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A generic classification of function germs with respect to the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence

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Abstract. We investigate several stabilities and a genericity of function germs with respect to the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence.

Key words: Legendrian Singularity, Contact Manifold, Mather theory, Singularity

1. Introduction

In [3], S. Izumiya introduced the equivalence relation 't-$\mathcal{P}$-$\mathcal{K}$-equivalence' of function germs in order to classify 'generic Legendrian unfoldings'. The classification list is given in [12] by V. M. Zakalyukin who classified quasi-homogeneous function germs.

In this paper we introduce a more general equivalence relation 'reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence' of function germs in $\mathfrak{M}(r; k+n+m)$ and give a generic classification in the case $r = 0$, $n \leq 5$, $m \leq 1$ and $r = 1$, $n \leq 3$, $m \leq 1$ respectively. Our one is for not only quasi-homogeneous function germs but also all smooth function germs. Our work in this paper will play an important role in a generic classification of bifurcations of wave fronts generated by a hypersurface germ with a boundary ([8], [9]).

Let $\mathbb{H}^r = \{(x_1, \ldots, x_r) \in \mathbb{R}^r | x_1 \geq 0, \ldots, x_r \geq 0\}$ be an r-corner. We consider a equivalence relation of the set $\mathcal{E}(r; k+n+m)$ of function germs on $(\mathbb{H}^r \times \mathbb{R}^{k+n+m}, 0)$. Function germs $F, G \in \mathcal{E}(r; k+n+m)$ are called *reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent* if there exist a diffeomorphism germ Φ on $(\mathbb{H}^r \times \mathbb{R}^{k+n+m}, 0)$ and a unit $\alpha \in \mathcal{E}(r; k+n+m)$ such that

(1) Φ can be written in the form:

$$\Phi(x, y, u, t) = \big(x_1\phi_1^1(x, y, u, t), \ldots, x_r\phi_1^r(x, y, u, t),$$
$$\phi_2(x, y, u, t), \phi_3(u, t), \phi_4(t)\big),$$

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(2) $G(x,y,u,t) = \alpha(x,y,u,t) \cdot F \circ \Phi(x,y,u,t)$ for all $(x,y,u,t) \in (\mathbb{H}^r \times \mathbb{R}^{k+n+m}, 0)$.

We investigate stabilities and a genericity of function germs under this equivalence relation. The main result is the following (Theorem 4.7):

Let $r = 0$, $n \leq 5$ or $r = 1$, $n \leq 3$ and U be a neighborhood of 0 in $\mathbb{H}^r \times \mathbb{R}^{k+n+1}$. Then there exists a residual set $O \subset C^\infty(U, \mathbb{R})$ with C^∞-topology such that for any $\tilde{F} \in O$ and $(0, y_0, u_0, t_0) \in U$, the function germ $F(x,y,u,t) \in \mathfrak{M}(r; k+n+1)$ given by $F(x,y,u,t) = \tilde{F}(x, y+y_0, u+u_0, t+t_0) - \tilde{F}(0, y_0, u_0, t_0)$ is *reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding* of $F|_{t=0}$ and *stably reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent* to one of the types:

In the case $r = 0, n \leq 5$: ${}^0A_l (0 \leq l \leq 5), {}^0D_4^\pm, {}^0D_5, {}^1A_l (1 \leq l \leq 6), {}^1D_4^\pm, {}^1D_5, {}^1D_6^\pm$, and ${}^1E_6$.
In the case $r = 1, n \leq 3$: ${}^0A_1, {}^0A_2, {}^0A_3, {}^0B_1, {}^0B_2, {}^0B_3, {}^0C_3^\pm, {}^1A_2, {}^1A_3, {}^1A_4, {}^1D_4^\pm, {}^1B_1, {}^1B_2, {}^1B_3, {}^1B_4, {}^1C_3^\pm, {}^1C_4$, and ${}^1F_4$.

This paper consists of three sections. In Section 2 we define notations and review stabilities of unfoldings under *the reticular $\mathcal{P}$-$\mathcal{K}$-equivalence relation.* In Section 3 we investigate stabilities of unfoldings under the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence relation. In Section 4 we give a generic classification of function germs under the equivalence relation.

2. Preliminaries

We denote by $\mathcal{E}(r; k_1, r; k_2)$ the set of all germs at 0 in $\mathbb{H}^r \times \mathbb{R}^{k_1}$ of smooth maps $\mathbb{H}^r \times \mathbb{R}^{k_1} \to \mathbb{H}^r \times \mathbb{R}^{k_2}$ and set $\mathfrak{M}(r; k_1, r; k_2) = \{f \in \mathcal{E}(r; k_1, r; k_2) | f(0) = 0\}$. We denote $\mathcal{E}(r; k_1, k_2)$ for $\mathcal{E}(r; k_1, 0; k_2)$ and denote $\mathfrak{M}(r; k_1, k_2)$ for $\mathfrak{M}(r; k_1, 0; k_2)$.

If $k_2 = 1$ we write simply $\mathcal{E}(r; k)$ for $\mathcal{E}(r; k, 1)$ and $\mathfrak{M}(r; k)$ for $\mathfrak{M}(r; k, 1)$. Then $\mathcal{E}(r; k)$ is an $\mathbb{R}$-algebra in the usual way and $\mathfrak{M}(r; k)$ is its unique maximal ideal. We also denote by $\mathcal{E}(k)$ for $\mathcal{E}(0; k)$ and $\mathfrak{M}(k)$ for $\mathfrak{M}(0; k)$.

