*Hokkaido Mathematical Journal Vol. 37 (2008) p. 611–625*

# Standing waves for a class of nonlinear Schrödinger equations with potentials in $L^\infty$

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(Received May 28, 2007; Revised December 18, 2007)

**Abstract.** We prove the existence of standing waves to the following family of nonlinear Schödinger equations:

$$\mathbf{i}\hbar\partial_t\psi = -\hbar^2\Delta\psi + V(x)\psi - \psi|\psi|^{p-2},\ (t, x) \in \mathbf{R}\times\mathbf{R}^n$$

provided that $\hbar > 0$ is small, $2 < p < 2n/(n-2)$ when $n \geq 3$, $2 < p < \infty$ when $n = 1, 2$ and $V(x) \in L^\infty(\mathbf{R}^n)$ is assumed to have a sublevel with positive and finite measure.

*Key words*: standing waves, minimization problems, compact perturbations.

In this paper we study the existence of standing waves to the following family of nonlinear Schrödinger equations:

$$\mathbf{i}\hbar\partial_t\psi = -\hbar^2\Delta\psi + V(x)\psi - \psi|\psi|^{p-2},\ (t, x) \in \mathbf{R}\times\mathbf{R}^n,\ n \geq 1 \tag{0.1}$$

where $2 < p < 2^* \equiv 2n/(n-2)$ for $n \geq 3$, $2 < p < \infty$ for $n = 1, 2$, $\hbar > 0$ is a constant sufficiently small and $V : \mathbf{R}^n \to \mathbf{R}$ is a measurable and bounded function that satisfies some other assumptions that will be specified in the sequel. In recent years, starting from the paper [11], a lot of attention has been devoted to the study of dispersive properties of the Schrödinger groups $e^{\mathbf{i}t(-\Delta+V(x))}$ (and also to the corresponding Cauchy problem (0.1)) under very weak regularity assumptions on $V(x)$. Those results have been the main motivation to analyse in this article the question of existence of standing waves to (0.1) for rough potentials $V(x)$. Moreover we think that the problem of the minimal regularity assumption to be required to $V(x)$, in order to guarantee the existence of standing waves for (0.1), is a mathematical question that has its own interest.

Since now on we shall look for solutions $\psi(t, x)$ to (0.1) of the form:

$$\psi(t, x) \equiv e^{-\mathbf{i}(Et/\hbar)}u(x),\ \text{where } E \in \mathbf{R} \tag{0.2}$$

with finite energy, i.e.

*2000 Mathematics Subject Classification* : 35J60, 35B20, 47J30.

$$\int_{\mathbf{R}^n} (\hbar^2|\nabla u|^2 + |V(x) - E||u|^2)dx < \infty.$$

Notice that the function $\psi(t, x)$ given in (0.2) satisfies (0.1) if and only if $u(x)$ solves the following elliptic equation:

$$-\hbar^2\Delta u + (V(x) - E)u = u|u|^{p-2},\ E \in \mathbf{R},\ x \in \mathbf{R}^n. \tag{0.3}$$

There exists an huge literature on the field and for this reason we cannot be exhaustive in the bibliography, however we would like to mention some papers connected with equation (0.3).

As far as we know the first result on the existence of solutions to (0.3) is ascribed to Floer and Weinstein in [7], where (0.3) is studied in the case $n = 1$, $p = 3$ and $V(x)$ is assumed to satisfy a suitable hypothesis, known in the literature as the $(V)_a$ condition.

After the seminal paper [7] many other contributions appeared in connection with equation (0.3), whose solutions are constructed by using various methods and by assuming different hypothesis on $V(x)$. In [1] the existence of solutions to (0.3) is treated via a Lyapunov–Schmidt finite dimensional reduction, under the assumption that the potential $V(x) \in C^2(\mathbf{R}^n)$ has a non–degenerate critical point. Let us mention [8] and [9] where the existence of solutions to (0.3) is treated via the concentration–compactness method and by assuming that the potential $V(x)$ is asymptotically flat in a suitable sense. In [12] the author proves the existence of standing waves provided that $V(x)$ satisfies the following condition:

$$V(x) \in C^1(\mathbf{R}^n) \text{ and } \inf_{\mathbf{R}^n} V(x) < \liminf_{x\to\infty} V(x). \tag{0.4}$$

Finally we quote [2] and [5] where the existence of solutions to (0.3) is treated under suitable assumptions on the potential $V(x)$. In fact the class of potentials analysed in [2] and [5] is very much related with the ones considered in this article. More precisely in [5] the authors prove the existence of a nontrivial solution $u \in H_0^1(\Omega)$ to the following critical nonlinear elliptic equation:

$$-\Delta u + b(x)u = u|u|^{2^*-2} \quad \text{in } \Omega \tag{0.5}$$

where $\Omega \subset \mathbf{R}^n$ is an open set, $n \geq 5$, $2^* = 2n/(n-2)$ and $b(x)$ is such that:

$$b(x) \in \mathcal{C}^1(\Omega) \cap L^{n/2}(\Omega) \text{ and } \Omega_- \equiv \{x \in \Omega \mid b(x) < 0\} \neq \emptyset.$$

Let us recall that in [10] the equation (0.5) is studied without any continuity assumption on $b(x)$.

