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Momentum operators with a winding gauge potential

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Abstract. Considered is a quantum system of N (≥ 2) charged particles moving in the plane $\mathbb{R}^2$ under the influence of a perpendicular magnetic field concenrated on the positions where the particle exsists. The gauge potential which gives this magnetic field is called a winding gauge potential. Properties of the momentum operators with a winding gauge potential are investigated. The momentum operators with a winding gauge potential are represented by the fibre direct integral of Arai's momentum operators [1]. Using this fibre direct integral decomposition, commutation properties of the momentum operators are investigated. A notion of local quantization of the magnetic flux is introduced to characterize the strong commutativity of the momentum operators. Aspects of the representation of the canonical commutation relations (CCR) are discussed. There is an interesting relation between the representation of the CCR with respect to this system and Arai's representation. Some applications of those results are also discussed.

Key words: momentum operators with a winding gauge potential, strong commutativity, representation of the CCR.

1. Introduction

In Ref. [1, 2, 3, 4], A. Arai investigated commutation properties of two dimensional momentum operators with a strongly singular gauge potential. In those papers, he showed some interesting results. Especially, there exsists a beautiful relation between representations of the canonical commutation relations (CCR) and the local quantization of the magnetic flux.

The main aim of this paper is to analyze a quantum system of N (≥ 2) particles moving in $\mathbb{R}^2$ under the influence of a perpendicular magnetic field concenrated on the positions where the particle exsists. The gauge potential which gives this magnetic field is said to be a winding gauge potential, and is strongly singular. We note that R.B. Laughlin et al. [5, 6] discussed this system in connection with the fractional statics gas (their discussions in [5, 6] are heuristic). Hence, to investigate this sytem is also important from a physical point of view.

We show that the momentum operators with a winding gauge potential can be represented as the direct integral of Arai's momentum operator. In this sense, our result is a natural extension of Arai's work [1, 2, 3, 4].

It is also important whether the magnetic flux is locally quantized or not. Indeed, we see that the local quantization of the magnetic flux is closely connected with the Schrödinger representation of the CCR's. As an application of those results, we study a class of Schrödinger operators with a winding gauge potential. Moreover, there are some other important properties about this system. In particular, we see that there is an interesting correspondence between bosons and fermions, so called the statistical transformation.

The outline of the present paper is as follows. In Section 2, we introduce a winding gauge potential and show the self-adjointness of the momentum operators with a winding gauge potential. We also investigate the commutation relations (in the strong sense) of the momentum operators with a winding gauge potential. To do this, we express the momentum operators as direct integrals of Arai's momentum operator. By using this expression, we prove that the momentum operators strongly commute if and only if the magnetic flux is locally quantized. In Section 3, we apply the preceding results to the theory of reperesentaion of the CCR. We show that the momentum operators with a winding gauge potential and the position operators fulfills the Weyl relation if and only if the magnetic flux is locally quantized. Furthermore, we discuss a relation between direct integral representation of Arai's representation of the CCR and our system. In Section 4, we define the Hamiltonian with a winding gauge potential and investigate the properties of this Hamiltonian. We note that formal discussion of this Hamiltonian is found in Ref. [5, 6]. Moreover, we introduce the statistical transformation and disscus some applications. This transformation gives the correspondence between bosons and fermions, and comes from the two dimensionality of the sytem.

2. Momentum operators with a winding gauge potential

2.1. Definition of the momentum operators with a winding gauge potential

We consider a quantum system of N (≥ 2) charged particles with charge $q \in \mathbb{R} \setminus \{0\}$, where each particle feels a perpendicular magnetic field B_j $(j = 1, \ldots, N)$ given by a real distribution of the form

$$B_j(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \gamma \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}_j), \quad \mathbf{r}_1, \ldots, \mathbf{r}_N \in \mathbb{R}^2, \ \mathbf{r}_j = (x_j, y_j),$$

where $\gamma \in \mathbb{R}$ and $\delta(\mathbf{r})$ is the Dirac's delta distribution. Gauge potentials $\mathbb{A}_j$ $(j = 1, \ldots, N)$ of the magnetic field B_j are defined to be $\mathbb{R}^2$-valued functions $\mathbb{A}_j = (A_{j1}, A_{j2})$ on the domain

$$\mathcal{M}_N := \left\{(\mathbf{r}_1, \ldots, \mathbf{r}_N) \in \mathbb{R}^{2N} \;\middle|\; \mathbf{r}_i \neq \mathbf{r}_j \quad (i \neq j)\right\}$$

such that

$$B_j = D_{x_j} A_{j2} - D_{y_j} A_{j1}$$

in the sense of distribution on $\mathbb{R}^{2N}$, where D_{x_j} and D_{y_j} denote the distributional partial differential operators in x_j and y_j, respectively.

We denote by Δ_j $(j = 1, \ldots, N)$ the 2 dimensional Laplacian

$$\Delta_j := D_{x_j}^2 + D_{y_j}^2.$$

Using the well-known formula

$$\Delta_j \log |\mathbf{r}_j - \mathbf{r}_k| = 2\pi \delta(\mathbf{r}_j - \mathbf{r}_k) \quad (k \neq j),$$

we see that the distribution

$$\phi_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) := \sum_{i<j} \frac{\gamma}{2\pi} \log |\mathbf{r}_i - \mathbf{r}_j|$$

satisfies

$$\Delta_j \phi_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) = B_j(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

From this fact, we can take as a gauge potential of the magnetic field

$$\mathbb{A}_j = (A_{j1}, A_{j2}) = (-D_{y_j}\phi_N, D_{x_j}\phi_N), \quad j = 1, \ldots, N.$$

Explicitely, we have

$$A_{j1}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = -\frac{\gamma}{2\pi} \sum_{i \neq j} \frac{y_j - y_i}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \tag{1}$$

$$A_{j2}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{\gamma}{2\pi} \sum_{i \neq j} \frac{x_j - x_i}{|\mathbf{r}_i - \mathbf{r}_j|^2}. \tag{2}$$

Definition 2.1 The gauge potential $\mathbb{A}_j = (A_{j1}, A_{j2})$ $(j = 1, \ldots N)$ given by (1) and (2) is called the *winding gauge potential*.

We use a system of units where the light speed c and the Planck constant $\hbar$ are equal to 1. Let

$$p_{j1} := -\mathrm{i}D_{x_j}, \quad p_{j2} := -\mathrm{i}D_{y_j} \quad (j = 1, \dots, N),$$

in $L^2(\mathbb{R}^{2N})$. The momentum operator $\mathbb{P}_j = (P_{j1}, P_{j2})$ with the gauge potential $\mathbb{A}_j$ is defined by

$$P_{j\alpha} := p_{j\alpha} - qA_{j\alpha}, \quad (j = 1, \dots, N, \ \alpha = 1, 2)$$

in $L^2(\mathbb{R}^{2N})$ with domain $\mathrm{dom}(P_{j\alpha}) = \mathrm{dom}(p_{j\alpha}) \cap \mathrm{dom}(A_{j\alpha})$.

Remark 2.2 We note that almost all discussions in this paper can be extended to following more general case:

$$B_j(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i \neq j} \gamma_{ij} \delta(\mathbf{r}_i - \mathbf{r}_j), \quad (\mathbf{r}_1, \dots, \mathbf{r}_j \in \mathbb{R}^2), \tag{3}$$

where $\gamma_{ij} \in \mathbb{R}$ $(i, j = 1, \dots, N, \ i \neq j)$, $\gamma_{ij} = \gamma_{ji}$. But to simplify our discussions, we only consider the case $\gamma_{ij} = \gamma$ $(i, j = 1, \dots, N)$.

2.2. Self-adjointness

Let

$$\mathcal{S}_1^{(N)} := \left\{(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N} \,\middle|\, \mathbf{r}_i = (x_i, y_i) \in \mathbb{R}^2, \ y_i \neq y_j \quad (i \neq j)\right\},$$
$$\mathcal{S}_2^{(N)} := \left\{(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N} \,\middle|\, \mathbf{r}_i = (x_i, y_i) \in \mathbb{R}^2, \ x_i \neq x_j \quad (i \neq j)\right\}$$

and

$$\psi_{j1}(\mathbf{r}_1, \dots, \mathbf{r}_N) := -\frac{\gamma}{2\pi} \sum_{i \neq j} \mathrm{Arctan}\Big(\frac{x_j - x_i}{y_j - y_i}\Big),$$
$$\psi_{j2}(\mathbf{r}_1, \dots, \mathbf{r}_N) := \frac{\gamma}{2\pi} \sum_{i \neq j} \mathrm{Arctan}\Big(\frac{y_j - y_i}{x_j - x_i}\Big).$$

Then it is easy to check that $\psi_{j\alpha} \in C^\infty(\mathcal{S}_\alpha^{(N)})$ and

$$A_{j1} = D_{x_j}\psi_{j1} \quad \text{on } \mathcal{S}_1^{(N)}, \tag{4}$$
$$A_{j2} = D_{y_j}\psi_{j2} \quad \text{on } \mathcal{S}_2^{(N)}. \tag{5}$$

Theorem 2.3 *For each $j = 1, \dots, N$ and $\alpha = 1, 2$, $P_{j\alpha}$ is essentially self-adjoint on $C^\infty(\mathcal{S}_\alpha^{(N)})$.*

Proof. Since $\psi_{j\alpha} \in C^\infty(\mathcal{S}_\alpha^{(N)})$, $\mathrm{e}^{\mathrm{i}q\psi_{j\alpha}}$ is a unitary operator such that

$$\mathrm{e}^{\mathrm{i}q\psi_{j\alpha}} C_0^\infty(\mathcal{S}_\alpha^{(N)}) = C_0^\infty(\mathcal{S}_\alpha^{(N)}).$$