We denote by $J^l(r+k, p)$ the set of l-jets at 0 of germs in $\mathcal{E}(r; k, p)$. There are natural projections:

$$\pi_l : \mathcal{E}(r; k, p) \longrightarrow J^l(r+k, p), \quad \pi^{l_1}_{l_2} : J^{l_1}(r+k, p) \longrightarrow J^{l_2}(r+k, p) \ (l_1 > l_2).$$

We write $j^l f(0)$ for $\pi_l(f)$ for each $f \in \mathcal{E}(r; k, p)$.

Let $(x, y) = (x_1, \ldots, x_r, y_1, \ldots, y_k)$ be a fixed coordinate system of $(\mathbb{H}^r \times$

$\mathbb{R}^k, 0)$. We denote by $\mathcal{B}(r;k)$ the group of diffeomorphism germs $(\mathbb{H}^r \times \mathbb{R}^k, 0) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$ of the form:

$$\phi(x,y) = \big(x_1\phi_1^1(x,y), \dots, x_r\phi_1^r(x,y),\ \phi_2^1(x,y), \dots, \phi_2^k(x,y)\big).$$

We denote by $\mathcal{B}_n(r;k+n)$ the group of diffeomorphism germs $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \to (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ of the form:

$$\begin{aligned}\phi(x,y,u) = \big(&x_1\phi_1^1(x,y,u), \dots, x_r\phi_1^r(x,y,u),\\ &\phi_2^1(x,y,u), \dots, \phi_2^k(x,y,u),\ \phi_3^1(u), \dots, \phi_3^n(u)\big).\end{aligned}$$

We denote by $\mathcal{B}_n^l(r;k+n)$ the Lie group of l-jets at 0 of germs in $\mathcal{B}_n(r;k+n)$. This group acts on $J^l(r+k+n,1)$ by the composition.

Lemma 2.1 (cf. [11, Corollary 1.8]) *Let B be a submodule of $\mathcal{E}(r;k+n+m)$, A_1 be a finitely generated $\mathcal{E}(m)$ submodule of $\mathcal{E}(r;k+n+m)$ generated d-elements, and A_2 be a finitely generated $\mathcal{E}(n+m)$ submodule of $\mathcal{E}(r;k+n+m)$. Suppose*

$$\begin{aligned}\mathcal{E}(r;k+n+m) = B + A_2 + A_1 + \mathfrak{M}(m)\mathcal{E}(r;k+n+m)\\ + \mathfrak{M}(n+m)^{d+1}\mathcal{E}(r;k+n+m).\end{aligned}$$

Then

$$\mathcal{E}(r;k+n+m) = B + A_2 + A_1,$$
$$\mathfrak{M}(n+m)^d\mathcal{E}(r;k+n+m) \subset B + A_2 + \mathfrak{M}(m)\mathcal{E}(r;k+n+m).$$

We recall the stabilities of n-dimensional unfolding under *reticular $\mathcal{P}$-$\mathcal{K}$-equivalence* which is developed in [7].

We say that $f_0, g_0 \in \mathcal{E}(r;k)$ are *reticular $\mathcal{K}$-equivalent* if there exist $\phi \in \mathcal{B}(r;k)$ and a unit $a \in \mathcal{E}(r;k)$ such that $g_0 = a \cdot f_0 \circ \phi$. We write $O_{r\mathcal{K}}(f_0)$ the orbit of f_0 under this equivalence relation.

Lemma 2.2 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ and $O^l_{r\mathcal{K}}(j^l f_0(0))$ be the submanifold of $J^l(r+k,1)$ consist of the image by π_l of the orbit of reticular $\mathcal{K}$-equivalence of f_0. Put $z = j^l f_0(0)$. Then*

$$T_z(O^l_{r\mathcal{K}}(z)) = \pi_l\left(\left\langle f_0, x_1\frac{\partial f_0}{\partial x_1}, \dots, x_r\frac{\partial f_0}{\partial x_r}\right\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\left\langle \frac{\partial f_0}{\partial y_1}, \dots, \frac{\partial f_0}{\partial y_k}\right\rangle\right).$$

We say that a function germ $f_0 \in \mathfrak{M}(r;k)$ is *reticular $\mathcal{K}$-l-determined* if all function germ which has same l-jet of f_0 is reticular $\mathcal{K}$-equivalent to f_0. If f_0 is reticular $\mathcal{K}$-l-determined for some l, then we say that f_0 is reticular $\mathcal{K}$-finitely determined.

We denote $x\frac{\partial f_0}{\partial x}$ for $\left(x_1\frac{\partial f_0}{\partial x_1}, \dots, x_r\frac{\partial f_0}{\partial x_r}\right)$ and $\frac{\partial f_0}{\partial y}$ for $\left(\frac{\partial f_0}{\partial y_1}, \dots, \frac{\partial f_0}{\partial y_k}\right)$, and denote other notations analogously.