In [2] it is studied the existence and multiplicity of solutions to the following family of subcritical elliptic equations, that are strictly related with (0.3):

$$-\Delta u + (\lambda b(x) + 1)u = u|u|^{p-2} \text{ on } \mathbf{R}^n, \tag{0.6}$$

for $\lambda > 0$ large and $1 < p < 2n/(n-2)$. Moreover in [2] the following assumptions are done on $b(x)$:

$b(x) \in \mathcal{C}(\mathbf{R}^n)$ and $b(x) \geq 0$,
the interior part of the set $\{x \in \mathbf{R}^n \mid b(x) = 0\}$ is non-empty,
there is $M_0 > 0$ such that $\{x \in \mathbf{R}^n \mid b(x) < M_0\}$ has finite measure.

The main novelty in our paper is that we prove the existence of solutions to (0.3) provided that $V(x) \in L^\infty(\mathbf{R}^n)$ and $V(x)$ has a sublevel with finite and positive measure. Notice that comparing our hypothesis with the ones in [2], [5] and [12], we assume neither the continuity of $V(x)$ nor the existence of a sublevel of $V(x)$ with finite measure and nontrivial interior part.

Let us underline that our approach bases on the one developed in [4], where (0.3) is studied on bounded domains and for critical nonlinearities, i.e. $p \equiv 2^*$. Neverthless, as it will be clear in the sequel, some modifications have to be done in order to show that the scheme developed in [4] still works for the general class of potentials treated in this paper.

As far as we know the class of potentials studied in this article has not been treated elsewhere.

In order to state more precisely our results let us fix some notations.

**Notations**

( i ) For every $2 < p < 2n/(n-2)$ when $n \geq 3$, and for every $2 < p < \infty$ when $n = 1, 2$, we shall denote by $M_p$ the following set:

$$M_p \equiv \left\{ u \in W^{1,2}(\mathbf{R}^n) \;\Big|\; \int_{\mathbf{R}^n} |u|^p \, dx = 1 \right\}; \tag{0.7}$$

(ii) for every measurable set $A \subset \mathbf{R}^n$, we shall denote by $\text{meas}(A)$ its Lebesgue measure;

(iii) given $b(x) \in L^\infty(\mathbf{R}^n)$ we shall denote by $\inf - \text{ess} b$ and $\sup - \text{ess} b$ the essential infimum and the essential supremum of $b(x)$.

Next we present the basic theorem of the paper. In fact it will allow us to prove existence of solutions to (0.3) when $p$ is subcritical under suitable assumptions on $V(x)$. Let us underline that in next theorem in order to simplify the notations we shall denote by $a(x)$ the potential $V(x) - E$ that appears in (0.3). In fact in Theorem 0.2 we shall go back to the original potential $V(x)$ that appears in (0.1) and (0.3).

**Theorem 0.1** *Assume that $a(x)\colon \mathbf{R}^n \to \mathbf{R}$ is a measurable function that satisfies the following conditions*:
(0.8) *there exist $m$, $M > 0$ such that $m \leq a(x) \leq M \quad \forall x \in \mathbf{R}^n$.*
(0.9) *there exists $m_0 > 0$ such that*

$$0 < \operatorname{meas}\{x \in \mathbf{R}^n \mid a(x) < m_0\} < \infty.$$

*Then then there exists a minimizer for*

$$I_{\hbar,a,p} \equiv \inf_{u \in M_p} \int_{\mathbf{R}^n} (\hbar^2 |\nabla u|^2 + a(x)|u|^2) dx, \tag{0.10}$$

*provided that $2 < p < 2n/(n-2)$ when $n \geq 3$, $2 < p < \infty$ when $n = 1, 2$ and $0 < \hbar < \hbar_0$ where $\hbar_0 = \hbar_0(a(x), p) > 0$ is a small number.*
*In particular there exists a nontrivial solution to*

$$-\hbar^2 \Delta u + a(x) u = u|u|^{p-2}, \ u \in W^{1,2}(\mathbf{R}^n) \tag{0.11}$$

