sum_{j,\,k=1}^{n} \|x_j \partial_k f\|_{X'_T} + \|\partial_t f\|_{X'_T} \biggr) + cT\|Gf\|_{Z_T}.
\end{aligned}
$$

Therefore, if $T > 0$ is sufficiently small,

$$\|Gf\|_{Z_T} \le c\biggl(\|f\|_{L^1_T(L^2)} + \|f(0)\|_2 + \|f\|_{L^\infty_T(L^2)} \\ + \||x|^2 f\|_{X'_T} + \sum_{j,\,k=1}^{n} \|x_j \partial_k f\|_{X'_T} + \|\partial_t f\|_{X'_T}\biggr) \tag{3.10}$$

for $f \in \mathcal{S}(\mathbf{R}^{n+1})$. By the density argument, (3.10) holds for any f satisfying the assumptions of this lemma. Since $\|f\|_{L^1_T(L^2)} \le T\|f\|_{L^\infty_T(L^2)}$, $\|f(0)\|_2 \le \|f\|_{L^\infty_T(L^2)}$, this implies (3.7) for $T > 0$ sufficiently small. In view of Lemma 2.5 and (3.8), it is easy to see that $\partial_t Gf \in C(I_T, L^2)$. Therefore we see that $Gf \in \bar{Z}_T$ if in addition $f \in C(I_T, L^2)$. □

Remark 3.4 Note that ΔGf does not always belong to the auxiliary space $L^\gamma_T(L^\rho)$ for f satisfying the assumptions of Lemma 3.3. On the other hand, for $\Sigma(1)$-solution u, $\nabla GF(u)$ belongs to the auxiliary space (cf. [16]).

To estimate the nonlinear term, we need the following two lemmas in Kato [11].

Lemma 3.4 *F maps L^{2p} into L^2 continuously, and maps any bounded set of L^{2p} into a bounded set of L^2.*

Let $\theta = 1 - l$. Recall that $l = (n/4)(1 - 1/p)$.

Lemma 3.5 *There exists $C > 0$ independent of T such that*

$$\|u(t) - u(s)\|_2 \le C|t - s|\,\|u\|_{Z_T},$$
$$\|u(t) - u(s)\|_{2p} \le C|t - s|^\theta \|u\|_{Z_T}$$

for any $u \in Z_T$ and t, $s \in I_T$.

We prove Theorem 1 by the contraction mapping argument. We introduce the following integral equation

$$u(t) = (U\phi)(t) - i(GF(u))(t). \tag{3.11}$$

We define the operator

$$K(u) = U\phi - iGF(u)$$

and the ball in Z_T

$$B_{T,R} = \{u \in Z_T \colon \|u\|_{Z_T} \le R,\ u(0) = \phi\}$$

for T, $R > 0$.

Remark 3.5 $B_{T,R}$ is a complete metric space in X_T metric. We can prove this property following, e.g., the proof of Proposition 6.6 in Kato [11].

Recall that we prove the case of $n \geq 4$.

Proposition 3.1 *Let* $\phi \in \Sigma(2)$. *Suppose that* $F(u)$ *satisfies* Assumptions (F1) *and* (F2) *with* $1 \leq p < 1 + 4/(n-4)$. K *maps* $B_{T,R}$ *into* $B_{T,R}$ *if* R *is sufficiently large and* T *is sufficiently small, depending only on* $\|\phi\|_{\Sigma(2)}$.

Proof. Let $u \in B_{T,R}$. By the Hölder inequality and (3.1), we have

$$\begin{aligned}
\||x|^2 F(u)\|_{X'_T} &\leq M_1 \||x|^2 u\|_{L^1_T(L^2)} + M_2 \||x|^2 |u|^p\|_{L^{\gamma'}_T(L^{\rho'})} \\
&\leq M_1 T \||x|^2 u\|_{L^\infty_T(L^2)} + M_2 T^\theta \||x|^2 |u|^p\|_{L^{\gamma}_T(L^{\rho'})} \\
&\leq M_1 T \||x|^2 u\|_{L^\infty_T(L^2)} + M_2 T^\theta \||x|^2 u\|_{L^{\gamma}_T(L^{\rho})} \|u\|^{p-1}_{L^\infty_T(L^{2p})} \\
&\leq M_1 T \|u\|_{Z_T} + c M_2 T^\theta \|u\|^p_{Z_T}.
\end{aligned}$$