By (4) and (5), we have

$$P_{j\alpha} = \mathrm{e}^{\mathrm{i}q\psi_{j\alpha}} p_{j\alpha} \mathrm{e}^{-\mathrm{i}q\psi_{j\alpha}} \quad \text{on } C_0^\infty(\mathcal{S}_\alpha^{(N)}).$$

On the other hand, it is easy to see that $p_{j\alpha}$ is essentially self-adjoint on $C_0^\infty(\mathcal{S}_\alpha^{(N)})$. Hence we have the desired result. □

2.3. Commutation relations of the momentum operators with the winding gauge potential

For $\mathbf{r} = (x, y) \in \mathbb{R}^2$, s, $t \in \mathbb{R}$, we introduce a path $C(\mathbf{r}; s, t)$ which is the rectangular curve:

$$\mathbf{r} \to \mathbf{r} + (s, 0) \to \mathbf{r} + (s, t) \to \mathbf{r} + (0, t) \to \mathbf{r}.$$

Let $D(\mathbf{r}; s, t)$ be the interior of $C(\mathbf{r}; s, t)$ and

$$\epsilon(s) := \begin{cases} 1 & (s \geq 0) \\ -1 & (s < 0) \end{cases}.$$

We denote the closure of $P_{j\alpha}$ by $\overline{P}_{j\alpha}$

Theorem 2.4 *For each s, $t \in \mathbb{R}$ and j, $k = 1, \ldots, N$, we have*

(i) $\mathrm{e}^{\mathrm{i}s\overline{P}_{j1}} \mathrm{e}^{\mathrm{i}t\overline{P}_{k2}} = \exp(-\mathrm{i}q\Phi_{j,k}^{(s,t)}) \mathrm{e}^{\mathrm{i}t\overline{P}_{k2}} \mathrm{e}^{\mathrm{i}s\overline{P}_{j1}}$,

(ii) $\mathrm{e}^{\mathrm{i}s\overline{P}_{j\alpha}} \mathrm{e}^{\mathrm{i}t\overline{P}_{k\alpha}} = \mathrm{e}^{\mathrm{i}t\overline{P}_{k\alpha}} \mathrm{e}^{\mathrm{i}s\overline{P}_{j\alpha}} \quad (\alpha = 1, 2)$,

where, for each $(\mathbf{r}_1, \ldots, \mathbf{r}_N) \in \mathbb{R}^{2N}$ with $\mathbf{r}_i = (x_i, y_i) \in \mathbb{R}^2$, we define

$$\Phi_{j,k}^{(s,t)}(\mathbf{r}_1, \ldots, \mathbf{r}_N) := \begin{cases} \gamma\epsilon(s)\epsilon(t)\#\{i \mid i \neq k,\ \mathbf{r}_i \in D(\mathbf{r}_k; s, t)\} & (j = k) \\ \gamma\epsilon(s)\epsilon(t)\#\{(j, k) \mid (x_k, y_j) \in D((x_j, y_k); s, t)\} & (j \neq k) \end{cases},$$

($\#A$ means the cardinality of the set A).

Definition 2.5 ([1]) We say that the magnetic flux associated with the winding gauge potential is *locally quantized* if $\Phi_{j,k}^{(s,t)}$ is a $2\pi\mathbb{Z}/q$-valued function for all $s, t \in \mathbb{R}$.

Corollary 2.6 *The momentum operators $\overline{P}_{j\alpha}$ strongly commute to each other if ond only if $\gamma/\theta_0 \in \mathbb{Z}$, where $\theta_0 := 2\pi/q$ the flux quanta, equivalently,* *the magnetic flux associated with the winding gauge potential is locally quantized.*

To prove Theorem 3.2, we need some preparations. Let

$$\mathbb{R}^{2N} = \mathbb{R}^2_1 \times \cdots \times \mathbb{R}^2_N,$$

where each $\mathbb{R}^2_i$ $(i = 1, \ldots, N)$ is a copy of $\mathbb{R}^2$. For each j, $k = 1, \ldots, N$, we define

$$\Omega_{jk} := \begin{cases} \mathbb{R}^2_1 \times \cdots \times \widehat{\mathbb{R}^2_j} \times \cdots \times \widehat{\mathbb{R}^2_k} \times \cdots \times \mathbb{R}^2_N & (j \neq k) \\ \mathbb{R}^2_1 \times \cdots \times \widehat{\mathbb{R}^2_j} \times \cdots \times \mathbb{R}^2_N & (j = k) \end{cases},$$

where $\widehat{\mathbb{R}^2_i}$ indicates the omission of $\mathbb{R}^2_i$.

Let $\omega_{jk} := (\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_N) \in \Omega_{jk}$ (if $j = k$, then ω_{jj} is given by $\omega_{jj} = (\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_N) \in \Omega_{jj}$). Then we define the multiplication operators $\tilde{A}_{l\alpha}(\omega_{jk})$ on $L^2(\mathbb{R}^2_j \times \mathbb{R}^2_k)$ $l = j$, k if $(j \neq k)$ and $\tilde{A}_{j\alpha}(\omega_{jj})$ on $L^2(\mathbb{R}^2_j)$ by

$$\begin{aligned} &\tilde{A}_{l\alpha}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k) \\ &\quad := A_{l\alpha}(\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{r}_j, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_{k-1}, \mathbf{r}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_N) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l = j, k,\ j \neq k), \\ &\tilde{A}_{j\alpha}(\omega_{jj})(\mathbf{r}_j) \\ &\quad := A_{j\alpha}(\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{r}_j, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_N), \end{aligned}$$

respectively. We set

$$\mathcal{H}_{jk} = \begin{cases} L^2(\mathbb{R}^2_j \times \mathbb{R}^2_k) & (j \neq k) \\ L^2(\mathbb{R}^2_j) & (j = k) \end{cases}.$$

Then relative to the direct integral decomposition

$$L^2(\mathbb{R}^{2N}) = \int^{\oplus}_{\Omega_{jk}} \mathcal{H}_{jk} \, \mathrm{d}\omega_{jk}, \tag{6}$$

we can represent the multiplication operators $A_{j\alpha}$, $A_{k\alpha}$ as

$$A_{j\alpha} = \int^{\oplus}_{\Omega_{jk}} \tilde{A}_{j\alpha}(\omega_{jk}) \, \mathrm{d}\omega_{jk}, \quad A_{k\alpha} = \int^{\oplus}_{\Omega_{jk}} \tilde{A}_{k\alpha}(\omega_{jk}) \, \mathrm{d}\omega_{jk}. \tag{7}$$

On the other hand, we set

$$\tilde{p}_{i1} := -\mathrm{i}D_{x_i}, \quad \tilde{p}_{i2} := -\mathrm{i}D_{y_i} \quad (i = j, k)$$

acting in $\mathcal{H}_{jk}$, where we also denote the distributional partial differential operators in x_i and y_i acting in $\mathcal{H}_{jk}$ by D_{x_i} and D_{y_i}. Then $\tilde{p}_{i\alpha}$ ($i = j$, k, $\alpha = 1$, 2) is a self-adjoint operator. Moreover, it is clear that

$$p_{j\alpha} = \int_{\Omega_{jk}}^{\oplus} \tilde{p}_{j\alpha}\,\mathrm{d}\omega_{jk}, \quad p_{k\alpha} = \int_{\Omega_{jk}}^{\oplus} \tilde{p}_{k\alpha}\,\mathrm{d}\omega_{jk} \tag{8}$$

for $\alpha = 1$, 2.

For each $\omega_{jk} \in \Omega_{jk}$, we define

$$\begin{aligned} &P_{j\alpha}(\omega_{jk}) := \tilde{p}_{j\alpha} - q\tilde{A}_{j\alpha}(\omega_{jk}), \\ &\mathrm{dom}(P_{j\alpha}(\omega_{jk})) := \mathrm{dom}(\tilde{p}_{j\alpha}) \cap \mathrm{dom}(\tilde{A}_{j\alpha}(\omega_{jk})) \end{aligned}$$

and

$$\begin{aligned} &P_{k\alpha}(\omega_{jk}) := \tilde{p}_{k\alpha} - q\tilde{A}_{k\alpha}(\omega_{jk}), \\ &\mathrm{dom}(P_{k\alpha}(\omega_{jk})) := \mathrm{dom}(\tilde{p}_{k\alpha}) \cap \mathrm{dom}(\tilde{A}_{k\alpha}(\omega_{jk})). \end{aligned}$$

Remark 2.7 If $j = k$, then the operator $P_{j\alpha}(\omega_{jj})$ is called *Arai's momentum operator* ([1]).

Now, we have a following useful lemma.