Lemma 2.3 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ and let*

$$\mathfrak{M}(r;k)^{l+1} \subset \mathfrak{M}(r;k)\left(\left\langle f_0, x\frac{\partial f_0}{\partial x}\right\rangle + \mathfrak{M}(r;k)\left\langle \frac{\partial f_0}{\partial y}\right\rangle\right) + \mathfrak{M}(r;k)^{l+2},$$

then f_0 is reticular $\mathcal{K}$-l-determined. Conversely if $f_0(x,y) \in \mathfrak{M}(r;k)$ is reticular $\mathcal{K}$-l-determined, then

$$\mathfrak{M}(r;k)^{l+1} \subset \left\langle f_0, x\frac{\partial f_0}{\partial x}\right\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\left\langle \frac{\partial f_0}{\partial y}\right\rangle.$$

Let $f(x,y,u) \in \mathfrak{M}(r;k+n_1)$, $g(x,y,v) \in \mathfrak{M}(r;k+n_2)$ be unfoldings of $f_0(x,y) \in \mathfrak{M}(r;k)$. We say that g *is reticular $\mathcal{P}$-$\mathcal{K}$-f_0-induced from* f if there exist $\Phi \in \mathfrak{M}(r;k+n_2,r;k+n_1)$ and $\alpha \in \mathcal{E}(r;k+n_2)$ satisfying the following conditions:

(1) $\Phi(x,y,0) = (x,y,0)$, $\alpha(x,y,0) = 1$ for all $(x,y) \in (\mathbb{H}^r \times \mathbb{R}^k, 0)$,
(2) Φ can be written in the form:

$$\Phi(x,y,v) = \big(x_1\phi_1^1(x,y,v), \dots, x_r\phi_1^r(x,y,v), \phi_2(x,y,v), \phi_3(v)\big),$$

(3) $g(x,y,v) = \alpha(x,y,v)\cdot f\circ\Phi(x,y,v)$ for all $(x,y,v) \in (\mathbb{H}^r \times \mathbb{R}^{k+n_2}, 0)$.
We denote $\Phi(x,y,v) = (x\phi_1(x,y,v), \phi_2(x,y,v), \phi_3(v))$.

We say that $f, g \in \mathcal{E}(r;k+n)$ are *reticular $\mathcal{P}$-$\mathcal{K}$-equivalent* if there exist $\Phi \in \mathcal{B}_n(r;k+n)$ and a unit $\alpha \in \mathcal{E}(r;k+n)$ such that $g = \alpha\cdot f\circ\Phi$. We call (Φ,α) a reticular $\mathcal{P}$-$\mathcal{K}$-isomorphism from f to g. We write $O_{r\mathcal{P}\text{-}\mathcal{K}}(f)$ the orbit of f under this equivalence relation.

Definition 2.4 We recall the definition of several stabilities of unfoldings under the reticular $\mathcal{P}$-$\mathcal{K}$-equivalence. Let $f(x, y, u) \in \mathfrak{M}(r; k+n)$ be an unfolding of $f_0(x, y) \in \mathfrak{M}(r; k)$.

We say that f is *reticular $\mathcal{P}$-$\mathcal{K}$-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{f} \in C^\infty(U, \mathbb{R})$ of f, there exists a neighborhood $N_{\tilde{f}}$ of $\tilde{f}$ in $C^\infty(U, \mathbb{R})$ with C^∞-topology such that for any element $\tilde{g} \in N_{\tilde{f}}$ the germ $\tilde{g}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, u_0)$ is reticular $\mathcal{P}$-$\mathcal{K}$-equivalent to f for some $(0, y_0, u_0) \in U$.

We say that f is *reticular $\mathcal{P}$-$\mathcal{K}$-versal* if any unfolding of f_0 is reticular $\mathcal{P}$-$\mathcal{K}$-f_0-induced from f.

We say that f is *reticular $\mathcal{P}$-$\mathcal{K}$-infinitesimally versal* if

$$\mathcal{E}(r; k) = \left\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y} \right\rangle_{\mathcal{E}(r;k)} + \left\langle \left.\frac{\partial f}{\partial u}\right|_{u=0} \right\rangle_{\mathbb{R}}.$$

We say that f is *reticular $\mathcal{P}$-$\mathcal{K}$-infinitesimally stable* if

$$\mathcal{E}(r; k+n) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u} \right\rangle_{\mathcal{E}(n)}.$$

We say that f is *reticular $\mathcal{P}$-$\mathcal{K}$-homotopically stable* if for any smooth path-germ $(\mathbb{R}, 0) \to \mathcal{E}(r; k+n), t \mapsto f^t$ with $f^0 = f$, there exists a smooth path-germ $(\mathbb{R}, 0) \to \mathcal{B}_n(r; k+n) \times \mathcal{E}(r; k+n), t \mapsto (\Phi_t, \alpha_t)$ with $(\Phi_0, \alpha_0) = (id, 1)$ such that each (Φ_t, α_t) is a reticular $\mathcal{P}$-$\mathcal{K}$-isomorphism from f^0 to f^t, that is $f^t = \alpha_t \cdot f^0 \circ \Phi_t$.

Theorem 2.5 *Let $f \in \mathfrak{M}(r; k+n)$ be an unfolding of $f_0 \in \mathfrak{M}(r; k)$. Then the following are equivalent.*

(1) *f is reticular $\mathcal{P}$-$\mathcal{K}$-stable.*
(2) *f is reticular $\mathcal{P}$-$\mathcal{K}$-versal.*
(3) *f is reticular $\mathcal{P}$-$\mathcal{K}$-infinitesimally versal.*
(4) *f is reticular $\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.*
(5) *f is reticular $\mathcal{P}$-$\mathcal{K}$-homotopically stable.*

For $f_0(x, y) \in \mathfrak{M}(r; k)$, if $a_1, \ldots, a_n \in \mathcal{E}(r; k)$ is a representative of a basis of the vector space

$$\mathcal{E}(r;k)/\left\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\right\rangle_{\mathcal{E}(r;k)},$$

then the function germ $f_0 + a_1u_1 + \cdots + a_nu_n \in \mathfrak{M}(r;k+n)$ is a reticular $\mathcal{P}$-$\mathcal{K}$-stable unfolding of f_0.