*provided that $\hbar > 0$ is small enough.*

**Remark 0.1** Let us emphasize that the homogeneity of the pure power nonlinearity $f(u) \equiv u|u|^{p-2}$ appearing in r.h.s. of (0.11), plays a crucial role in order to deduce the existence of a non trivial solution to (0.11) once the existence of a minimizer to (0.10) is proved. However, we think that our argument can be useful also to treat (0.11) with more general nonlinearities $f(u)$. Of course in this case the existence of solutions has to be deduced via the mountain-pass theorem, following the approach in [12]. In fact the argument involved in the proof of lemma 3.1 (that is essentially a comparison result for the minima of suitable funtionals), can be adapted in order to prove a comparison result for the mountain–pass values of suitable functionals, that is one of the key points in [12].

As a consequence of the previous theorem we deduce an existence result of standing waves to (0.1) under suitable assumptions on $V(x)$ and for subcritical nonlinearities.

**Theorem 0.2** *Assume that $V(x) \in L^\infty(\mathbf{R}^n)$ satisfies the following condition:*
(0.12) *there exists $k_0 \in \mathbf{R}$ such that*

$$0 < \mathrm{meas}\{x \in \mathbf{R}^n \mid V(x) < k_0\} < \infty.$$

*Then for every $E \in (-\infty, \inf - \mathrm{ess}V(x))$ and $2 < p < 2n/(n-2)$, there exists $\hbar_0 = \hbar_0(E, p) > 0$ such that*

$$-\hbar^2 \Delta u + (V(x) - E)u = u|u|^{p-2}, \ u \in W^{1,2}(\mathbf{R}^n)$$

*has a nontrivial solution, for $0 < \hbar < \hbar_0$. In particular*

$$\psi(t, x) = e^{-\mathbf{i}Et/\hbar} u(x)$$

*is a standing wave solution to* (0.1) *for $2 < p < 2n/(n-2)$ when $n \geq 3$, $2 < p < \infty$ when $n = 1, 2$.*

**Remark 0.2** Notice that Theorem 0.2 follows trivially from Theorem 0.1, where we choose $a(x) \equiv V(x) - E$. In fact this function satisfies all the assumptions of Theorem 0.1, provided that $V(x)$ and $E$ are as in Theorem 0.2.

**Remark 0.3** Notice that if $V(x)$ is any bounded function that satisfies (0.4), then $V(x)$ satisfies also the assumptions of Theorem 0.2. In fact in this case the hypothesis (0.12) is fulfilled by any real number $k_0$ such that

$$\inf_{\mathbf{R}^n} V(x) < k_0 < \liminf_{x \to \infty} V(x).$$

**Remark 0.4** Notice that there exist potentials $V(x) \in C^\infty(\mathbf{R}^n)$ that are bounded, satisfy condition (0.12), but (0.4) is not fulfilled by $V(x)$. For example consider any regular and bounded function $V(x)$ whose range belongs to $[-1, 1]$ and such that:

$$V(x) \equiv -1 \text{ for } x \in \cup_{k=1}^\infty B(x_k, 2^{-(k+2)})$$
$$V(x) \equiv 1 \text{ for } x \in \mathbf{R}^n \setminus \cup_{k=1}^\infty B(x_k, 2^{-(k+1)})$$

where $x_k = (k, 0, \ldots, 0) \in \mathbf{R}^n$. Hence the hypothesis done on $V(x)$ in Theorem 0.2 is weaker than (0.4), also in the case that $V(x)$ is bounded and regular.

The paper is organized as follows: Section 1 is devoted to present in an abstract and generalized version a fundamental result contained in [4]. In fact this generalization will be a basic tool for the proof of Theorem 0.1;

in Section 2 we shall recall some well–known facts connected with the best constant in the Sobolev embedding on $\mathbf{R}^n$ and finally in Section 3 we shall prove Theorem 0.1.

## 1. The abstract approach

In this section we present a suitable version of a Lemma proved in [4], that will be very useful along the proof of Theorem 0.1.

Next we fix some notations and we give some definitions.

We shall denote by $\mathcal{H}$ a generic Hilbert space endowed with the norm $\|\cdot\|_{\mathcal{H}}$ and we shall assume that there exists a continuous and dense inclusion

$$\mathcal{H} \subset X, \tag{1.1}$$

where $X$ is a Banach space endowed with the norm $\|\cdot\|_X$.