Lemma 2.8 *Let $P_{j\alpha}(\omega_{jk})$ and $P_{k\alpha}(\omega_{jk})$ be as above. Then, for all $\omega_{jk} \in \Omega_{jk}$ and $\alpha = 1$, 2, $P_{j\alpha}(\omega_{jk})$ and $P_{k\alpha}(\omega_{jk})$ are essentially self-adjoint.*

Proof. If $j = k$, then we can apply [1, Theorem 3.2]. Hence we only prove the assertion in the case $j \neq k$. For each $\omega_{jk} = (\mathbf{a}_1, \ldots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_N) \in \Omega_{jk}$ ($\mathbf{a}_i = (a_{i1}, a_{i2}) \in \mathbb{R}_i^2$), let

$$\begin{aligned} \tilde{S}_{j1}^{(N)}(\omega_{jk}) := \bigl\{(\mathbf{r}_j, \mathbf{r}_k) \in \mathbb{R}_j^2 \times \mathbb{R}_k^2 \bigm| y_j \neq y_k,\ y_j \neq a_{i2}, \\ (i = 1, \ldots, \hat{j}, \ldots, \hat{k}, \ldots, N)\bigr\}. \end{aligned}$$

We introduce the function $\tilde{\psi}_{j1}(\omega_{jk})$ on $\mathbb{R}_j^2 \times \mathbb{R}_k^2$ by

$$\begin{aligned} &\tilde{\psi}_{j1}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k) \\ &\quad := \psi_{j1}(\mathbf{a}_1, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_k, \ldots, \mathbf{a}_N) \end{aligned}$$

$$= -\frac{\gamma}{2\pi}\Big\{\mathrm{Arctan}\Big(\frac{x_j - x_k}{y_j - y_k}\Big) + \sum_{i \neq j,\, k} \mathrm{Arctan}\Big(\frac{x_j - a_{i1}}{y_j - a_{i2}}\Big)\Big\}$$

for each $(\mathbf{r}_j, \mathbf{r}_k) \in \tilde{S}_{j1}^{(N)}(\omega_{jk})$. Then it is easy to check that $\tilde{\psi}_{j1}(\omega_{jk}) \in C^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$ and

$$D_{x_j}\tilde{\psi}_{j1}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k) = \tilde{A}_{j1}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k) \quad ((\mathbf{r}_j, \mathbf{r}_k) \in \tilde{S}_{j1}^{(N)}(\omega_{jk})).$$

Hence $\mathrm{e}^{\mathrm{i}q\tilde{\psi}_{j1}(\omega_{jk})}$ is a unitary operator such that

$$\mathrm{e}^{\mathrm{i}q\tilde{\psi}_{j1}(\omega_{jk})} C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk})) = C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$$

and

$$P_{j1}(\omega_{jk}) = \mathrm{e}^{\mathrm{i}q\tilde{\psi}_{j1}(\omega_{jk})} \tilde{p}_{j\alpha} \mathrm{e}^{-\mathrm{i}q\tilde{\psi}_{j1}(\omega_{jk})}$$

on $C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$. Since $\tilde{p}_{j1}$ is essentially self-adjoint on $C_0^\infty(\tilde{S}_{j1}^{(N)}(\omega_{jk}))$, we have the desired result. By the similar way, we can prove the assertions about $P_{j2}(\omega_{jk})$, $P_{k1}(\omega_{jk})$ and $P_{k2}(\omega_{jk})$. □

We denote the closure of $P_{j\alpha}(\omega_{jk})$ (resp. $P_{k\alpha}(\omega_{jk})$) by $\overline{P}_{j\alpha}(\omega_{jk})$ (resp. $\overline{P}_{k\alpha}(\omega_{jk})$).

To state the next result, we introduce some basic definitions of operator-valued measurable mappings. Let (Λ, μ) be a measure space and $\mathcal{X}$ be a Hilbert space. We denote the set of bounded operators on $\mathcal{X}$ by $\mathcal{B}(\mathcal{X})$ and the set of linear operators on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$. For a self-adjoint operator T, we denote its spectral measure by $E_T(J)$, where $J \in \mathbb{B}^1$ (the Borel field of $\mathbb{R}$).

For a $\mathcal{B}(\mathcal{X})$-valued mapping $B\colon \Lambda \ni \lambda \to B(\lambda) \in \mathcal{B}(\mathcal{X})$, we say that the mapping $\lambda \to B(\lambda)$ is *measurable* if the mapping $\lambda \to \langle \phi, B(\lambda)\phi \rangle_{\mathcal{X}}$ is measurable for each $\phi \in \mathcal{X}$.

Let $A\colon \Lambda \ni \lambda \to A(\lambda) \in \mathcal{L}(\mathcal{X})$ be a $\mathcal{L}(\mathcal{X})$-valued mapping. We call that the mapping $\lambda \to A(\lambda)$ is *self-adjoint mapping* if $A(\lambda)$ is self-adjoint for μ-a.e. λ.

Let $\lambda \to A(\lambda)$ be a self-adjoint mapping. If the $\mathcal{B}(\mathcal{X})$-valued mapping $\lambda \to E_{A(\lambda)}(J)$ is measurable for each $J \in \mathbb{B}^1$, we say that the mapping $\lambda \to A(\lambda)$ is *measurable*. For a measurable self-adjoint mapping $\lambda \to A(\lambda)$, we can define a self-adjoint operator acting in $\int_\Lambda^\oplus \mathcal{X}\, \mathrm{d}\mu(\lambda)$ by

$$
\begin{aligned}
\operatorname{dom}(A) := \Bigg\{ \Psi \in \int_\Lambda^\oplus \mathcal{H}\, \mathrm{d}\mu(\lambda) \,\Bigg|\, & \Psi(\lambda) \in \operatorname{dom}(A(\lambda)) \ \mu\text{-a.e. } \lambda, \\
& \lambda \to A(\lambda)\Psi(\lambda) \text{ is measurable}, \\
& \int_\Lambda \|A(\lambda)\Psi(\lambda)\|_{\mathcal{H}}^2 \, \mathrm{d}\mu(\lambda) < \infty \Bigg\}, \\
(A\Psi)(\lambda) := A(\lambda)\Psi(\lambda) \quad & \mu\text{-a.e. } \lambda, \Psi \in \operatorname{dom}(A).
\end{aligned}
$$

The operator A is said to be the *direct integral of* $A(\lambda)$ and written by

$$
A = \int_\Lambda^\oplus A(\lambda)\, \mathrm{d}\mu(\lambda).
$$

More details on these objects, see Appendix A.

Proposition 2.9 *For each j, $k = 1, \ldots, N$, $\alpha = 1, 2$, the mapping $\omega_{jk} \to \overline{P}_{i\alpha}(\omega_{jk})$ $(i = j,\ k,\ \alpha = 1,\ 2)$ is a measurable self-adjoint mapping and*

$$
\overline{P}_{j\alpha} = \int_{\Omega_{jk}}^\oplus \overline{P}_{j\alpha}(\omega_{jk})\, \mathrm{d}\omega_{jk}, \quad \overline{P}_{k\alpha} = \int_{\Omega_{jk}}^\oplus \overline{P}_{k\alpha}(\omega_{jk})\, \mathrm{d}\omega_{jk}.
$$

Proof. By Theorem 2.3 and Lemma 2.8, $P_{j\alpha}$ and $P_{j\alpha}(\omega_{jk})$ are essentially self-adjoint. Combing this with (7) and (8), we have the desired result by Theorem A.7. □

Proof of Theorem 2.4. By Propostion 2.9, we have

$$
\exp(\mathrm{i}t\overline{P}_{l\alpha}) = \int_{\Omega_{jk}}^\oplus \exp(\mathrm{i}t\overline{P}_{l\alpha}(\omega_{jk}))\, \mathrm{d}\omega_{jk} \quad (l = j,\ k)
$$

for each $t \in \mathbb{R}$. Hence it suffices to disscus the commutation relations in the theorem at each fibre.

If $j = k$, then we can apply [1, Theorem 2.1] and obtain

$$
\begin{aligned}
&\exp(\mathrm{i}s\overline{P}_{j1}(\omega_{jj}))\exp(\mathrm{i}t\overline{P}_{j2}(\omega_{jj})) \\
&\quad = \exp\big(-\mathrm{i}q\Phi_{j,j}^{(s,t)}(\omega_{jj})\big)\exp(\mathrm{i}t\overline{P}_{j2}(\omega_{jj}))\exp(\mathrm{i}s\overline{P}_{j1}(\omega_{jj})),
\end{aligned}
$$

where $\Phi_{j,j}^{(s,t)}(\omega_{jj})$ is a multiplication operator defined by

$$
\begin{aligned}
&\Phi_{j,j}^{(s,t)}(\omega_{jj})(\mathbf{r}_j) \\
&\quad := \gamma\epsilon(s)\epsilon(t)\#\big\{i \mid i \neq j,\ \omega_{jj}(i) \in D(\mathbf{r}_j; s, t)\big\},
\end{aligned}
$$

where $\omega_{jj} = (\omega_{jj}(1), \dots, \omega_{jj}(N-1)) \in \Omega_{jj}$. Hence we have the desired result in this case.

Next we prove the assertion in the case $j \neq k$. We can apply the Trotter product formula (e.g., [8, Theorem VIII 31]) to each $\overline{P}_{l\alpha}(\omega_{jk})$ $(l = j,\, k)$ to obtain

$$\begin{aligned}&\exp(\mathrm{i}t\overline{P}_{l\alpha}(\omega_{jk}))\\&\quad = \operatorname*{s-lim}_{n\to\infty}\bigl(\exp(\mathrm{i}t\tilde{p}_{l\alpha}/n)\exp(-\mathrm{i}tq\tilde{A}_{l\alpha}(\omega_{jk})/n)\bigr)^n \quad (l = j,\, k)\end{aligned}$$

for each $t \in \mathbb{R}$. Using the fact that

$$(\mathrm{e}^{\mathrm{i}t\tilde{p}_{j1}}\Psi)(\mathbf{r}_j, \mathbf{r}_k) = \Psi(\mathbf{r}_j + (t,0), \mathbf{r}_k) \text{ a.e. } (\mathbf{r}_j, \mathbf{r}_k) \in \mathbb{R}^2_j \times \mathbb{R}^2_k,\ s \in \mathbb{R},$$

we can show that

$$\begin{aligned}&(\exp(\mathrm{i}t\overline{P}_{j1}(\omega_{jk}))\Psi)(\mathbf{r}_j, \mathbf{r}_k)\\&\quad = \exp\left(-\mathrm{i}q\int_0^t \tilde{A}_{j1}(\omega_{jk})(\mathbf{r}_j + (x'_j, 0), \mathbf{r}_k)\,\mathrm{d}x'_j\right)\Psi(\mathbf{r}_j + (t,0), \mathbf{r}_k)\end{aligned}$$

for a.e. $(\mathbf{r}_j, \mathbf{r}_k) \in \mathbb{R}^2_j \times \mathbb{R}^2_k$ and each $\Psi \in L^2(\mathbb{R}^2_j \times \mathbb{R}^2_k)$. Similarly we have