Proposition 2.6 *Let $f_0 \in \mathfrak{M}(r;k)$. Then f_0 has a reticular $\mathcal{P}$-$\mathcal{K}$-stable unfolding if and only if f_0 is reticular $\mathcal{K}$-finitely determined.*

3. Reticular t-$\mathcal{P}$-$\mathcal{K}$-stabilities of unfoldings

The right-left-(n,m)-stabilities of m-dimensional unfoldings of n-dimensional unfoldings of function germs is studied by G. Wassermann in [11]. In this section we study *stabilities* of m-dimensional unfoldings of n-dimensional unfoldings of function germs under *the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence* which should be called reticular (n,m)-$\mathcal{K}$-equivalence in G. Wassermann's notation.

Lemma 3.1 *Let $f(x,y,u) \in \mathcal{E}(r;k+n)$ and set $z = j^lf(0)$. Let $O^l_{r\mathcal{P}\text{-}\mathcal{K}}(z)$ be the submanifold of $J^l(r+k+n,1)$ consist of the image by π_l of the orbit of reticular $\mathcal{P}$-$\mathcal{K}$-equivalence of f_0. Then*

$$T_z\big(O^l_{r\mathcal{P}\text{-}\mathcal{K}}(z)\big) = \pi_l\left(\left\langle f, x\frac{\partial f}{\partial x}\right\rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(r;k+n)\left\langle\frac{\partial f}{\partial y}\right\rangle + \mathfrak{M}(n)\left\langle\frac{\partial f}{\partial u}\right\rangle\right). \tag{1}$$

Here we give the definitions of stabilities of unfoldings under the equivalence relation 'reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence' and prove that these definitions are all equivalent.

Let $F(x,y,u,t) \in \mathfrak{M}(r;k+n+m_1)$ and $G(x,y,u,s) \in \mathfrak{M}(r;k+n+m_2)$ be unfoldings of $f(x,y,u) \in \mathfrak{M}(r;k+n)$.

A reticular t-$\mathcal{P}$-$\mathcal{K}$-f-morphism from G to F is a pair (Φ,α), where $\Phi \in \mathfrak{M}(r;k+n+m_2,r;k+n+m_1)$ and α is a unit of $\mathcal{E}(r;k+n+m_2)$, satisfying the following conditions:

(1) Φ can be written in the form: $\Phi(x,y,u,s) = (x\phi_1(x,y,u,s), \phi_2(x,y,u,s), \phi_3(u,s), \phi_4(s))$,
(2) $\Phi|_{\mathbb{H}^r\times\mathbb{R}^{k+n}} = id_{\mathbb{H}^r\times\mathbb{R}^{k+n}}$, $\alpha|_{\mathbb{H}^r\times\mathbb{R}^{k+n}} \equiv 1$
(3) $G(x,y,u,s) = \alpha(x,y,u,s)\cdot F\circ\Phi(x,y,u,s)$ for all $(x,y,u,s) \in (\mathbb{H}^r\times\mathbb{R}^{k+n+m_2},0)$.

If there exists a reticular t-$\mathcal{P}$-$\mathcal{K}$-f-morphism from F to G, we say that *G is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from F*. If $m_1 = m_2$ and Φ is invertible, we call (Φ, α) *a reticular t-$\mathcal{P}$-$\mathcal{K}$-f-isomorphism from F to G* and we say that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-equivalent to G.

Let U be a neighborhood of 0 in $\mathbb{R}^{r+k+n+m}$ and let $F : U \to \mathbb{R}$ be a smooth function and q be a non-negative integer. We define the smooth map germ

$$j_1^q F : U \longrightarrow J^q(r+k+n, 1)$$

as the follow: For $(x, y, u, t) \in U$ we set $j_1^q F(x, y, u, t)$ by the l-jet of the function germ $\tilde{F}_{(x,y,u,t)} \in \mathfrak{M}(r; k+n)$ at 0, where $\tilde{F}_{(x,y,u,t)}$ is given by $\tilde{F}_{(x,y,u,t)}(x', y', u') = F(x + x', y + y', u + u', t) - F(x, y, u, t)$.

Theorem 3.2 *Let U be a neighborhood of 0 in $\mathbb{R}^{r+k+n+m}$ and A be a smooth submanifold of $J^q(r+k+n, 1)$. We define*

$$T_A = \big\{F \in C^\infty(U, \mathbb{R}) |\ j_1^q F|_{x=0} \textit{ is transversal to } A\big\}.$$

Then T_A is dense in $C^\infty(U, \mathbb{R})$.

The transversality we used is a slightly different for the ordinary one [10], however we can also prove this theorem by the method which is the same as the ordinary method.

Definition 3.3 We define stabilities of unfoldings. Let $F(x, y, u, t) \in \mathfrak{M}(r; k+n+m)$ be an unfolding of $f(x, y, u) \in \mathfrak{M}(r; k+n)$.

Let q be a non-negative integer and $z = j^q f(0)$. We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f* if the $j_1^q F|_{x=0}$ at 0 is transversal to $O^q_{r\mathcal{P}\text{-}\mathcal{K}}(z)$.