We assume also that the following property is satisfied:

(1.2) if $h_k \in \mathcal{H}$ is such that $h_k \rightharpoonup \bar{h}$ in $\mathcal{H}$, then up to a subsequence

$$\|h_k\|_X^2 \leq \|h_k - \bar{h}\|_X^2 + \|\bar{h}\|_X^2 + o(1),$$

where $\lim_{k\to\infty} o(1) = 0$.

A typical example of spaces $\mathcal{H}$ and $X$ that satisfy (1.2) are the spaces $W^{1,2}(\mathbf{R}^n)$ and $L^p(\mathbf{R}^n)$ for $2 \leq p \leq 2n/(n-2)$. The proof of this fact follows from the Brézis and Lieb lemma, see [3].

Next we give a definition.

**Definition 1.1** Let $T\colon \mathcal{H} \to \mathbf{R}$ be a map (possibly nonlinear). We shall say that $T$ is *weakly continuous* if for every sequence $h_k \in \mathcal{H}$ the following implication is satisfied:

$$h_k \rightharpoonup \bar{h} \text{ in } \mathcal{H} \Rightarrow \lim_{k\to\infty} T(h_k) = T(\bar{h}).$$

Let us give an explicit example of operator that is weakly compact in the sense of Definition 1.1.

**Proposition 1.1** *Let $K \subset \mathbf{R}^n$, with $n \geq 1$, be a measurable subset such that meas$(K) < \infty$ and $a(x) \in L^\infty(\mathbf{R}^n)$. Then the operator*

$$T\colon W^{1,2}(\mathbf{R}^n) \to \int_K a(x)|u|^2 dx \in \mathbf{R}$$

*is weakly continuous.*

*Proof.* Let $u_k \in W^{1,2}(\mathbf{R}^n)$ be such that $u_k \rightharpoonup \bar{u}$ in $W^{1,2}(\mathbf{R}^n)$. Then due to the compactness of the Sobolev embedding on bounded sets we get:

$$u_k \to \bar{u} \text{ in } L^2(B_R(0)) \ \forall\ 0 < R < \infty.$$

In particular this implies that

$$\lim_{k\to\infty} \int_{K\cap B_R(0)} a(x)|u_k|^2 dx = \int_{K\cap B_R(0)} a(x)|\bar{u}|^2 dx, \tag{1.3}$$

for every $0 < R < \infty$.

Since $\operatorname{meas}(K) < \infty$ we deduce that

$$\lim_{R\to\infty} \operatorname{meas}(K \cap (\mathbf{R}^n \setminus B_R(0)) = 0. \tag{1.4}$$

On the other hand the Sobolev embedding implies:

$$\|u_k\|_{L^{p(n)}(\mathbf{R}^n)} < C \quad \forall k \in \mathbf{N}, \tag{1.5}$$

where $p(n) > 2$ and $0 < C < \infty$.

By using the Hölder inequality we get:

$$\begin{aligned} &\int_{K\cap(\mathbf{R}^n\setminus B_R(0))} |a(x)||u_k|^2 dx \\ &\le \|a\|_{L^\infty(\mathbf{R}^n)} \|u_k\|^2_{L^{p(n)}(\mathbf{R}^n)} |\operatorname{meas}(K \cap (\mathbf{R}^n \setminus B_R(0))|^{1/q_n}, \end{aligned}$$

where $2/p_n + 1/q_n = 1$. By combining this inequality with (1.4) and (1.5) we deduce:

$$\lim_{R\to\infty} \left( \sup_{k\in\mathbf{N}} \int_{K\cap(\mathbf{R}^n\setminus B_R(0))} a(x)|u_k|^2 dx \right) = 0, \tag{1.6}$$

that in conjunction with (1.3) implies:

$$\lim_{k\to\infty} \int_K a(x)|u_k|^2 dx = \int_K a(x)|\bar{u}|^2.$$

□

Next we state an abstract result whose proof follows step by step an argument used in [4]. However this abstract formulation will be useful for our purpose in the sequel.

We shall denote by $B_X$ the following set:

$$B_X \equiv \{x \in X \mid \|x\|_X = 1\}.$$

**Proposition 1.2** *Let $\mathcal{H}$ and $X$ be an Hilbert space and a Banach space that satisfy* (1.1) *and* (1.2).
*Let $T\colon \mathcal{H} \to \mathbf{R}$ be a map such that*:
(1.7) *$T$ is weakly continuous* (*see definition* (1.1));
(1.8) $T(\lambda h) = \lambda^2 T(h) \quad \forall \lambda \geq 0,\ h \in \mathcal{H}$;
(1.9) *there exists $C > 0$ such that*

$$|T(h)| \leq C \ \forall h \in \mathcal{H} \cap B_X.$$

*Then there exists a minimizer $h_0 \in \mathcal{H} \cap B_X$ for the following minimization problem*:

$$I_T \equiv \inf_{h \in \mathcal{H} \cap B_X} (\|h\|^2_{\mathcal{H}} + T(h)),$$

*provided that*

$$I_T < I_0 \equiv \inf_{h \in \mathcal{H} \cap B_X} \|h\|^2_{\mathcal{H}}. \tag{1.10}$$