$$\begin{aligned}&(\exp(\mathrm{i}t\overline{P}_{k2}(\omega_{jk}))\Psi)(\mathbf{r}_j, \mathbf{r}_k)\\&\quad = \exp\left(-\mathrm{i}q\int_0^t \tilde{A}_{k2}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k + (0, y'_k))\,\mathrm{d}y'_k\right)\Psi(\mathbf{r}_j, \mathbf{r}_k + (0,t))\end{aligned}$$

for a.e. $(\mathbf{r}_j, \mathbf{r}_k) \in \mathbb{R}^2_j \times \mathbb{R}^2_k$ and each $\Psi \in L^2(\mathbb{R}^2_j \times \mathbb{R}^2_k)$. Using these formulas, we obtain

$$\begin{aligned}&\exp(\mathrm{i}s\overline{P}_{j1}(\omega_{jk}))\exp(\mathrm{i}t\overline{P}_{k2}(\omega_{jk}))\\&\quad = \exp\bigl(-\mathrm{i}q\Phi^{(s,t)}_{j,k}(\omega_{jk})\bigr)\exp(\mathrm{i}t\overline{P}_{k2}(\omega_{jk}))\exp(\mathrm{i}s\overline{P}_{j1}(\omega_{jk})),\end{aligned}$$

where

$$\begin{aligned}&\Phi^{(s,t)}_{j,k}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k)\\&\quad = \int_0^s \tilde{A}_{j1}(\omega_{jk})(\mathbf{r}_j + (x'_j, 0), \mathbf{r}_k)\,\mathrm{d}x'_j\\&\qquad + \int_0^t \tilde{A}_{k2}(\omega_{jk})(\mathbf{r}_j + (s,0), \mathbf{r}_k + (0, y'_k))\,\mathrm{d}y'_k\\&\qquad - \int_0^t \tilde{A}_{k2}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k + (0, y'_k))\,\mathrm{d}y'_k\end{aligned}$$

$$-\int_0^s \tilde{A}_{j1}(\omega_{jk})(\mathbf{r}_j+(x_j',0),\mathbf{r}_k+(0,t))\,\mathrm{d}x_j'$$

for a.e. $(\mathbf{r}_j,\mathbf{r}_k)\in\mathbb{R}_j^2\times\mathbb{R}_k^2$. To calculate $\Phi_{j,k}^{(s,t)}(\omega_{jk})$, we introduce some notations:

$$a_{jk,1}(\mathbf{r}_j,\mathbf{r}_k):=-\frac{\gamma}{2\pi}\frac{y_j-y_k}{|\mathbf{r}_j-\mathbf{r}_k|^2},\quad a_{jk,2}(\mathbf{r}_j,\mathbf{r}_k):=\frac{\gamma}{2\pi}\frac{x_k-x_j}{|\mathbf{r}_j-\mathbf{r}_k|^2},$$
$$b_{jk,1}(\omega_{jk})(\mathbf{r}_j,\mathbf{r}_k):=-\frac{\gamma}{2\pi}\sum_{i\neq j,\,k}\frac{y_j-\omega_{jk}(i)_2}{|\mathbf{r}_j-\omega_{jk}(i)|^2},$$
$$b_{jk,2}:=\frac{\gamma}{2\pi}\sum_{i\neq j,\,k}\frac{x_k-\omega_{jk}(i)_1}{|\mathbf{r}_k-\omega_{jk}(i)|^2},$$

where we use the notations $\omega_{jk}=(\omega_{jk}(1),\ldots,\widehat{\omega_{jk}(j)},\ldots,\widehat{\omega_{jk}(k)},\ldots,\omega_{jk}(N))\in\Omega_{jk}$, $\omega_{jk}(i)=(\omega_{jk}(i)_1,\omega_{jk}(i)_2)\in\mathbb{R}_i^2$. Then it is clear that

$$\tilde{A}_{j1}(\omega_{jk})=a_{jk,1}+b_{jk,1}(\omega_{jk}),\quad \tilde{A}_{k2}(\omega_{jk})=a_{jk,2}+b_{jk,2}(\omega_{jk}).$$

Next, we introduce the new coordinate:

$$\mathbf{r}_{jk}:=(x_k,y_j),\quad \overline{\mathbf{r}}_{jk}:=(x_j,y_k)$$

for each $\mathbf{r}_j=(x_j,y_j)\in\mathbb{R}_j^2$, $\mathbf{r}_k=(x_k,y_k)\in\mathbb{R}_k^2$. Then we can regard $a_{jk,\alpha}$ ($\alpha=1,\,2$) as the function with two variables $\mathbf{r}_{jk}$, $\overline{\mathbf{r}}_{jk}$:

$$\tilde{a}_{jk,\alpha}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk})=a_{jk,\alpha}(\mathbf{r}_j,\mathbf{r}_k),\quad \alpha=1,\,2.$$

On the one hand, we have

$$\begin{aligned}
&\int_0^s a_{jk,1}(\mathbf{r}_j+(x_j',0),\mathbf{r}_k)\,\mathrm{d}x_j'+\int_0^t a_{jk,2}(\mathbf{r}_j+(s,0),\mathbf{r}_k+(0,y_k'))\,\mathrm{d}y_k'\\
&-\int_0^t a_{jk,2}(\mathbf{r}_j,\mathbf{r}_k+(0,y_k'))\,\mathrm{d}y_k'-\int_0^s a_{jk,1}(\mathbf{r}_j+(x_j',0),\mathbf{r}_k+(0,t))\,\mathrm{d}x_j'\\
&=\int_{C(\overline{\mathbf{r}}_{jk};s,t)}\tilde{\mathbf{a}}_{jk}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk}')\cdot\mathrm{d}\overline{\mathbf{r}}_{jk}'\quad(\tilde{\mathbf{a}}_{jk}:=(\tilde{a}_{jk,1},\tilde{a}_{jk,2}))\\
&=\int_{D(\overline{\mathbf{r}}_{jk};s,t)}(D_{x_j'}\tilde{a}_{jk,1}-D_{y_k'}\tilde{a}_{jk,2})\,\mathrm{d}\overline{\mathbf{r}}_{jk}'\quad\text{(by Green's theorem)}\\
&=-\gamma\epsilon(s)\epsilon(t)\#\big\{(j,k)\,\big|\,\mathbf{r}_{jk}\in D(\overline{\mathbf{r}}_{jk};s,t)\big\},
\end{aligned}$$

where we use the fact

$$D_{x_j}\tilde{a}_{jk,1}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk})-D_{y_k}\tilde{a}_{jk,2}(\mathbf{r}_{jk},\overline{\mathbf{r}}_{jk})=-\gamma\delta(\mathbf{r}_{jk}-\overline{\mathbf{r}}_{jk}).$$

On the other hand, we obtain

$$\begin{aligned}
&\int_0^s b_{jk,1}(\omega_{jk})(\mathbf{r}_j + (x'_j, 0), \mathbf{r}_k)\,\mathrm{d}x'_j \\
&+ \int_0^t b_{jk,2}(\omega_{jk})(\mathbf{r}_j + (s, 0), \mathbf{r}_k + (0, y'_k))\,\mathrm{d}y'_k \\
&- \int_0^t b_{jk,2}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k + (0, y'_k))\,\mathrm{d}y'_k \\
&- \int_0^s b_{jk,1}(\omega_{jk})(\mathbf{r}_j + (x'_j, 0), \mathbf{r}_k + (0, t))\,\mathrm{d}x'_j \\
&= 0.
\end{aligned}$$

Combining these facts, we have

$$\Phi^{(s,t)}_{j,k}(\omega_{jk})(\mathbf{r}_j, \mathbf{r}_k) = -\gamma\epsilon(s)\epsilon(t)\#\big\{(j,k)\;\big|\;\mathbf{r}_{jk} \in D(\overline{\mathbf{r}}_{jk}; s, t)\big\}.$$

Therefore we have the assertion about $\overline{P}_{j1}$, $\overline{P}_{k2}$. By a similar way, we have the assertions about $\overline{P}_{j1}$, $\overline{P}_{k1}$ and $\overline{P}_{j2}$, $\overline{P}_{k2}$. □

3. Representation of the CCR

3.1. Schrödinger representation and local quantization of the magnetic flux

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ and $\{p_j, q_j\}_{j=1}^n$ be a set of self-adjoint operators on $\mathcal{H}$. The set

$$\pi := \big\{\mathcal{H},\, \mathcal{D},\, \{p_j, q_j\}_{j=1}^n\big\}$$

is called a *representation of the CCR with n degree of freedom on $\mathcal{D}$* if it satisfies

(i) $\mathrm{dom}(p_j)$, $\mathrm{dom}(q_j) \subset \mathcal{D}$ $(j = 1, \ldots, n)$;
(ii) $p_j\mathcal{D} \subset \mathcal{D}$, $q_j\mathcal{D} \subset \mathcal{D}$ $(j = 1, \ldots, n)$;
(iii) $\{p_j, q_j\}_{j=1}^n$ satisfy the canonical commutation relations (CCR) on $\mathcal{D}$:

$$\begin{aligned}
&[p_j, q_k]\psi = -\mathrm{i}\delta_{jk}\psi, \\
&[p_j, p_k]\psi = 0 = [q_j, q_k]\psi \quad (j, k = 1, \ldots, N).
\end{aligned}$$

for all ψ in $\mathcal{D}$.

We often write $\mathcal{D}(\pi)$ for $\mathcal{D}$. Our main aim of this section is to investigate the momentum operators with the winding gauge potential from the viewpoint of the representation of the CCR.

For a representation of the CCR $\pi := \{\mathcal{H},\, \mathcal{D},\, \{p_j, q_j\}_{j=1}^n\}$, we introduce the strong commutant of π as follows:

$$(\pi)'_{\mathrm{s}} := \big\{T \in \mathcal{B}(\mathcal{H}) \,\big|\, T\mathcal{D} \subset \mathcal{D},\ p_jT\phi = Tp_j\phi,\\ q_jT\phi = Tq_j\phi,\ \phi \in \mathcal{D},\quad j = 1, \ldots, n\big\},$$

where we denote the set of all bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. We say that π is *irreducible* if $(\pi)'_{\mathrm{s}} = \mathbb{C}I$.