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of f* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n+m}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ in $C^\infty(U, \mathbb{R})$ with C^∞-topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n+m}}$ at $(0, y_0, u_0, t_0)$ is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to F for some $(0, y_0, u_0, t_0) \in U$.

We say that F is *a reticular t-$\mathcal{P}$-$\mathcal{K}$-versal unfolding of f* if any unfolding of f is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from F.

We say that F is *a reticular t-$\mathcal{P}$-$\mathcal{K}$-universal unfolding of f* if m is minimal in reticular t-$\mathcal{P}$-$\mathcal{K}$-versal unfoldings of f.

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally versal* if

$$\mathcal{E}(r;k+n) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u} \right\rangle_{\mathcal{E}(n)} + \left\langle \left.\frac{\partial F}{\partial t}\right|_{t=0} \right\rangle_{\mathbb{R}}.$$

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable* if

$$\begin{aligned}&\mathcal{E}(r;k+n+m)\\ &\quad= \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial F}{\partial u} \right\rangle_{\mathcal{E}(n+m)} + \left\langle \frac{\partial F}{\partial t} \right\rangle_{\mathcal{E}(m)}. \qquad (2)\end{aligned}$$

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable* if for any smooth path-germ $(\mathbb{R},0) \to \mathcal{E}(r;k+n+m), \tau \mapsto F_\tau$ with $F_0 = F$, there exists a smooth path-germ $(\mathbb{R},0) \to \mathcal{B}(r,k+n+m) \times \mathcal{E}(r;k+n+m), \tau \mapsto (\Phi_\tau, \alpha_\tau)$ with $(\Phi_0, \alpha_0) = (id, 1)$ such that each (Φ_τ, α_τ) is a reticular t-$\mathcal{P}$-$\mathcal{K}$-isomorphism and $F_\tau = \alpha_\tau \cdot F_0 \circ \Phi_\tau$ for $\tau \in (\mathbb{R},0)$.

For a function germ $f(x,y,u) \in \mathcal{E}(r;k+n)$, we define that

$$T_e(r\mathcal{P}\text{-}\mathcal{K})(f) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u} \right\rangle_{\mathcal{E}(n)},$$

and define that $r\mathcal{P}$-$\mathcal{K}$-cod$f = \dim_{\mathbb{R}} \mathcal{E}(r;k+n)/T_e(r\mathcal{P}\text{-}\mathcal{K})(f)$.

Lemma 3.4 *Let $F(x,y,u,t) \in \mathcal{E}(r;k+n+m)$ be an unfolding of $f(x,y,u) \in \mathfrak{M}(r;k+n)$ and q be a non-negative integer.*

The function germ F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal if and only if

$$\mathcal{E}(r;k+n) = T_e(r\mathcal{P}\text{-}\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial t}|_{t=0} \right\rangle_{\mathbb{R}} + \mathfrak{M}(r;k+n)^{q+1}.$$

We remark that if F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal then F is also reticular t-$\mathcal{P}$-$\mathcal{K}$-q'-transversal for any $q' \leq q$.

Proof of the lemma. By an immediate calculation, we have

$$\begin{aligned}T(j_1^q F|_{x=0})(T_0\mathbb{R}^{k+n+m}) &= \left\langle j^q\frac{\partial f}{\partial y}(0), j^q\frac{\partial f}{\partial u}(0), j^q\frac{\partial F}{\partial t}\Big|_{t=0}(0)\right\rangle_{\mathbb{R}}\\ &= \pi_q\left(\left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial u}, \frac{\partial F}{\partial t}\Big|_{t=0}\right\rangle_{\mathbb{R}}\right)\end{aligned}$$

Therefore

$$\begin{aligned}&F \text{ is a reticular } t\text{-}\mathcal{P}\text{-}\mathcal{K}\text{-}q\text{-transversal}\\
&\Leftrightarrow J^q(r+k+n,1) = T_{j^q f(0)}\big(O^q_{r\mathcal{P}\text{-}\mathcal{K}}(j^q f(0))\big) + T(j_1^q F|_{x=0})(T_0\mathbb{R}^{k+n+m})\\
&\Leftrightarrow J^q(r+k+n,1) = \pi_q\left(\left\langle f, x\frac{\partial f}{\partial x}\right\rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(r;k+n)\left\langle \frac{\partial f}{\partial y}\right\rangle\right.\\
&\qquad\qquad \left. + \mathfrak{M}(n)\left\langle \frac{\partial f}{\partial u}\right\rangle\right) + \pi_q\left(\left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial u}, \frac{\partial F}{\partial t}\Big|_{t=0}\right\rangle_{\mathbb{R}}\right)\\
&\Leftrightarrow J^q(r+k+n,1) = \pi_q\left(\left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)}\right.\\
&\qquad\qquad \left. + \left\langle \frac{\partial F}{\partial t}|_{t=0}\right\rangle_{\mathbb{R}}\right)\\
&\Leftrightarrow \mathcal{E}(r;k+n) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)} + \left\langle \frac{\partial F}{\partial t}|_{t=0}\right\rangle_{\mathbb{R}}\\
&\qquad\qquad + \mathfrak{M}(r;k+n)^{q+1}. \qquad \Box\end{aligned}$$

Proposition 3.5 *Let $F, G \in \mathfrak{M}(r;k+n+m)$ and q be a non-negative integer. Suppose that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to G. If F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal, then G is also reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal.*

Theorem 3.6 (cf. [11, Theorem 3.6]) *Let $f(x,y,u) \in \mathfrak{M}(r;k+n)$ be an unfolding of $f_0(x,y) \in \mathfrak{M}(r;k)$ and $F(x,y,u,t) \in \mathfrak{M}(r;k+n+m)$ be an unfolding of f. Suppose f_0 is reticular $\mathcal{K}$-finitely determined. Choose an integer l such that*

$$\mathfrak{M}(r;k)^{l+1} \subset \left\langle f_0, x\frac{\partial f_0}{\partial x}\right\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\left\langle \frac{\partial f_0}{\partial y}\right\rangle. \tag{3}$$

Let $q \geq lm + l + m$. Then the following are equivalent.