*Proof.* Notice that due to (1.9) we have $I_T > -\infty$, hence there exists $h_k \in \mathcal{H}$ such that:

$$\|h_k\|_X = 1 \text{ and } \|h_k\|^2_{\mathcal{H}} + T(h_k) = I_T + o(1). \tag{1.11}$$

Due again to (1.9) it is easy to deduce that $h_k$ is bounded in $\mathcal{H}$ and then up to a subsequence we can assume that $h_k \rightharpoonup \bar{h}$ in $\mathcal{H}$ and then

$$\|h_k\|^2_{\mathcal{H}} = \|\bar{h}\|^2_{\mathcal{H}} + \|h_k - \bar{h}\|^2_{\mathcal{H}} + o(1). \tag{1.12}$$

Moreover we are assuming that $T$ is weakly continuous and then

$$T(h_k) = T(\bar{h}) + o(1). \tag{1.13}$$

*First step*: $\bar{h} \neq 0$

Due to the definition of $I_0$, (1.11) and (1.13) we get

$$I_0 + T(\bar{h}) \leq \|h_k\|^2_{\mathcal{H}} + T(h_k) + o(1) \leq I_T + o(1)$$

and this chain of inequalities clearly mplies $T(\bar{h}) \leq I_T - I_0 < 0$.

In particular $T(\bar{h}) < 0$ and since by (1.8) we have $T(0) = 0$, we deduce that $\bar{h} \neq 0$.

*Second step*: *$\bar{h}/\|\bar{h}\|_X$ is minimizer for $I_T$*

By combining (1.2), (1.11), (1.12), (1.13) with the definition of $I_0$ we get:

$$\begin{aligned}
&I_0\|h_k-\bar{h}\|_X^2+\|\bar{h}\|_{\mathcal{H}}^2+T(\bar{h})\leq\|h_k-\bar{h}\|_{\mathcal{H}}^2+\|\bar{h}\|_{\mathcal{H}}^2+T(\bar{h})\\
&=\|h_k\|_{\mathcal{H}}^2+T(h_k)+o(1)=I_T=I_T\|h_k\|_X^2\\
&\leq I_T\|h_k-\bar{h}\|_X^2+I_T\|\bar{h}\|_X^2+o(1)\leq I_0\|h_k-\bar{h}\|_X^2+I_T\|\bar{h}\|_X^2+o(1)
\end{aligned}$$

where at the last step we have used the assumption $I_T < I_0$.

Since $\lim_{k\to\infty} o(1) = 0$ it is easy to deduce that the previous chain of inequalities imply:

$$\|\bar{h}\|_{\mathcal{H}}^2+T(\bar{h})\leq I_T\|\bar{h}\|_X^2$$

and then due to (1.8) we can deduce that $\bar{h}/\|\bar{h}\|_X$ is a minimizer for $I_T$.

□

## 2. Some preliminary facts

For every $\lambda > 0$, $\hbar > 0$, $2 < p < 2n/(n-2)$ when $n \geq 3$ and $2 < p < \infty$ when $n = 1, 2$, we introduce the numbers $m_{\hbar,\lambda,p}$ defined as follows:

$$m_{\hbar,\lambda,p}\equiv\inf_{u\in M_p}\int_{\mathbf{R}^n}(\hbar^2|\nabla u|^2+\lambda|u|^2)dx, \tag{2.1}$$

(recall that $M_p$ is defined in (0.7)).

Following the notation of Theorem 0.1 we have

$$m_{\hbar,\lambda,p}=I_{\hbar,a,p}\text{ where }a(x)\equiv\lambda.$$

The minimization problem (2.1) has been extensively studied in [8]. As a consequence of the results proved in [8] we can state the following proposition that will be useful in the sequel.