We have used $1/\gamma' - 1/\gamma = \theta > 0$ and $(p-1)/2p + 1/\rho = 1/\rho'$. Similarly we have

$$\begin{aligned}
\|x_j \partial_k F(u)\|_{X'_T} &\leq M_1 T \|u\|_{Z_T} + c M_2 T^\theta \|u\|^p_{Z_T}, \\
\|\partial_t F(u)\|_{X'_T} &\leq M_1 T \|u\|_{Z_T} + c M_2 T^\theta \|u\|^p_{Z_T}.
\end{aligned}$$

By Remark 2.3, we have for $z_1,\ z_2 \in \mathbf{C}$,

$$\begin{aligned}
|F_1(z_1) - F_1(z_2)| &\leq M_1 |z_1 - z_2|, \\
|F_2(z_1) - F_2(z_2)| &\leq M_2 |z_1 - z_2| (|z_1|^{p-1} + |z_2|^{p-1}).
\end{aligned}$$

From Lemma 3.5, (3.1) and the Hölder inequality, we have

$$\begin{aligned}
\|F_1(u(t)) - F_1(u(s))\|_2 &\leq M_1 \|u(t) - u(s)\|_2 \\
&\leq c M_1 |t-s| \, \|u\|_{Z_T}
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\|F_2(u(t)) - F_2(u(s))\|_2 &\leq M_2 \||u(t) - u(s)|(|u(t)|^{p-1} + |u(s)|^{p-1})\|_2 \\
&\leq M_2 \|u(t) - u(s)\|_{2p} (\|u(t)\|^{p-1}_{2p} + \|u(s)\|^{p-1}_{2p}) \\
&\leq c M_2 |t-s|^\theta \|u\|^p_{Z_T}
\end{aligned} \tag{3.13}$$

for any t, $s \in I_T$. Therefore from Lemma 3.4, we obtain

$$\begin{aligned}\|F(u)\|_{L_T^\infty(L^2)} &\leq \|F(\phi)\|_{L_T^\infty(L^2)} + \|F(u) - F(\phi)\|_{L_T^\infty(L^2)} \\ &\leq c\|\phi\|_{\Sigma(2)} + cM_1T\|u\|_{Z_T} + cM_2T^\theta\|u\|_{Z_T}^p,\end{aligned} \tag{3.14}$$

where $u(0) = \phi$. From these estimates, if $T > 0$ is small enough, we can apply Lemma 3.3 with $f = F(u)$. Then we have

$$\|GF(u)\|_{Z_T} \leq c\|\phi\|_{\Sigma(2)} + cM_1T\|u\|_{Z_T} + cM_2T^\theta\|u\|_{Z_T}^p$$

for T sufficiently small. By Lemma 3.2, we see that if $T > 0$ is small enough,

$$\|K(u)\|_{Z_T} \leq c\|\phi\|_{\Sigma(2)} + cM_1T\|u\|_{Z_T} + cM_2T^\theta\|u\|_{Z_T}^p. \tag{3.15}$$

The fact $u \in B_{T,R}$ implies

$$\|K(u)\|_{Z_T} \leq c\|\phi\|_{\Sigma(2)} + cM_1TR + cM_2T^\theta R^p.$$

Hence we can choose $R > 0$ sufficiently large and $T > 0$ sufficiently small so that

$$c\|\phi\|_{\Sigma(2)} + cM_1TR + cM_2T^\theta R^p \leq R.$$

It follows from (3.15) that

$$\|K(u)\|_{Z_T} \leq R.$$

□

Proposition 3.2 *Suppose that $F(u)$ satisfies* Assumptions (F1) *and* (F2) *with $1 \leq p < 1 + 4/(n-4)$. If R is sufficiently large and T is sufficiently small, K maps $B_{T,R}$ onto itself and it is contraction in the metric of X_T over $B_{T,R}$, where R and T depend only on $\|\phi\|_{\Sigma(2)}$.*