(a) *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.*
(b) *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally versal.*
(c)
$$\mathcal{E}(r;k+n) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u} \right\rangle_{\mathcal{E}(n)} + \left\langle \frac{\partial F}{\partial t}\bigg|_{t=0} \right\rangle_{\mathbb{R}} + \mathfrak{M}(n)^{m+1}\mathcal{E}(r;k+n) + \mathfrak{M}(r;k+n)^{q+1}$$

Proof. It is enough to prove (c)$\Rightarrow$(a). Since $f|_{u=0} = f_0$ it follows that $\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y} \rangle_{\mathcal{E}(r;k)} \subset \langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(n)\mathcal{E}(r;k+n)$. Since $\mathfrak{M}(r;k+n)^{l+1} \subset \mathfrak{M}(r;k)^{l+1} + \mathfrak{M}(n)\mathcal{E}(r;k+n)$ it follows that $\mathfrak{M}(r;k+n)^{q+1} \subset \mathfrak{M}(r;k+n)^{(l+1)(m+1)} \subset \langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(n)^{m+1}\mathcal{E}(r;k+n)$. Therefore we may drop the term $\mathfrak{M}(r;k+n)^{q+1}$ from the right-hand side of (c). Then the following holds:

$$\mathcal{E}(r;k+n+m) = \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial F}{\partial u} \right\rangle_{\mathcal{E}(n+m)} + \left\langle \frac{\partial F}{\partial t} \right\rangle_{\mathcal{E}(m)} + \mathfrak{M}(n+m)^{m+1}\mathcal{E}(r;k+n+m) + \mathfrak{M}(m)\mathcal{E}(r;k+n+m).$$

Then the assumption of Lemma 2.1 holds for $B = \langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle_{\mathcal{E}(r;k+n+m)}$, $A_2 = \langle \frac{\partial F}{\partial u} \rangle_{\mathcal{E}(n+m)}$, $A_1 = \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}(m)}$ and $m = d$. Hence we have (a). □

The following two lemma's can be proved by almost parallel methods of the corresponding assertions in [11].

Lemma 3.7 (cf. [11, Corollary 3.7]) *Let $F(x,y,u,t) \in \mathfrak{M}(r;k+n+m_1)$ and $G(x,y,u,t,s) \in \mathfrak{M}(r;k+n+m_1+m_2)$ and suppose $G|_{s=0} = F$. If F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable, then G is also reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.*

Lemma 3.8 (cf. [11, Theorem 3.8]) *Let $F, G \in \mathfrak{M}(r;k+n+m)$. If F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable and if F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to G, then G is also reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.*

Lemma 3.9 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ be a reticular $\mathcal{K}$-l-determined function germ. Let $q \geq lm+l+m$. If $F(x,y,u,t) \in \mathfrak{M}(r;k+n+m)$ unfold $f(x,y,u) \in \mathfrak{M}(r;k+n)$ and f_0, and if F is a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal, then the following holds*:

$$\mathfrak{M}(r;k+n)^{q+1} \subset \left\langle f, x\frac{\partial f}{\partial x}\right\rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(r;k+n)\left\langle \frac{\partial f}{\partial y}\right\rangle + \mathfrak{M}(n)\left\langle \frac{\partial f}{\partial u}\right\rangle.$$

Proof. By Lemma 2.3, we have that $\mathfrak{M}(r;k)^{l+1} \subset \langle f_0, x\frac{\partial f_0}{\partial x}\rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k)\langle \frac{\partial f_0}{\partial y}\rangle$. It follows as the proof of Lemma 3.6 that

$$\mathfrak{M}(r;k+n)^{q+1} \subset \left\langle f, x\frac{\partial f}{\partial x}\right\rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(r;k+n)\left\langle \frac{\partial f}{\partial y}\right\rangle + \mathfrak{M}(n)^{m+1}\mathcal{E}(r;k+n). \tag{4}$$

Therefore we have that

$$\mathfrak{M}(r;k+n)^{q+1} \subset \left\langle F, x\frac{\partial F}{\partial x}\right\rangle_{\mathcal{E}(r;k+n+m)} + \mathfrak{M}(r;k+n+m)\left\langle \frac{\partial F}{\partial y}\right\rangle$$
$$+ \mathfrak{M}(n+m)^{m+1}\mathcal{E}(r;k+n+m) + \mathfrak{M}(m)\mathcal{E}(r;k+n+m).$$