Proposition 3.1 *Let Q_{j1}, Q_{j2} $(j = 1, \ldots, N)$ be the multiplication operators by the coordinate functions x_j and y_j, respectively. Then*

$$\pi_{\mathbb{A}} := \Big\{L^2(\mathbb{R}^{2N}), C_0^\infty(\mathcal{M}_N), \big\{\overline{P}_{j\alpha}, Q_{j\alpha} \,\big|\, j = 1, \ldots, N,\ \alpha = 1, 2\big\}\Big\}$$

is an irreducible representation of the CCR of $2N$ degree of freedom.

Proof. It is not difficult to show that $\pi_{\mathbb{A}}$ is a representation of the CCR. So we only prove the irrudicibility of $\pi_{\mathbb{A}}$.

Let T be an element in $(\pi_{\mathbb{A}})'_{\mathrm{s}}$. Then, for each $\phi \in C_0^\infty(\mathcal{M}_N)$, we have

$$TQ_{j\alpha}\phi = Q_{j\alpha}T\phi \quad (j = 1, \ldots, n,\ \alpha = 1, 2).$$

Since each $Q_{j\alpha}$ is essentially self-adjoint on $C_0^\infty(\mathcal{M}_N)$, we can conclude that $TQ_{j\alpha} \subset Q_{j\alpha}T$. Hence, we obtain

$$T\mathrm{e}^{\mathrm{i}sQ_{j\alpha}} = \mathrm{e}^{\mathrm{i}sQ_{j\alpha}}T$$

for all $s \in \mathbb{R}$. Therefore, there exists a function $F \in L^\infty(\mathbb{R}^{2N})$ such that

$$T\phi = F\phi$$

for each $\phi \in L^2(\mathbb{R}^{2N})$. Since T commutes with $\overline{P}_{j\alpha}$ on $C_0^\infty(\mathcal{M}_N)$, T also commutes with $p_{j\alpha}$ on $C_0^\infty(\mathcal{M}_N)$. Hence we have

$$\mathrm{e}^{\mathrm{i}sp_{j\alpha}}T = T\mathrm{e}^{\mathrm{i}sp_{j\alpha}}$$

for all $s \in \mathbb{R}$. Since $\mathrm{e}^{\mathrm{i}sp_{j\alpha}}$ is the translation, we can conclude that F is constant almost everywhere. □

As for the CCRs in the Weyl form, we have the following result.

Theorem 3.2 *The set $\big\{\overline{P}_{j\alpha}, Q_{j\alpha} \,\big|\, j = 1, \ldots, N,\ \alpha = 1, 2\big\}$ of self-adjoint operators fullfills the Weyl relations if and only if $\gamma/\theta_0 \in \mathbb{Z}$, i.e., the magnetic flux is locally quantized.*

Proof. In a similar way to the proof of Theorem 2.4, we can prove

$$\exp(\mathrm{i}sQ_{j\alpha})\exp(\mathrm{i}t\overline{P}_{k\beta}) = \exp(-\mathrm{i}st\delta_{jk}\delta_{\alpha\beta})\exp(\mathrm{i}t\overline{P}_{k\beta})\exp(\mathrm{i}sQ_{j\alpha})$$

for all $s, t \in \mathbb{R},\ j, k = 1, \dots, N,\ \alpha, \beta = 1, 2$. Combining these facts with Theorem 2.4, we have the desired assertion. $\square$

Let $\pi_1 = \{\mathcal{H}_{\pi_1},\ \mathcal{D}(\pi_1),\ \{p_{\pi_1 j},\ q_{\pi_1 j}\}_{j=1}^n\}$ and $\pi_2 = \{\mathcal{H}_{\pi_2},\ \mathcal{D}(\pi_2),\ \{p_{\pi_2 j}, q_{\pi_2 j}\}_{j=1}^n\}$ be representations of the CCR of n degree of freedom. If there exists a unitary operator U from $\mathcal{H}_{\pi_1}$ onto $\mathcal{H}_{\pi_2}$ such that

$$Up_{\pi_1}U^* = p_{\pi_2}, \quad Uq_{\pi_1}U^* = q_{\pi_2},$$

for each $j = 1, \dots, n$, then we say that π_1 and π_2 are *unitarily equivalent.*

To state a corollary of the above theorem, we need some notations. Let $p_{j\alpha}$ be the free momentum operators defined in the preveous section. Then it is not difficult to check that $p_{j\alpha}$, $Q_{j\alpha}$ are self-adjoint and satisfy the CCR on $C_0^\infty(\mathcal{M}_N)$. Hence

$$\pi_{\mathrm{S}} = \Big\{L^2(\mathbb{R}^{2N}), C_0^\infty(\mathcal{M}_N), \big\{p_{j\alpha}, Q_{j\alpha} \bigm| j = 1, \dots, N,\ \alpha = 1, 2\big\}\Big\}$$

is an irreducible representation of the CCR. π_{S} is said to be the *Schrödinger representation on* $C_0^\infty(\mathcal{M}_N)$.

Corollary 3.3 *$\pi_{\mathbb{A}}$ is unitarily equivalent to π_{S} if and only if $\gamma/\theta_0 \in \mathbb{Z}$. Furtheremore, the unitary operator which gives this unitary equivalence has the follwing formula*:

$$U_N(\theta_0, \gamma) = \exp\Big(\mathrm{i}\frac{\gamma}{\theta_0}\eta_N\Big),$$
$$\eta_N(\mathbf{r}_1, \dots, \mathbf{r}_N) := \frac{1}{i}\sum_{k<j}\log\frac{z_k - z_j}{|z_k - z_j|},$$

where $z_k = x_k + \mathrm{i}y_k$. That is, for each $j = 1, \dots, N$, $\alpha = 1, 2$,

$$U_N(\theta_0, \gamma)p_{j\alpha}U_N(\theta_0, \gamma)^* = \overline{P}_{j\alpha}, \quad U_N(\theta_0, \gamma)Q_{j\alpha}U_N(\theta_0, \gamma)^* = Q_{j\alpha}.$$

Proof. The first half immediately follows from Proposition 3.1, Theorem 3.2 and [8, Theorem VIII 14].

Note that η_N is real valued. Indeed, since

$$z_k - z_j = |z_k - z_j|\mathrm{e}^{\mathrm{i}\theta(z_k, z_j)},$$

where θ_{z_k,z_j} is a real valued function defined by

$$\theta(z_k, z_j) = \frac{1}{\mathrm{i}} \log \frac{z_k - z_j}{|z_k - z_j|},$$

we can conclude that

$$\eta_N(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i<j} \theta(z_i - z_j)$$

is also real valued. Thus $\mathrm{e}^{\mathrm{i}\theta_0\eta_N}$ is a unitary operator. It is easy to see that $\mathrm{e}^{\mathrm{i}\theta_0\eta_N} C_0^\infty(\mathcal{S}_1^{(N)}) = C_0^\infty(\mathcal{S}_1^{(N)})$. By a direct calculation, we can check that

$$D_{x_k}\eta_N(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{2\pi}{\gamma} A_{k1}(\mathbf{r}_1, \dots, \mathbf{r}_N), \quad (\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathcal{S}_1^{(N)}.$$

Hence we have

$$\mathrm{e}^{\mathrm{i}(\gamma/\theta_0)\eta_N} p_{k1} \mathrm{e}^{-\mathrm{i}(\gamma/\theta_0)\eta_N} = P_{k1} \quad \text{on } C_0^\infty(\mathcal{S}_1^{(N)}).$$

By a similar way, we can show the assertion about P_{k2}. □

3.2. Fibre direct integral representation of the CCR and Arai's representation

For each subset $I \subset \{1, \dots, N\}$, we introduce

$$\pi_{\mathbb{A}}(I) := \Big\{ L^2(\mathbb{R}^{2N}), C_0^\infty(\mathcal{M}_N), \big\{ \overline{P}_{i\alpha}, Q_{i\alpha} \;\big|\; i \in I,\ \alpha = 1, 2 \big\} \Big\}.$$

Then it is clear that $\pi_{\mathbb{A}}(I)$ is a representation of the CCR of $2 \times \#I$ degree of freedom ($\#I$ means the cardinality of the set I). In this subsection, we derive the direct integral decomposition of $\pi_{\mathbb{A}}(I)$.

Let $I = \{i_1, \dots, i_k\} \subset \{1, \dots, N\}$ $(i_1 < i_2 < \cdots < i_k)$ be fixed. Then we introduce

$$\Omega_I := \mathbb{R}_1^2 \times \cdots \times \widehat{\mathbb{R}}_{i_1}^2 \times \cdots \times \widehat{\mathbb{R}}_{i_k}^2 \times \cdots \times \mathbb{R}_N^2,$$
$$\mathbb{R}^2(I) := \mathbb{R}_{i_1}^2 \times \cdots \times \mathbb{R}_{i_k}^2.$$

For each $\omega_I = (\mathbf{a}_1, \dots, \widehat{\mathbf{a}}_{i_1}, \dots, \widehat{\mathbf{a}}_{i_k}, \dots, \mathbf{a}_N) \in \Omega_I$, we define the multiplication operator $\tilde{A}_{i\alpha}(\omega_I)$ $(i \in I)$ on $L^2(\mathbb{R}^2(I))$ by

$$\tilde{A}_{i\alpha}(\omega_I)(\mathbf{r}_{i_1}, \dots, \mathbf{r}_{i_k}) := A_{i\alpha}(\mathbf{a}_1, \dots, \mathbf{r}_{i_1}, \dots, \mathbf{r}_{i_k}, \dots, \mathbf{a}_N).$$