Proof. Let u, $v \in B_{T,R}$. By the definition of K, we see

$$K(u) - K(v) = -i(GF(u) - GF(v)).$$

Then as in the proof of Proposition 3.1, we have

$$\|K(u) - K(v)\|_{X_T} \leq (cM_1T + cM_2T^\theta R^{p-1})\|u - v\|_{X_T}. \tag{3.16}$$

We can choose R sufficiently large and T sufficiently small, depending only on $\|\phi\|_{\Sigma(2)}$, so that

$$cM_1T + cM_2T^\theta R^{p-1} \leq \frac{1}{2}.$$

The proof of this proposition completes. □

Now we prove Theorem 1.

Proof of Theorem 1. First we consider the subcritical case, that is, we assume that $n \geq 4$ and that $1 \leq p < 1 + 4/(n-4)$. We show the existence result. By Remark 3.5, Propositions 3.1 and 3.2, if R is sufficiently large and T is sufficiently small, K has a unique fixed point u in $B_{T,R}$. Namely, u is a unique solution of (3.11) in $B_{T,R}$. By (3.12) and (3.13), we see that $F(u) \in C(I_T, L^2)$. Therefore $u \in \bar{Z}_T$ follows from Lemmas 3.2 and 3.3. Since $F(u) \in C(I_T, L^2)$, u is a solution of (1.1)–(1.2) by Lemmas 2.3 and 2.4.

We next prove the uniqueness of the solution. Let u, $v \in C(I_T, \Sigma(2))$ be solutions of (1.1) with $u(0) = v(0) = \phi$. Then u, v satisfy the integral equation (3.11). Therefore, as in the proof of Proposition 3.2,

$$\begin{aligned}\|u - v\|_{X_T} &= \|GF(u) - GF(v)\|_{X_T} \\ &\leq \left\{cM_1T + cM_2T^\theta\big(\|u\|^{p-1}_{L^\infty_T(\Sigma(2))} + \|v\|^{p-1}_{L^\infty_T(\Sigma(2))}\big)\right\}\|u - v\|_{X_T}.\end{aligned}$$

We can choose $T > 0$ sufficiently small, depending only on $\|\phi\|_{\Sigma(2)}$, so that

$$cM_1T + cM_2T^\theta\big(\|u\|^{p-1}_{L^\infty_T(\Sigma(2))} + \|v\|^{p-1}_{L^\infty_T(\Sigma(2))}\big) \leq \frac{1}{2}.$$

Hence, if $T > 0$ sufficiently small, we have

$$\|u - v\|_{X_T} = 0.$$

We next show that $\partial_t u \in \bar{\mathcal{X}}_T$. From (3.8), we note that

$$\begin{aligned}i\partial_t u &= H(t)U\phi + i\partial_t GF(u) \\ &= H(t)U\phi + iG\partial_t F(u) - iUF(\phi) - \frac{1}{2}(\Delta GF(u) - G\Delta F(u)) \\ &\quad + G\{(A \cdot \nabla - i\tilde{V})F(u)\} - (A \cdot \nabla - i\tilde{V})GF(u). \end{aligned} \tag{3.17}$$

Since terms in RHS of (3.17) except the 1st, the 4th and the last terms are images of U or G, they are in $\mathcal{X}_T$. For any (q, r) satisfying (1.4), we can see that, in the exactly same way as the proof of (3.5), (3.7),

$$\|H(t)U\phi\|_{L^r_T(L^q)} \leq c\|\phi\|_{\Sigma(2)},$$

and that

$$\|\Delta GF(u) - G\Delta F(u)\|_{L^r_T(L^q)}, \ \|(A \cdot \nabla - i\tilde{V})GF(u)\|_{L^r_T(L^q)}$$

$$\le C\bigg(\|F(u)\|_{L^\infty_T(L^2)} + \||x|^2F(u)\|_{X'_T} + \sum_{j,\,k=1}^{n} \|x_j\partial_k F(u)\|_{X'_T} + \|\partial_t F(u)\|_{X'_T}\bigg).$$

We have used Lemma 2.5 for the estimate of the first and the last terms, and (3.9) for that of the 4th term, respectively. Thus we obtain $\partial_t u \in \mathcal{X}_T$. Since $u \in \bar{Z}_T$, we have $\partial_t u \in \bar{\mathcal{X}}_T$.