This means that

$$\begin{aligned}
&\mathcal{E}(r;k+n+m) \\
&\quad \subset \mathcal{E}(r;k+n) + \mathfrak{M}(m)\mathcal{E}(r;k+n+m) \\
&\quad \subset \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)} + \left\langle \left.\frac{\partial F}{\partial t}\right|_{t=0}\right\rangle_{\mathbb{R}} \\
&\qquad + \mathfrak{M}(r;k+n)^{q+1} + \mathfrak{M}(m)\mathcal{E}(r;k+n+m) \\
&\quad \subset \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial F}{\partial u}\right\rangle_{\mathcal{E}(n+m)} + \left\langle \frac{\partial F}{\partial t}\right\rangle_{\mathcal{E}(m)} \\
&\qquad + \mathfrak{M}(n+m)^{m+1}\mathcal{E}(r;k+n+m) + \mathfrak{M}(m)\mathcal{E}(r;k+n+m).
\end{aligned}$$

We apply $B = \langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\rangle_{\mathcal{E}(r;k+n+m)}$, $A_2 = \langle \frac{\partial F}{\partial u}\rangle_{\mathcal{E}(n+m)}$, $A_1 = \langle \frac{\partial F}{\partial t}\rangle_{\mathcal{E}(m)}$ and $m = d$ for Lemma 2.1. Then we have that

$$\begin{aligned}
&\mathfrak{M}(n+m)^{m}\mathcal{E}(r;k+n+m) \\
&\quad \subset \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial F}{\partial u}\right\rangle_{\mathcal{E}(n+m)} + \mathfrak{M}(m)\mathcal{E}(r;k+n+m).
\end{aligned}$$

Restrict this equation on $t = 0$, then we have that

$$\mathfrak{M}(n)^m\mathcal{E}(r;k+n)\subset\left\langle f,x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle_{\mathcal{E}(r;k+n)}+\left\langle\frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)}.$$

From this equation and the equation (4), we have the result. □

Let q be a non-negative integer. We say that a function germ $f\in\mathfrak{M}(r;k+n)$ is *reticular $\mathcal{P}$-$\mathcal{K}$-q-determined* if all function germ which has same q-jet of f is reticular $\mathcal{P}$-$\mathcal{K}$-equivalent to f.

Lemma 3.10 *Let $f(x,y,u)\in\mathfrak{M}(r;k+n)$ and q be a non-negative integer. If*

$$\mathfrak{M}(r;k+n)^q\subset\left\langle f,x\frac{\partial f}{\partial x}\right\rangle_{\mathcal{E}(r;k+n)}+\mathfrak{M}(r;k+n)\left\langle\frac{\partial f}{\partial y}\right\rangle+\mathfrak{M}(n)\left\langle\frac{\partial f}{\partial u}\right\rangle$$
$$+\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q,\tag{5}$$

then f is reticular $\mathcal{P}$-$\mathcal{K}$-q-determined.

Proof. Let a germ $g(x,y,u)\in\mathcal{E}(r;k+n)$ with the same q-jet of f be given. We have to show that there exists a germ $\phi\in\mathcal{B}_n(r;k+n)$ and $\alpha\in\mathcal{E}(r;k+n)$ such that g has the form $g(x,y,u)=\alpha(x,y,u)f\circ\phi(x,y,u)$. By the restriction of (5) to $u=0$, we have that $f(x,y,0)\in\mathcal{E}(r;k)$ is reticular $\mathcal{K}$-q-determined by Lemma 2.3. It follows that there exist $\phi'(x,y)\in\mathcal{B}(r;k)$ and a unit $a\in\mathcal{E}(r;k)$ such that $f(x,y,0)=a(x,y)g(\phi'(x,y),0)$. Therefore we may assume that $f(x,y,0)=g(x,y,0)$. Hence we may assume that $f-g\in\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q$.

Define the one-parameter family F connect f and g by $F(x,y,u,\tau)=(1-\tau)f(x,y,u)+\tau g(x,y,u),\tau\in[0,1]$ and set $F_{\tau_0}\in\mathcal{E}(r;k+n+1)$ by $F_{\tau_0}(x,y,u,\tau)=F(x,y,u,\tau_0+\tau)$ for $\tau_0\in[0,1]$.

By using the same methods of the Mather theorem (see [10, p. 37]), we need only to show that

$$\frac{\partial F_{\tau_0}}{\partial\tau}\in\mathfrak{M}(n)\left\langle F_{\tau_0},x\frac{\partial F_{\tau_0}}{\partial x}\right\rangle_{\mathcal{E}(r;k+n+1)}$$
$$+\mathfrak{M}(n)\mathfrak{M}(r;k+n)\left\langle\frac{\partial F_{\tau_0}}{\partial y}\right\rangle_{\mathcal{E}(r;k+n+1)}+\mathfrak{M}(n)^2\left\langle\frac{\partial F_{\tau_0}}{\partial u}\right\rangle_{\mathcal{E}(n+1)}$$