Relative to the direct integral decomposition

$$L^2(\mathbb{R}^{2N}) = \int_{\Omega_I}^{\oplus} L^2(\mathbb{R}^2(I))\,\mathrm{d}\omega_I,$$

it is not difficult to see that

$$A_{i\alpha} = \int_{\Omega_I}^{\oplus} \tilde{A}_{i\alpha}(\omega_I)\,\mathrm{d}\omega_I, \quad p_{i\alpha} = \int_{\Omega_I}^{\oplus} \tilde{p}_{i\alpha}\,\mathrm{d}\omega_I$$

for each $\alpha = 1, 2$, $i \in I$, where we denote the self-adjoint operators $-\mathrm{i}D_{x_i}$, $-\mathrm{i}D_{y_i}$ ($i \in I$) acting in $L^2(\mathbb{R}^2(I))$ by $\tilde{p}_{i1}$ and $\tilde{p}_{i2}$, respectively. For each $\omega_I \in \Omega_I$, we define

$$\begin{aligned} &P_{i\alpha}(\omega_I) := \tilde{p}_{i\alpha} - q\tilde{A}_{i\alpha}(\omega_I),\\ &\mathrm{dom}(P_{i\alpha}(\omega_I)) := \mathrm{dom}(\tilde{p}_{i\alpha}) \cap \mathrm{dom}(\tilde{A}_{i\alpha}(\omega_I)). \end{aligned}$$

Then, by a similar way to the proof of Lemma 2.8, we can prove that for each $\omega_I \in \Omega_I$, $P_{i\alpha}(\omega_I)$ is essentially self-adjoint and measurable. Moreover,

$$\overline{P}_{i\alpha} = \int_{\Omega_I}^{\oplus} \overline{P}_{i\alpha}(\omega_I)\,\mathrm{d}\omega_I \quad (i \in I,\ \alpha = 1, 2), \tag{9}$$

where we denote the closure of $P_{i\alpha}(\omega_I)$ by $\overline{P}_{i\alpha}(\omega_I)$.

Let $I = \{i_1, \ldots, i_k\}$ ($k \le N$) be a subset of $\{1, \ldots, N\}$. For each $\omega = (\mathbf{a}_1, \ldots, \mathbf{a}_{N-k}) \in \Omega(I)$, we introduce

$$\begin{aligned} \mathcal{M}_N(\omega) := \big\{(\mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_k}) \in \mathcal{M}_k \;\big|\; &\mathbf{r}_{i_m} \neq \mathbf{a}_j \\ &(m = 1, \ldots, k,\ j = 1, \ldots, N-k)\big\}. \end{aligned}$$

The following proposition can be easily proven.

Proposition 3.4 *For each $\omega \in \Omega_I$,*

$$\pi_{\mathbb{A}}^I(\omega) := \Big\{L^2(\mathbb{R}^2(I)), C_0^\infty(\mathcal{M}_N(\omega)), \big\{\overline{P}_{i\alpha}(\omega), Q_{i\alpha} \,\big|\, i \in I,\ \alpha = 1, 2\big\}\Big\}$$

is a representation of the CCR of $\#I \times 2$ degree of freedom.

Remark 3.5 If $\#I = 1$, then $\pi_{\mathbb{A}}^I(\omega)$ is called *Arai's representation* [1].

Let $\mathcal{K} = \int_\Lambda^\oplus \mathcal{H}\,\mathrm{d}\mu(\lambda)$ be the direct intgral of $\mathcal{H}$ over a measure space (Λ, μ). Suppose that

$$\pi = \big\{\mathcal{K}, \mathcal{D}, \{p_i, q_i \mid i = 1, \ldots, N\}\big\}$$

be a representation of the CCR of N degree of freedom. If for μ-a.e. $\lambda \in \Lambda$, there exsists a representation of the CCR of N degree of freedom

$$\pi_\lambda = \{\mathcal{H}, \mathcal{D}_\lambda, \{p_j(\lambda), q_j(\lambda) \mid j = 1, \ldots, N\}\}$$

such that

(i) for each $j = 1, \ldots, N$, the self-adjoint mappings $\lambda \in \Lambda \to p_j(\lambda)$, $q_j(\lambda)$ are measurable and

$$p_j = \int_\Lambda^\oplus p_j(\lambda)\, \mathrm{d}\mu(\lambda), \quad q_j = \int_\Lambda^\oplus q_j(\lambda)\, \mathrm{d}\mu(\lambda),$$

(ii) for all $\Psi \in \mathcal{D}$, $\Psi(\lambda) \in \mathcal{D}_\lambda$ μ-a.e.,

then we say that π is *decomposable* or *direct integal of* $\{\pi_\lambda\}_{\lambda\in\Lambda}$ and write

$$\pi = \int_\Lambda^\oplus \pi_\lambda\, \mathrm{d}\mu(\lambda).$$

Theorem 3.6 *Let $\pi_{\mathbb{A}}$ be the representation of the CCR defined in the preceeding subsection. Then, for each $I \subset \{1, \ldots, N\}$, we have*

$$\pi_{\mathbb{A}}(I) = \int_{\Omega_I}^\oplus \pi_{\mathbb{A}}^I(\omega_I)\, \mathrm{d}\omega_I.$$

Especially, if $\#I = 1$, then $\pi_{\mathbb{A}}(I)$ is a direct integral of Arai's representations $\{\pi_{\mathbb{A}}^I(\omega)\}_{\omega\in\Omega_I}$.

Proof. By using (9), we can easily prove this theorem. □

Theorem 3.7 *Let I be a subset of $\{1, \ldots, N\}$. Suppose that ω, $\omega' \in \Omega_I$. Then we have the following:*

(i) *If the magnetic flux is locally quantized, then $\pi_{\mathbb{A}}^I(\omega)$ and $\pi_{\mathbb{A}}^I(\omega')$ are unitarily equivalent to each other for all ω, $\omega' \in \Omega_I$.*

(ii) *If the magnetic flux is not locally quantized, then $\pi_{\mathbb{A}}^I(\omega)$ and $\pi_{\mathbb{A}}^I(\omega')$ are unitarily equivalent if and only if $\omega = \omega'$.*

Proof. We will prove this theorem for the case $N = 2$, $I = \{1\}$. To extend the proof for this simple case to the proof for more general case is not difficult. In this simple case, it is clear that $p_{11} = -\mathrm{i}D_x$, $p_{12} = -\mathrm{i}D_y$ and

$$A_{11}(\omega)(\mathbf{r}) = -\frac{\gamma}{2\pi}\frac{y - \omega_2}{|\mathbf{r} - \omega|^2}, \quad A_{12}(\omega)(\mathbf{r}) = \frac{\gamma}{2\pi}\frac{x - \omega_1}{|\mathbf{r} - \omega|^2},$$

where $\mathbf{r} = (x, y) \in \mathbb{R}_1^2$, $\omega = (\omega_1, \omega_2) \in \Omega_I = \mathbb{R}_2^2$. Hence, $\pi_{\mathbb{A}}^I(\omega)$ is Arai's representation.

(i) If the magnetic flux is locally quantized, then applying [1, Theorem 4.2], $\pi^I_{\mathbb{A}}(\omega)$ is unitarily equivalent to the Schrödinger representation. Thus we have the desired result.

(ii) If $\omega = \omega'$, then it is clear that $\pi^I_{\mathbb{A}}(\omega)$ and $\pi^I_{\mathbb{A}}(\omega')$ are unitarily equivalent. Conversely, for $\omega,\,\omega' \in \Omega_I,\ \omega \neq \omega'$, assume that $\pi^I_{\mathbb{A}}(\omega)$ and $\pi^I_{\mathbb{A}}(\omega')$ are unitarily equivalent to each other. Then there exists a unitary operator U satisfying

$$U\overline{P}_{1\alpha}(\omega)U^{-1} = \overline{P}_{1\alpha}(\omega'), \tag{10}$$

$$UQ_{1\alpha}U^{-1} = Q_{1\alpha} \tag{11}$$

for each $\alpha = 1,\,2$. It follows from (11) that, for each $F \in L^\infty(\mathbb{R}^2_1)$,

$$UFU^{-1} = F.$$

Combing this with (10), we have

$$\overline{Up_{1\alpha}U^{-1} - qA_{1\alpha}(\omega)} = \overline{p_{1\alpha} - qA_{1\alpha}(\omega')}.$$

Using this equation, it is not hard to prove that

$$Up_{1\alpha}U^{-1} = \overline{p_{1\alpha} - q(A_{1\alpha}(\omega') - A_{1\alpha}(\omega))}. \tag{12}$$

We denote the R.H.S. of equation (12) by $\overline{P}_{1\alpha}(\omega,\omega')$. It is clear that $\{Up_{1\alpha}U^{-1}, Q_{1\alpha} \mid \alpha = 1,2\}$ satisfies the Weyl relations. On the other hand, applying [1, Theorem 2.1], we obtain

$$\mathrm{e}^{\mathrm{i}s\overline{P}_{11}(\omega,\omega')}\mathrm{e}^{\mathrm{i}t\overline{P}_{12}(\omega,\omega')} = \mathrm{e}^{-\mathrm{i}q\Phi_{s,t}(\omega,\omega')}\mathrm{e}^{\mathrm{i}t\overline{P}_{12}(\omega,\omega')}\mathrm{e}^{\mathrm{i}s\overline{P}_{11}(\omega,\omega')}$$

for all $s,\,t \in \mathbb{R}$, where

$$\Phi_{s,t}(\omega,\omega')(\mathbf{r}) := \begin{cases} \epsilon(s)\epsilon(t)\gamma & (\omega \notin D(\mathbf{r};s,t),\ \omega' \in D(\mathbf{r};s,t)) \\ -\epsilon(s)\epsilon(t)\gamma & (\omega \in D(\mathbf{r};s,t),\ \omega' \notin D(\mathbf{r};s,t)) \\ 0 & (\text{otherwise}) \end{cases}.$$

Since the magnetic flux is not locally quantized, $\{\overline{P}_{1\alpha}(\omega,\omega'), Q_{1\alpha} \mid \alpha = 1,2\}$ does not satisfy the Weyl relations. Hence we have a contradiction. $\square$

Remark 3.8 Suppose that the magnetic flux is *not* locally quantized. Then, by Theorem 3.7 (ii), we can conclude that there exist uncountably many representations of the CCR which are inequivalent to each other.