**Proposition 2.1** *Let $\lambda > 0$ and $2 < p < 2n/(n-2)$ when $n \geq 3$, $2 < p < \infty$ when $n = 1, 2$, then there exists $\omega_{\lambda,p} \in M_p$ such that*

$$\int_{\mathbf{R}^n}(|\nabla\omega_{\lambda,p}|^2+\lambda|\omega_{\lambda,p}|^2)dx=m_{1,\lambda,p}>0. \tag{2.2}$$

*For every $\hbar > 0$ we have*

$$\int_{\mathbf{R}^n}(\hbar^2|\nabla\omega_{\hbar,\lambda,p}|^2+\lambda|\omega_{\hbar,\lambda,p}|^2)dx=m_{\hbar,\lambda,p}, \tag{2.3}$$

*where* $\omega_{\hbar,\lambda,p} = \hbar^{-n/p}\omega_{\lambda,p}(x/\hbar) \in M_p$, *and in particular*

$$m_{\hbar,\lambda,p} = \hbar^{n(1-2/p)} m_{1,\lambda,p}. \tag{2.4}$$

*Moreover the following inequality holds for every* $\hbar > 0$*:*

$$m_{\hbar,\lambda_1,p} < m_{\hbar,\lambda_2,p} \text{ when } 0 < \lambda_1 < \lambda_2 < \infty. \tag{2.5}$$

*Proof.* The proof of (2.2) can be found in [8]. The identities (2.3) and (2.4) follow from an easy rescaling argument.

In order to prove (2.5) notice that due to the definition of $m_{\hbar,\lambda_1,p}$ and due to the assumption $\lambda_1 < \lambda_2$ we get:

$$\begin{aligned} m_{\hbar,\lambda_1,p} &\le \int_{\mathbf{R}^n} (\hbar^2 |\nabla \omega_{\hbar,\lambda_2,p}|^2 + \lambda_1 |\omega_{\hbar,\lambda_2,p}|^2) dx \\ &< \int_{\mathbf{R}^n} (\hbar^2 |\nabla \omega_{\hbar,\lambda_2,p}|^2 + \lambda_2 |\omega_{\hbar,\lambda_2,p}|^2) dx = m_{\hbar,\lambda_2,p}, \end{aligned}$$

where $\omega_{\hbar,\lambda_2,p}$ is the function that appears in (2.3) for $\lambda = \lambda_2$. $\square$

## 3. Proof of Theorem 0.1

Along this section the numbers $m_0, m, M > 0$ are the ones that appear in the assumptions of Theorem 0.1. Recall also that $M_p$ is the manifold defined in (0.7), while $I_{\hbar,a,p}$ and $m(\hbar,\, \lambda,\, p)$ are the quantities introduced in (0.10) and (2.1).

For each $\hbar > 0$ we introduce the Hilbert space $\mathcal{H}_\hbar \equiv W^{1,2}(\mathbf{R}^n)$ endowed with the norm:

$$\|u\|^2_{\mathcal{H}_\hbar} \equiv \int_{\mathbf{R}^n} (\hbar^2 |\nabla u|^2 dx + \max\{m_0,\, a(x)\} |u|^2) dx, \tag{3.1}$$

and the operator $T$

$$T \colon \mathcal{H}_\hbar \ni u \to \int_{\{a(x)<m_0\}} (a(x) - m_0)|u|^2 dx \in \mathbf{R}. \tag{3.2}$$

Notice that we have the following identity:

$$\int_{\mathbf{R}^n} (\hbar^2 |\nabla u|^2 + a(x)|u|^2) dx = \|u\|^2_{\mathcal{H}_\hbar} + T(u).$$

Next we shall verify that the assumptions of Proposition 1.2 are fulfilled for the choice:

(3.3) $\mathcal{H} \equiv \mathcal{H}_\hbar,\ X \equiv L^p(\mathbf{R}^n)$
with $2 < p < 2n/(n-2)$, $n \geq 3$, $2 < p < \infty$ when $n = 1, 2$ and $T$ as in (3.2).

In fact condition (1.8) is trivially satisfied, while (1.9) follows from the following computation:

$$|T(u)| \leq 2 \sup_{\mathbf{R}^n} |a(x)| (\text{meas}\{x \in \mathbf{R}^n \mid a(x) < m_0\})^1/q \|u\|^2_{L^p(\mathbf{R}^n)}$$

where $2/p + 1/q = 1$, and then, due to the assumption (0.9), we get

$$|T(u)| < C \quad \forall u \in M_p = \Big\{ u \in \mathcal{H}_\hbar \Bigm| \int_{\mathbf{R}^n} |u|^p dx = 1 \Big\}.$$

On the other hand the operator $T$ is *weakly continuous* due to Proposition 1.1, while the Brézis and Lieb lemma (see [3]) implies that condition (1.2) is satisfied for the choice of $\mathcal{H}$ and $X$ done in (3.3).

We are then in position to apply Proposition 1.2 and to deduce that Theorem 0.1 will follow from the next lemma.