Finally we show the continuous dependence on the initial datum. Assume that $\phi \in \Sigma(2)$ and that $u \in C(I_{T_0}, \Sigma(2))$ is a solution of (1.1) with $u(0) = \phi$. We also assume that $\phi_m \in \Sigma(2)$ for $m = 1, 2, \dots,$ and that $\phi_m \to \phi$ as $m \to \infty$ in $\Sigma(2)$. By standard continuation argument, it is sufficient to prove that $u_m \to u$ in Z_T for $T > 0$ sufficiently small depending only on $\|\phi\|_{\Sigma(2)}$. Let

$$K_m(v) = U\phi_m - iGF(v)$$

for $m = 1, 2, \dots$. By the same argument as above, we can show that if $T > 0$ is sufficiently small and $R > 0$ is sufficiently large, depending only on $\|\phi\|_{\Sigma(2)}$, u is a unique fixed point of K in $B_{T,R}$, and K_m has a unique fixed point u_m in $B_{T,R}$ for m sufficiently large. Then u_m is a unique solution of (1.1) in $C(I_T, \Sigma(2))$ with $u_m(0) = \phi_m$. As in the proof of Proposition 3.2, we see that

$$\begin{aligned}\|u_m - u\|_{X_T} &= \|K_m(u_m) - K(u)\|_{X_T}\\ &\le \|U\phi_m - U\phi\|_{X_T} + \|GF(u_m) - GF(u)\|_{X_T}\\ &\le c\|\phi_m - \phi\|_2 + (cM_1T + cM_2T^\theta R^{p-1})\|u_m - u\|_{X_T}.\end{aligned}$$

This implies that if $T > 0$ is small enough,

$$\|u_m - u\|_{X_T} \le c\|\phi_m - \phi\|_{\Sigma(2)}.$$

We obtain that $u_m \to u$ in $X_T \subset L^\infty_T(L^2)$. On the other hand, since $\|u_m(t)\|_{H^2} \le \|u_m\|_{Z_T} \le R$, it follows that $u_m \to u$ in $L^\infty_T(H^{2l})$ for any $l < 1$. By Lemmas 3.2 and 3.3, we see that

$$\begin{aligned}&\|u_m - u\|_{Z_T}\\ &\quad\le c\|\phi_m - \phi\|_{\Sigma(2)}\\ &\qquad + c\bigg(\|F(u_m) - F(u)\|_{L^\infty_T(L^2)} + \||x|^2(F(u_m) - F(u))\|_{X'_T}\end{aligned}$$

$$+ \sum_{j,k=1}^{n} \|F'(u_m)x_j\partial_k(u_m-u)\|_{X'_T}$$
$$+ \sum_{j,k=1}^{n} \|(F'(u_m)-F'(u))x_j\partial_k u\|_{X'_T}$$
$$+ \|F'(u_m)(\partial_t u_m - \partial_t u)\|_{X'_T} + \|(F'(u_m)-F'(u))\partial_t u\|_{X'_T}\Big).$$