Corollary 3.9 *Suppose that the magnetic flux is not locally quantized. Then, for each subset $I \subset \{1, \ldots, N\}$, $\pi_{\mathbb{A}}(I)$ is a direct integral of mutually inequivalent representations of the CCR.*

4. Applications

4.1. Schrödinger operators for systems with the winding gauge potential

Let $V(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ be a real valued Borel measurable function on $\mathbb{R}^{2N}$. Here, we investigate the Schrödinger operator on $L^2(\mathbb{R}^{2N})$ defined by

$$H = \sum_{j=1}^{N} \left(-\mathrm{i}D_{x_j} + \frac{\gamma q}{2\pi} \sum_{k \neq j} \frac{y_j - y_k}{|\mathbf{r}_j - \mathbf{r}_k|^2} \right)^2 + \sum_{j=1}^{N} \left(-\mathrm{i}D_{y_j} - \frac{\gamma q}{2\pi} \sum_{k \neq j} \frac{x_j - x_k}{|\mathbf{r}_j - \mathbf{r}_k|^2} \right)^2 + V(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

For this purpose, we introduce an operator H_0 defined by

$$H_0 := -\Delta + V,$$

where Δ is the $2N$ dimensional Laplacian.

We assume the following conditions:

(A.1) H_0 is essentially self-adjoint.

(A.2) $\gamma/\theta_0 \in \mathbb{Z}$.

Theorem 4.1 *Under the assumption* (A.1) *and* (A.2)*, we have the following.*

(i) *H is essentially self-adjoint.*

(ii) *$\sigma(\overline{H}) = \sigma(\overline{H_0})$, where $\sigma(A)$ denots the spectrum of the linear operator A.*

(iii) *$\sigma_{\mathrm{p}}(\overline{H}) = \sigma_{\mathrm{p}}(\overline{H_0})$, where $\sigma_{\mathrm{p}}(A)$ denotes the point spectrum of A. Especially, if $E \in \sigma_{\mathrm{p}}(\overline{H_0})$, then*

$$\ker(H - E) = \left\{ \prod_{k<j} (z_k - z_j)^{\gamma/\theta_0} |z_k - z_j|^{-\gamma/\theta_0} \Psi \,\middle|\, \Psi \in \ker(H_0 - E) \right\}.$$

Proof. (i) The Hamiltonian H can be expressed as

$$H = \sum_{j=1}^{N} \mathbb{P}_j^2 + V,$$

where $\mathbb{P}_j := (P_{j1}, P_{j2})$. Hence, by Cororally 3.3, we have

$$H = U_N(\theta_0, \gamma) H_0 U_N(\theta_0, \gamma)^*$$

on $\mathrm{dom}(H)$. Hence

$$\overline{H} = U_N(\theta_0, \gamma)\overline{H}_0 U_N(\theta_0, \gamma)^*. \tag{13}$$

Since $\overline{H}_0$ is self-adjoint, it follows that $\overline{H}$ is self-adjoint. Parts (ii) and (iii) follow from (13). □

4.2. Statistical transformation

Throughout this subsection, we assume the following condition:

$$\frac{\gamma}{\theta_0} \in \mathbb{Z}. \tag{14}$$

Under this condition, the representation of the CCR

$$\pi_{\mathbb{A}} = \Big\{ L^2(\mathbb{R}^{2N}), C_0^\infty(\mathcal{M}_N), \big\{ \overline{P}_{j\alpha}, Q_{j\alpha} \;\big|\; j = 1, \ldots, N,\ \alpha = 1, 2 \big\} \Big\}$$

is unitarily equivalent to the Schrödinger representation π_{S} by Cororally 3.3. Hence, the system satisfying (14) seems to be trivial at first glance. But there are some interesting structures in this system.

Let $\mathcal{H} := L^2(\mathbb{R}^2)$. For each $N \geq 2$, it is well-known that $L^2(\mathbb{R}^{2N}) = \otimes^N \mathcal{H}$. We introduce the following closed subspaces of $\otimes^N \mathcal{H}$:

$$\otimes_{\mathrm{s}}^N \mathcal{H} := S_N(\otimes^N \mathcal{H}),$$
$$\otimes_{\mathrm{as}}^N \mathcal{H} := A_N(\otimes^N \mathcal{H}),$$

where we denote by S_N (resp. A_N) the symmetrizer (resp. the antisymmetrizer) on $\otimes^N \mathcal{H}$.

Proposition 4.2 *Suppose that the condtion* (14) *is satisfied. Then we have*

(i) *if γ/θ_0 is even, then*

$$U_N(\theta_0, \gamma) S_N = S_N U_N(\theta_0, \gamma), \quad U_N(\theta_0, \gamma) A_N = A_N U_N(\theta_0, \gamma);$$

(ii) *if γ/θ_0 is odd, then*

$$U_N(\theta_0, \gamma) A_N = S_N U_N(\theta_0, \gamma).$$

Hence, the unitary operator $U_N(\theta_0, \gamma)$ gives a natural correspondence between $\otimes_{\mathrm{s}}^N \mathcal{H}$ and $\otimes_{\mathrm{as}}^N \mathcal{H}$.

Proof. Let S_N be the group of permutations of a set of cardinality N. For each $\phi_1, \dots, \phi_N \in \mathcal{H}$ and $\sigma \in \mathsf{S}_N$, we define

$$U_\sigma \phi_1 \otimes \cdots \otimes \phi_N := \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(N)}.$$

Then it is easy to see that U_σ can be extended to a unitary operator on $\otimes^N \mathcal{H}$. We denote it by the same symbol U_σ.

For each $(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathcal{M}_N$, we have

$$U_N(\theta_0, \gamma)(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i<j} \left(\frac{z_i - z_j}{|z_i - z_j|} \right)^{\gamma/\theta_0}.$$

Hence if γ/θ_0 is even, then for each $\Phi \in C_0^\infty(\mathcal{M}_N)$, we have

$$U_N(\theta_0, \gamma) U_\sigma \Phi = U_\sigma U_N(\theta_0, \gamma) \Phi.$$

Since $C_0^\infty(\mathcal{M}_N)$ is dense in $\otimes^N \mathcal{H}$, we obtain

$$U_N(\theta_0, \gamma) U_\sigma = U_\sigma U_N(\theta_0, \gamma).$$

On the other hand, if γ/θ_0 is odd, we can easily check that

$$U_N(\theta_0, \gamma) U_\sigma = \operatorname{sgn}(\sigma) U_\sigma U_N(\theta_0, \gamma).$$

From these facts, we have the desired results. $\square$

Let A be a self-adjoint operator acting in $\otimes^N \mathcal{H}$. We denote the pure point spectrum of A by $\sigma_{\mathrm{p}}(A)$. Then we introduce the closed subspaces of $\otimes^N \mathcal{H}$ by

$$\mathcal{H}(A) := \bigoplus_{\lambda \in \sigma_{\mathrm{p}}(A)} \ker(A - \lambda)$$

and

$$\mathcal{H}_{\mathrm{s}}(A) := S_N \mathcal{H}(A), \quad \mathcal{H}_{\mathrm{as}}(A) := A_N \mathcal{H}(A).$$

It is clear that

$$\mathcal{H}_{\mathrm{s}}(A) = \bigoplus_{\lambda \in \sigma_{\mathrm{p}}(A)} \ker_{\mathrm{s}}(A - \lambda),$$

$$\mathcal{H}_{\mathrm{as}}(A) = \bigoplus_{\lambda \in \sigma_{\mathrm{p}}(A)} \ker_{\mathrm{as}}(A - \lambda),$$

where $\ker_{\mathrm{s}}(A - \lambda) := S_N \ker(A - \lambda)$ and $\ker_{\mathrm{as}}(A - \lambda) := A_N \ker(A - \lambda)$.

Proposition 4.3 *Let $\overline{H}$ and $\overline{H}_0$ be the Schrödinger operators defined in the preceding subsection. Suppose that the conditions* (A.1) *are satisfied. Moreover, if θ_0/γ is odd, we have the following.*

(i) *For each $\lambda \in \sigma_{\mathrm{p}}(\overline{H}_0)$,*

$$\ker_{\mathrm{s}}(\overline{H}-\lambda) = U_N(\theta_0,\gamma)\ker_{\mathrm{as}}(\overline{H}_0-\lambda),$$
$$\ker_{\mathrm{as}}(\overline{H}-\lambda) = U_N(\theta_0,\gamma)\ker_{\mathrm{s}}(\overline{H}_0-\lambda).$$

(ii)
$$\mathcal{H}_{\mathrm{s}}(\overline{H}) = U_N(\theta_0,\gamma)\mathcal{H}_{\mathrm{as}}(\overline{H}_0),$$
$$\mathcal{H}_{\mathrm{as}}(\overline{H}) = U_N(\theta_0,\gamma)\mathcal{H}_{\mathrm{s}}(\overline{H}_0).$$

Proof. These are simple applications of Theorem 4.1 and Proposition 4.2. □

A. Decomposable self-adjoint operators

In this appendix, we summarize some basic properties of decomposable self-adjoint operators.

Let (Λ,μ) be a measure space and $\mathcal{H}$ be a Hilbert space. We denote the set of bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ and the set of linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. It is clear that $\mathcal{B}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$. For a self-adjoint operator T, we denote its spectral measure by $E_T(J)$, where $J \in \mathbb{B}^1$ (the Borel field of $\mathbb{R}$).

For a $\mathcal{B}(\mathcal{H})$-valued mapping $B\colon \Lambda \ni \lambda \to B(\lambda) \in \mathcal{B}(\mathcal{H})$, we say that the mapping $\lambda \to B(\lambda)$ is *measurable* if the mapping $\lambda \to \langle \phi, B(\lambda)\phi\rangle_{\mathcal{H}}$ is measurable for each $\phi \in \mathcal{H}$.