**Lemma 3.1** *Let* $a(x)$, $n$, $p$ *be as in* Theorem 0.1. *Then there exists a constant* $\hbar_0 = \hbar_0(a(x), p, n) > 0$ *such that*

$$I_{\hbar,a,p} < \inf_{u \in M_p} \|u\|^2_{\mathcal{H}_\hbar}$$

*when* $0 < \hbar < \hbar_0$.

*Proof.* Since $a(x)$ is bounded by assumption, we also have $a(x) \in L^1_{\text{loc}}(\mathbf{R}^n)$. We can then use the Lebesgue derivation theorem (see [6]) in order to deduce:

$$\lim_{\delta \to 0} \frac{\int_{B_\delta(x_0)} |a(x) - a(x_0)| dx}{\delta^n} = 0 \quad \forall x_0 \in \mathbf{R}^n \setminus \mathcal{N},$$

where *meas*$\mathcal{N} = 0$. On the other hand by assumption we have

$$\text{meas}\{x \in \mathbf{R}^n \mid m \leq a(x) < m_0\} > 0,$$

and in particular it implies that

$$\{x \in \mathbf{R}^n \mid m \leq a(x) < m_0\} \cap (\mathbf{R}^n \setminus \mathcal{N}) \neq \emptyset,$$

or equivalently there exists $\bar{x} \in \mathbf{R}^n$ such that:

$$\lim_{\delta \to 0} \frac{\int_{B_\delta(\bar{x})} |a(x) - a(\bar{x})| dx}{\delta^n} = 0 \text{ with } m \leq a(\bar{x}) < m_0.$$

Up to a traslation we can assume that $\bar{x} = 0$, then

$$\lim_{\delta\to 0} \frac{\int_{B_\delta(0)} |a(x) - m_1| dx}{\delta^n} = 0 \text{ with } m \le m_1 < m_0, \tag{3.4}$$

where we have used the notation $m_1 = a(0)$.

Next we shall need the function $\omega_{m_1,p} \in M_p$ defined in Proposition 2.1 and associated to the value $\lambda = m_1$, together with the rescaled functions

$$\omega_{\hbar,m_1,p} = \hbar^{-n/p} \omega_{m_1,p}\Big(\frac{x}{\hbar}\Big) \in M_p.$$

Notice that by using Proposition 2.1 and the definition of $I_{\hbar,a,p}$ we have:

$$\begin{aligned} I_{\hbar,a,p} &\le \int_{\mathbf{R}^n} (\hbar^2 |\nabla \omega_{\hbar,m_1,p}|^2 + a(x)|\omega_{\hbar,m_1,p}|^2) dx \\ &= \int_{\mathbf{R}^n} (\hbar^2 |\nabla \omega_{\hbar,m_1,p}|^2 + m_1 |\omega_{\hbar,m_1,p}|^2 + (a(x) - m_1)|\omega_{\hbar,m_1,p}|^2) dx \\ &= \hbar^{n(1-2/p)} \int_{\mathbf{R}^n} (|\nabla \omega_{m_1,p}|^2 + m_1 |\omega_{m_1,p}|^2 + (a(\hbar x) - m_1)|\omega_{m_1,p}|^2) dx \\ &= \hbar^{n(1-2/p)} m_{1,m_1,p} + \hbar^{n(1-2/p)} R(\hbar), \end{aligned} \tag{3.5}$$

where

$$R(\hbar) \equiv \int_{\mathbf{R}^n} (a(\hbar x) - m_1)|\omega_{m_1,p}|^2 dx$$

and $m_1 > 0$ is the number that appears in (3.4).

*Estimate for $R(\hbar)$*

Let us fix $R > 0$ such that

$$\int_{|x|>R} |\omega_{m_1,p}|^2 dx < \frac{1}{4M}(m_{1,m_0,p} - m_{1,m_1,p}) \tag{3.6}$$

where $M$, $m_0 > 0$ are the constants that appear in the assumptions of Theorem 0.1.

Notice that combination of the trivial inequality $m_1 < m_0$ with (2.5) in Proposition 2.1 implies:

$$m_{1,m_1,p} < m_{1,m_0,p}. \tag{3.7}$$

On the other hand the function $\omega_{m_1,p}$ is solution of an elliptic equation and then an easy bootstrap argument implies that

$$\omega_{m_1,p} \in L^\infty(\mathbf{R}^n).$$

Then we can can perform the following estimates, where $R > 0$ is the number given in (3.6):

$$\begin{aligned}|R(\hbar)| &\leq \int_{|x|<R} |a(\hbar x) - m_1||\omega_{m_1,p}|^2 dx \\ &\quad + \int_{|x|>R} |a(\hbar x) - m_1||\omega_{m_1,p}|^2 dx \\ &\leq \Big(\int_{|x|<R} |a(\hbar x) - m_1| dx\Big) \|\omega_{m_1,p}\|^2_{L^\infty(\mathbf{R}^n)} + 2M \|\omega_{m_1,p}\|^2_{L^2(|x|>R)} \\ &= \frac{R^n}{R^n \hbar^n} \Big(\int_{|x|<R\hbar} |a(x) - m_1| dx\Big) \|\omega_{m_1,p}\|^2_{L^\infty(\mathbf{R}^n)} + 2M \|\omega_{m_1,p}\|^2_{L^2(|x|>R)}.\end{aligned}$$