As in the proof of Proposition 3.1, we have

$$\begin{aligned}
&\|F(u_m)-F(u)\|_{L^\infty_T(L^2)} \le C\|u_m-u\|_{Z_T},\\
&\||x|^2(F(u_m)-F(u))\|_{X'_T} \le M_1 T\||x|^2(u_m-u)\|_{L^\infty_T(L^2)}\\
&\qquad\qquad + M_2 T^\theta R^{p-1}\||x|^2(u_m-u)\|_{L^\gamma_T(L^\rho)}\\
&\qquad\qquad \le (M_1T+M_2T^\theta R^{p-1})\||x|^2(u_m-u)\|_{X_T},\\
&\|F'(u_m)x_j\partial_k(u_m-u)\|_{X'_T} \le (M_1T+M_2T^\theta R^{p-1})\|x_j\partial_k(u_m-u)\|_{X_T},\\
&\|F'(u_m)(\partial_t u_m-\partial_t u)\|_{X'_T} \le (M_1T+M_2T^\theta R^{p-1})\|\partial_t u_m-\partial_t u\|_{X_T}.
\end{aligned}$$

It remains to prove

$$(F'(u_m)-F'(u))\partial_t u \to 0, \tag{3.18}$$
$$(F'(u_m)-F'(u))x_j\partial_k u \to 0, \tag{3.19}$$

as $m\to\infty$ in X'_T. By Lemma 2.5 and the Hölder inequality, it is easily seen

$$\begin{aligned}
&\|(F'(u_m)-F'(u))\partial_t u\|_{X'_T}\\
&\quad \le \|(F'_1(u_m)-F'_1(u))\partial_t u\|_{L^1_T(L^2)} + \|(F'_2(u_m)-F'_2(u))\partial_t u\|_{L^{\gamma'}_T(L^{\rho'})}\\
&\quad \le T\|(F'_1(u_m)-F'_1(u))\partial_t u\|_{L^\infty_T(L^2)}\\
&\qquad + T^\theta\|(F'_2(u_m)-F'_2(u))\|_{L^\infty_T\left(L^{2p/(p-1)}\right)}\|\partial_t u\|_{X_T}.
\end{aligned}$$

Then noting Remark 2.3, we see that

$$\|(F'_1(u_m)-F'_1(u))\partial_t u\|_{L^\infty_T(L^2)} \to 0,$$

as $m\to\infty$, by Remark 4.3 in Kato [11], the dominated convergence theorem and the fact $\partial_t u \in X_T \subset L^\infty_T(L^2)$, and by Lemma 4.2 in Kato [11] and the fact that $u_m \to u$ in $L^\infty_T(H^{2l})$ for any $l<1$, we have

$$\|F'_2(u_n)-F'_2(u)\|_{L^\infty_T\left(L^{2p/(p-1)}\right)} \to 0,$$

as $m \to \infty$. These imply (3.18). Similarly, we can prove (3.19) since $x_j \partial_k u \in X_T \subset L^\infty_T(L^2)$.

In the critical case, namely, when $n \geq 5$ and $p = 1 + 4/(n-4)$, we have $\theta = 0$. If $C_1 \|\phi\|_{\Sigma(2)} < \delta$ for $C_1 > 0$ appeared in Lemma 3.2 and sufficiently small $\delta > 0$, then for $R > 0$ satisfying $cM_2 R^{p-1} < 1/2$ and $T > 0$ depending only on δ, we have

$$\delta + cM_1 TR + cM_2 R^p \leq R.$$

Thus for these R and T, the assertions in Propositions 3.1 and 3.2 hold, and we have K has a unique fixed point u in $B_{T,R}$. We can prove the remained parts in the same way as above. We remark that $u_m \to u$ in $L^\infty_T(H^2)$ in this case. We omit the details. □

4. Proof of Theorem 2

Before the proof of Theorem 2, We introduce the following results of Yajima [30].

Lemma 4.1 (Yajima [30]) *Assume* Assumptions (A) *and* (V). *Let $T > 0$ be sufficiently small, $\mu > 1/2$ and $\rho > 0$. There exists a constant $C > 0$, depending on μ and ρ, such that for $s \in \mathbf{R}$ and $f \in \mathcal{S}(\mathbf{R}^n)$*

$$\int_{s-T}^{s+T} \|\langle x\rangle^{-\mu-\rho}\langle D_x\rangle^{\rho} U(t,s) f\|_2^2 \, dt \leq C \|\langle D_x\rangle^{\rho-1/2} f\|_2^2.$$

Using Lemma 4.1, we prove Theorem 2.