Let $A\colon \Lambda \ni \lambda \to A(\lambda) \in \mathcal{L}(\mathcal{H})$ be a $\mathcal{L}(\mathcal{H})$-valued mapping. We call that the mapping $\lambda \to A(\lambda)$ is *self-adjoint mapping* if $A(\lambda)$ is self-adjoint for μ-a.e. λ.

Definition A.1 Let $\lambda \to A(\lambda)$ be a self-adjoint mapping. If the $\mathcal{B}(\mathcal{H})$-valued mapping $\lambda \to E_{A(\lambda)}(J)$ is measurable for each $J \in \mathbb{B}^1$, we say that the mapping $\lambda \to A(\lambda)$ is *measurable.*

Lemma A.2 *Suppose that $\lambda \to A(\lambda)$ is a measurable self-adjoint mapping. Then, for each $F \in L^\infty(\mathbb{R})$, the $\mathcal{B}(\mathcal{H})$-valued mapping $\lambda \to F(A(\lambda))$ is measurable, where $F(A(\lambda))$ is given by the operational calculas.*

Proof. Since we can easily prove the assertion for characteristic functions, we have the desired result. □

Theorem A.3 *Suppose that $\lambda \to A(\lambda)$ is a self-adjoint mapping. Then the following three statements are equivalent to each other*:

(i) $\lambda \to A(\lambda)$ *is measurable.*

(ii) $\lambda \to \mathrm{e}^{\mathrm{i}tA(\lambda)}$ *is a $\mathcal{B}(\mathcal{H})$-valued measurable mapping for each $t \in \mathbb{R}$.*

(iii) $\lambda \to R_z(A(\lambda))$ *is a $\mathcal{B}(\mathcal{H})$-valued measurable mapping for each $z \in \mathbb{C} \setminus \mathbb{R}$, where $R_z(T) = (T - z)^{-1}$.*

Proof. (i)$\Rightarrow$(ii): Use Lemma A.2.

(ii)$\Rightarrow$(iii): Use the following well-known formula:

$$R_z(A(\lambda)) = \begin{cases} -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}sA(\lambda)}\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s & (\operatorname{Im} z < 0) \\ \mathrm{i}\int_{-\infty}^0 \mathrm{e}^{\mathrm{i}sA(\lambda)}\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s & (\operatorname{Im} z > 0) \end{cases}.$$

(iii)$\Rightarrow$(i): By Stone's formula, the mapping $\lambda \to E_{A(\lambda)}((a,b))$ is measurable for each a, $b \in \mathbb{R}$, $a < b$. Hence, by the limiting argument, we can conclude (i). □

Theorem A.4 *Suppose that $\lambda \to A(\lambda)$ is a self-adjoint mapping. Then the following three statements are equivalent to each other*:

(i) $\lambda \to A(\lambda)$ *is measurable.*

(ii) $\lambda \to \mathrm{e}^{\mathrm{i}A(\lambda)}$ *is a $\mathcal{B}(\mathcal{H})$-valued measurable mapping.*

(iii) $\lambda \to R_\mathrm{i}(A(\lambda))$ *is a $\mathcal{B}(\mathcal{H})$-valued measurable mapping.*

Proof. Theorem A.3 (ii)$\Rightarrow$(ii): Clear.

(ii)$\Rightarrow$Theorem A.3 (ii): For each $t \in \mathbb{Q}$, we can easily show that the mapping $\lambda \to \mathrm{e}^{\mathrm{i}tA(\lambda)}$ is measurable. Hence, we can extend the measurability of $\mathrm{e}^{\mathrm{i}tA(\lambda)}$ ($t \in \mathbb{Q}$) to that of $\mathrm{e}^{\mathrm{i}tA(\lambda)}$ ($t \in \mathbb{R}$) by the strong continuity of $\mathrm{e}^{\mathrm{i}sA(\lambda)}$ in $s \in \mathbb{R}$.

Theorem A.3 (iii)$\Rightarrow$(iii): Clear.

(iii)$\Rightarrow$Theorem A.3 (iii): For each $z \in \mathbb{C}\setminus\mathbb{R}$ such that $|z - \mathrm{i}| \cdot \|R_\mathrm{i}(A(\lambda))\| < 1$, it is well-known that

$$R_z(A(\lambda)) = \sum_{n=0}^{\infty} (\mathrm{i} - z)^n R_\mathrm{i}(A(\lambda))$$

in operator norm topology. Hence, for such z, the mapping $\lambda \to R_z(A(\lambda))$ is measurable. Repeating this argument, we can conclude Theorem A.3 (iii). □

Let $\lambda \to A(\lambda)$ be a measurable self-adjoint mapping. Then we define an operator A acting in $\int_\Lambda^\oplus \mathcal{H}\,\mathrm{d}\mu(\lambda)$ by

$$\begin{aligned}
&\mathrm{dom}(A)\\
&\quad := \left\{ \Psi \in \int_\Lambda^\oplus \mathcal{H}\,\mathrm{d}\mu(\lambda) \,\middle|\, \Psi(\lambda) \in \mathrm{dom}(A(\lambda)) \ \mu\text{-a.e. } \lambda, \right.\\
&\qquad\qquad \left. \lambda \to A(\lambda)\Psi(\lambda) \text{ is measurable, } \int_\Lambda \|A(\lambda)\Psi(\lambda)\|_{\mathcal{H}}^2\,\mathrm{d}\mu(\lambda) < \infty \right\},\\
&(A\Psi)(\lambda)\\
&\quad := A(\lambda)\Psi(\lambda) \quad \mu\text{-a.e. } \lambda, \ \Psi \in \mathrm{dom}(A).
\end{aligned}$$

Definition A.5 The operator A is said to be the *direct integral of* $A(\lambda)$ and written by

$$A = \int_\Lambda^\oplus A(\lambda)\,\mathrm{d}\mu(\lambda).$$

Proposition A.6 *Suppose that $\lambda \to A(\lambda)$ is measurable self-adjoint mapping. Then, the operator $\int_\Lambda^\oplus A(\lambda)\,\mathrm{d}\mu(\lambda)$ is self-ajoint.*

Proof. See [9, Theorem XIII. 85]. □

Theorem A.7 *Suppose that $\lambda \to A(\lambda)$ and $\lambda \to B(\lambda)$ are measurable self-adjoint mappings. Let*

$$A = \int_\Lambda^\oplus A(\lambda)\,\mathrm{d}\mu(\lambda), \quad B = \int_\Lambda^\oplus B(\lambda)\,\mathrm{d}\mu(\lambda).$$

Suppose that $C(\lambda) := A(\lambda) + B(\lambda)$ is essentially self-ajoint for μ-a.e. λ. Then we have the following:

(i) *$\lambda \to \overline{C(\lambda)}$ is a measurable self-adjoint mapping.*

(ii) *Let $C := \int_\Lambda^\oplus \overline{C(\lambda)}\,\mathrm{d}\mu(\lambda)$. If $A + B$ is essentially self-adjoint, then*

$$C = \overline{A + B}.$$

Proof. (i) By Trotter's product formula, we have

$$\mathrm{e}^{\mathrm{i}t\overline{C(\lambda)}} = \operatorname*{s-lim}_{N\to\infty} \left(\mathrm{e}^{\mathrm{i}tA(\lambda)/N}\mathrm{e}^{\mathrm{i}tB(\lambda)/N}\right)^N.$$

Since $\lambda \to \mathrm{e}^{\mathrm{i}tA(\lambda)/N}\mathrm{e}^{\mathrm{i}tB(\lambda)/N}$ is a $\mathcal{B}(\mathcal{H})$-valued measurable mapping, we have the desired result.

(ii) For each $\Psi \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$, we have

$$\|\overline{C(\lambda)}\Psi(\lambda)\|_{\mathcal{H}}^2 \le 2(\|A(\lambda)\Psi(\lambda)\|_{\mathcal{H}}^2 + \|B(\lambda)\Psi(\lambda)\|_{\mathcal{H}}^2) \quad \mu\text{-a.e. } \lambda.$$

Hence, we obtain

$$\|C\Psi\|^2 \le 2(\|A\Psi\|^2 + \|B\Psi\|^2) < \infty,$$

i.e., $\mathrm{dom}(A) \cap \mathrm{dom}(B) \subset \mathrm{dom}(C)$. Moreover,

$$(C\Psi)(\lambda) = \overline{C(\lambda)}\Psi(\lambda) = A(\lambda)\Psi(\lambda) + B(\lambda)\Psi(\lambda)$$

for μ-a.e. λ. Thus we conclude that $C \supset A + B$. Since C, $\overline{A+B}$ are self-adjoint, we have the desired result. □

Corollary A.8 *Suppose that $\lambda \to A(\lambda)$ and $\lambda \to B(\lambda)$ are measurable self-adjoint mappings. Let*

$$A = \int_\Lambda^\oplus A(\lambda)\,\mathrm{d}\mu(\lambda), \quad B = \int_\Lambda^\oplus B(\lambda)\,\mathrm{d}\mu(\lambda).$$

If A and B strongly commute, then $C(\lambda) = A(\lambda) + B(\lambda)$ is essentially self-adjoint for μ-a.e. λ and $\lambda \to \overline{C(\lambda)}$ is a measurable self-adjoint mapping. Moreover

$$\int_\Lambda^\oplus \overline{C(\lambda)}\,\mathrm{d}\mu(\lambda) = \overline{A+B}.$$

Proof. Since A and B strongly commute, $A+B$ is essentially self-adjoint. On the other hand, we can easily check that $A(\lambda)$ and $B(\lambda)$ strongly commute μ-a.e. λ. Hence, $C(\lambda)$ is essentially self-adjoint for μ-a.e. λ. Combining these facts with Theorem A.7, we have the desired result. □

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