By combining this chain of inequalities with (3.4) and (3.6) we get

$$|R(\hbar)| \leq R^n o(1) + \frac{1}{2}(m_{1,m_0,p} - m_{1,m_1,p}) \tag{3.8}$$

where

$$\lim_{\hbar\to 0} o(1) = 0.$$

By combining (3.5) with (3.8) we deduce:

$$\begin{aligned}I_{\hbar,a,p} &\leq \hbar^{n(1-2/p)} m_{1,m_1,p} + \hbar^{n(1-2/p)} R^n o(1) \\ &+ \frac{1}{2}\hbar^{n(1-2/p)}(m_{1,m_0,p} - m_{1,m_1,p}).\end{aligned} \tag{3.9}$$

On the other hand (3.7) implies that for $\hbar > 0$ small enough we have:

$$\begin{aligned}&\hbar^{n(1-2/p)} m_{1,m_1,p} + \hbar^{n(1-2/p)} R^n o(1) + \frac{1}{2}\hbar^{n(1-2/p)}(m_{1,m_0,p} - m_{1,m_1,p}) \\ &< \hbar^{n(1-2/p)} m_{1,m_0,p} = m_{\hbar,m_0,p},\end{aligned}$$

that in conjunction with (3.9) implies:

$$I_{\hbar,a,p} < m_{\hbar,m_0,p}. \tag{3.10}$$

On the other hand looking at the definition of $\|\cdot\|_\hbar$ in (3.1) we have

$$\int_{\mathbf{R}^n} (\hbar^2 |\nabla u|^2 + m_0 |u|^2) dx \leq \|u\|^2_{\mathcal{H}_h} \quad \forall u \in M_p$$

and then

$$m_{\hbar,m_0,p} \leq \inf_{M_p} \|u\|^2_{\mathcal{H}_h}$$

that can be combined with (3.10) in order to give the desired result. $\square$

## References


[ 1 ] Ambrosetti A., Badiale M. and Cingolani S., *Semiclassical states of nonlinear Schrödinger equations*. Arch. Rational Mech. Anal. (3) **140** (1997), 285–300.

[ 2 ] Bartsch T. and Wang Z.-Q., *Multiple positive solutions for a nonlinear Schrödinger equation*. Z. Angew. Math. Phys. (3) **51** (2000), 366–384.

[ 3 ] Brézis H. and Lieb E., *A relation between pointwise convergence of functions and convergence of functionals*. Proc. Amer. Math. Soc. (33) **88** (1983), 486–490.

[ 4 ] Brézis H. and Nirenberg L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*. Comm. Pure Appl. Math. (4) **36** (1983), 437–477.

[ 5 ] Chen S. and Li Y., *Nontrivial solution for a semilinear elliptic equation in unbounded domain with critical Sobolev exponent*. J. Math. Anal. Appl. (2) **272** (2002), 393–406.

[ 6 ] Evans L. and Gariepy R.F., *Measure theory and fine properties of functions*. Studies in Advanced Mathematics, 1992.

[ 7 ] Floer A. and Weinstein A., *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*. J. Funct. An. **69** (1986), 397–408.

[ 8 ] Lions P.L., *The concentration–compactness principle in the calculus of variations: the locally compact case. Part* I. Ann. Inst. H. Poincaré Analyse Non Linéaire (2) **1** (1984), 109–145.

[ 9 ] Lions P.L., *The concentration–compactness principle in the calculus of variations: the locally compact case. Part* II. Ann. Inst. H. Poincaré Analyse non Linéaire (4) **1** (1984), 223–283.

[10] Prinari F. and Visciglia N., *On a Minimization problem involving the ciritcal Sobolev exponent*. Adv. Nonl. Studies **7** (2007), 551–564.

[11] Rodnianski I. and Schlag W., *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*. Invent. Math. (3) **155** (2004), 451–513.

[12] Rabinowitz P.H., *On a class of nonlinear Schrödinger equations.* Z.A.M.P. **43** (1992), 27–42.

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