Proof of Theorem 2. By (3.11), (1.3), Lemma 2.4 and (3.8), we have

$$\begin{aligned}
&\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2} u \\
&\quad = \langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2} U\phi - i\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2} GF(u) \\
&\quad = \langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2} U\phi + 2\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2} UF(\phi) \\
&\qquad + 2\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2} G\partial_t F(u) \\
&\qquad + i\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}(\Delta GF(u) - G\Delta F(u)) \\
&\qquad - \langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2} G[2A\cdot\nabla F(u) - 2i\tilde{V}F(u) + iF(u)] \\
&\qquad + 2i\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2} F(u).
\end{aligned} \tag{4.1}$$

First we estimate the 1st and 2nd term in the RHS of (4.1). By Lemma 4.1, it is easily seen that

$$\int_{I_T} \|\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{5/2}(U\phi)(t)\|_2^2\,dt \le c\|\langle D_x\rangle^2\phi\|_2^2 < \infty,$$
$$\int_{I_T} \|\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}(UF(\phi))(t)\|_2^2\,dt \le c\|F(\phi)\|_2^2 < \infty.$$

We have used (3.1) and Lemma 3.4 in the second estimate. To estimate the 3rd, the 4th and the 5th terms in the RHS of (4.1), we need the following lemma (cf. [2, 15, 16, 17, 21, 22]).

Lemma 4.2 *For $\mu > 1/2$, there exists $C > 0$, depending on μ such that*

$$\left(\int_{I_T} \|\langle x\rangle^{-\mu-1/2}\langle D_x\rangle^{1/2}(Gf)(t)\|_2^2\,dt\right)^{1/2} \le C\|f\|_{L_T^1(L^2)},$$

Proof. Let $g \in C_0^\infty(I_T \times \mathbf{R}^n)$. By Lemma 4.1 and the Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\int_{I_T} \big(\langle x\rangle^{-\mu-1/2}\langle D_x\rangle^{1/2}(Gf)(t), g(t)\big)\,dt\right| \\
&\le \int_{I_T}\int_0^t \big|\big(\langle x\rangle^{-\mu-1/2}\langle D_x\rangle^{1/2}U(t,s)f(s), g(t)\big)\big|\,ds\,dt \\
&\le \int_{I_T}\int_{I_T} \|\langle x\rangle^{-\mu-1/2}\langle D_x\rangle^{1/2}U(t,s)f(s)\|_2\|g(t)\|_2\,dt\,ds \\
&\le \left(\int_{I_T}\left(\int_{I_T} \|\langle x\rangle^{-\mu-1/2}\langle D_x\rangle^{1/2}U(t,s)f(s)\|_2^2\,dt\right)^{1/2} ds\right)\|g\|_{L^2(I_T\times\mathbf{R}^n)} \\
&\le c\left(\int_{I_T} \|f(s)\|_2\,ds\right)\|g\|_{L^2(I_T\times\mathbf{R}^n)},
\end{aligned}$$

where $(\,\cdot\,,\,\cdot\,)$ is the $L^2(\mathbf{R}^n)$ scalar product. By the duality argument, we have this lemma. □

For the 4th term in the RHS of (4.1), we have

$$\begin{aligned}
&\left(\int_{I_T} \big\|\langle x\rangle^{-\mu-5/2}\langle D_x\rangle^{1/2}\big((\Delta GF(u))(t) - (G\Delta F(u))(t)\big)\big\|_2^2\,dt\right)^{1/2} \\
&\quad \le c\Big(\|GF(u)\|_{L_T^1(L^2)} \vee \||x|^2 GF(u)\|_{L_T^1(L^2)}
\end{aligned}$$

$$\vee \sum_{j,\,k=1}^{n} \|x_j\partial_k GF(u)\|_{L^1_T(L^2)} \vee \|\Delta GF(u)\|_{L^1_T(L^2)}\Big)$$

$$\le cT\|GF(u)\|_{Z_T}$$

$$\le cT\Big(\|F(u)\|_{L^\infty_T(L^2)} + \||x|^2F(u)\|_{X'_T} + \sum_{j,\,k=1}^{n}\|x_j\partial_k F(u)\|_{X'_T} + \|\partial_t F(u)\|_{X'_T}\Big).$$

Since $u \in Z_T$, it is clear that the RHS of above inequality is finite.

According to Lemma 4.2, to estimate the $L^2(I_T \times \mathbf{R}^n)$-norm of the 3rd term in the RHS of (4.1), it is sufficient to show

$$\partial_t F(u) = F'(u)\partial_t u \in L^1_T(L^2), \tag{4.2}$$

and to estimate the $L^2(I_T \times \mathbf{R}^n)$-norm of the 5th term in the RHS of (4.1), it is enough to prove

$$F(u) \in L^1_T(L^2), \tag{4.3}$$

$$|x|^2 F(u) \in L^1_T(L^2), \tag{4.4}$$

$$x_j\partial_k F(u) = F'(u)(x_j\partial_k u) \in L^1_T(L^2). \tag{4.5}$$

On the other hand, to estimate the $L^2(I_T \times \mathbf{R}^n)$-norm of the 6th term in the RHS of (4.1), it is sufficient to prove

$$F(u) \in L^2(I_T \times \mathbf{R}^n), \tag{4.6}$$

$$\nabla F(u) = F'(u)\nabla u \in L^2(I_T \times \mathbf{R}^n). \tag{4.7}$$

(4.3) and (4.6) follow from (3.14). We show (4.2). When $n \le 3$, since $u \in C(I_T, \Sigma(2)) \hookrightarrow L^\infty(I_T \times \mathbf{R}^n)$ and $\partial_t u \in L^\infty_T(L^2)$, we have $\partial_t F(u) \in L^1_T(L^2)$. When $n \ge 4$, since $u \in L^\infty_T(H^2)$, we see that $u \in L^\infty_T(L^q)$ for $2 \le q < \infty$ when $n = 4$ and for $2 \le q \le 2n/(n-4)$ when $n \ge 5$. For proving in the case of $n \ge 5$, we note that there exist the real constants a, b satisfying

$$\frac{p-1}{a} + \frac{1}{b} = \frac{1}{2},$$

$$\frac{1}{2} - \frac{2}{n} \le \frac{1}{a} \le \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{n} \le \frac{1}{b} \le \frac{1}{2}.$$

Therefore we obtain that, by the Hölder inequality and the fact $\partial_t u \in \mathcal{X}_T$,

$$\|F'(u)\partial_t u\|_{L^1_T(L^2)}$$

$$
\begin{aligned}
&\le cM_1T\|\partial_t u\|_{L^\infty_T(L^2)} + cM_2T^{1/2}\||u|^{p-1}\partial_t u\|_{L^2(I_T\times\mathbf{R}^n)}\\
&\le cM_1T\|\partial_t u\|_{L^\infty_T(L^2)} + cM_2T^{1/2}\|u\|^{p-1}_{L^\infty_T(L^a)}\|\partial_t u\|_{L^2_T(L^b)}\\
&\le cM_1T\|\partial_t u\|_{L^\infty_T(L^2)} + cM_2T^{1-1/r}\|u\|^{p-1}_{L^\infty_T(L^a)}\|\partial_t u\|_{L^r_T(L^b)}\\
&< \infty,
\end{aligned}
$$

where r is a constant such that (b, r) is an admissible pair, and when $n \ge 5$ we can estimate in the similar way. We have (4.2).

By (3.11), (3.4) and (3.3), we see that $|x|^2u$ and $x_j\partial_k u$ are represented as the sum of the images of U or G. Thus $|x|^2u$, $x_j\partial_k u \in \mathcal{X}_T$. In the exactly same way as the proof of (4.2), we can show (4.4) and (4.5).

We can also show (4.7) in the similar way to (4.2). Proof of Theorem 2 is completed. □

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