Then we have that

$$
\begin{aligned}
&\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1)\\
&\quad=\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q(\mathcal{E}(r;k+n)+\mathfrak{M}(1)\mathcal{E}(r;k+n+1))\\
&\quad=\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q+\mathfrak{M}(1)\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1)\\
&\quad\subset\mathfrak{M}(n)\bigg(\Big\langle f,x\frac{\partial f}{\partial x}\Big\rangle_{\mathcal{E}(r;k+n)}+\mathfrak{M}(r;k+n)\Big\langle\frac{\partial f}{\partial y}\Big\rangle\\
&\qquad\qquad+\mathfrak{M}(n)\Big\langle\frac{\partial f}{\partial u}\Big\rangle+\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\bigg)\\
&\qquad+\mathfrak{M}(1)\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1)\\
&\quad\subset\mathfrak{M}(n)\Big\langle f,x\frac{\partial f}{\partial x}\Big\rangle_{\mathcal{E}(r;k+n+1)}+\mathfrak{M}(n)\mathfrak{M}(r;k+n)\Big\langle\frac{\partial f}{\partial y}\Big\rangle_{\mathcal{E}(r;k+n+1)}\\
&\qquad+\mathfrak{M}(n)^2\Big\langle\frac{\partial f}{\partial u}\Big\rangle_{\mathcal{E}(n+1)}+\mathfrak{M}(n+1)\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1)\\
&\quad\subset\mathfrak{M}(n)\Big\langle F_{\tau_0},x\frac{\partial F_{\tau_0}}{\partial x}\Big\rangle_{\mathcal{E}(r;k+n+1)}+\mathfrak{M}(n)\mathfrak{M}(r;k+n)\Big\langle\frac{\partial F_{\tau_0}}{\partial y}\Big\rangle_{\mathcal{E}(r;k+n+1)}\\
&\qquad+\mathfrak{M}(n)^2\Big\langle\frac{\partial F_{\tau_0}}{\partial u}\Big\rangle_{\mathcal{E}(n+1)}+\mathfrak{M}(n+1)\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1).
\end{aligned}
$$

By the assumption (5), we have the first inclusion. For the last inclusion, observe that

$$
\begin{aligned}
x_i\frac{\partial F_{\tau_0}}{\partial x_i}-x_i\frac{\partial f}{\partial x_i}&=(\tau_0+\tau)x_i\frac{\partial}{\partial x_i}(g-f)\in\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q,\\
\frac{\partial F_{\tau_0}}{\partial y_i}-\frac{\partial f}{\partial y_i}&=(\tau_0+\tau)\frac{\partial}{\partial y_i}(g-f)\in\mathfrak{M}(n)\mathfrak{M}(r;k+n)^{q-1},\\
\frac{\partial F_{\tau_0}}{\partial u_i}-\frac{\partial f}{\partial u_i}&=(\tau_0+\tau)\frac{\partial}{\partial u_i}(g-f)\in\mathfrak{M}(r;k+n)^q.
\end{aligned}
$$

Since $\mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1)$ is a finitely generated $\mathcal{E}(r;k+n+1)$-module, we have by Malgrange preparation theorem (see [11, p. 60 Theorem 1.6, Corollary 1.7]) that

$$
\begin{aligned}
&\frac{\partial F_{\tau_0}}{\partial \tau} = g - f \\
&\in \mathfrak{M}(n)\mathfrak{M}(r;k+n)^q \subset \mathfrak{M}(n)\mathfrak{M}(r;k+n)^q\mathcal{E}(r;k+n+1) \\
&\subset \mathfrak{M}(n)\left\langle F_{\tau_0}, x\frac{\partial F_{\tau_0}}{\partial x}\right\rangle_{\mathcal{E}(r;k+n+1)} \\
&\quad + \mathfrak{M}(n)\mathfrak{M}(r;k+n)\left\langle \frac{\partial F_{\tau_0}}{\partial y}\right\rangle_{\mathcal{E}(r;k+n+1)} + \mathfrak{M}(n)^2\left\langle \frac{\partial F_{\tau_0}}{\partial u}\right\rangle_{\mathcal{E}(n+1)} \qquad \square
\end{aligned}
$$

Lemma 3.11 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ be a reticular $\mathcal{K}$-l-determined function germ. Let $f(x,y,u) \in \mathfrak{M}(r;k+n)$ unfold f_0 and suppose $m = r\mathcal{P}$-$\mathcal{K}$-*cod$f$ *is a finite number. Let $q \geq lm+l+m$ and let $F(x,y,u,t), G(x,y,u,t) \in \mathfrak{M}(r;k+n+m)$ be reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f. Then F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-f-equivalent.*

Proof. By using analogous methods of the Mather theorem (see [10, the proof of p. 68 Lemma 3.16]), we need only to prove the following assertion: *Suppose that $E_\tau(x,y,u,t) = (1-\tau)F(x,y,u,t) + \tau G(x,y,u,t) \in \mathcal{E}(r;k+n+m+1)$ is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f for all $\tau \in [0,1]$ and define $E_{\tau_0} \in \mathcal{E}(r;k+n+m+1)$ by $E_{\tau_0}(x,y,t,u,\tau) = (1-\tau_0-\tau)F(x,y,u,t) + (\tau_0+\tau)G(x,y,u,t)$ for $\tau_0 \in [0,1]$. Then for all $\tau \in [0,1]$, the following holds*

$$
\begin{aligned}
\mathcal{E}(r;k+n+m+1) = &\left\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x}, \frac{\partial E_{\tau_0}}{\partial y}\right\rangle_{\mathcal{E}(r;k+n+m+1)} \\
&+ \left\langle \frac{\partial E_{\tau_0}}{\partial u}\right\rangle_{\mathcal{E}(n+m+1)} + \left\langle \frac{\partial E_{\tau_0}}{\partial t}\right\rangle_{\mathcal{E}(m+1)}.
\end{aligned}
$$

Proof of this assertion Fix $\tau_0 \in [0,1]$. Since E_{τ_0} is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal, we have

$$
\begin{aligned}
\mathcal{E}(r;k+n) = &\left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)} \\
&+ \left\langle \frac{\partial E_{\tau_0}}{\partial t}|_{t=0}\right\rangle_{\mathbb{R}} + \mathfrak{M}(r;k+n)^{q+1}.
\end{aligned}
$$

By Lemma 3.9, we have that

$$\mathfrak{M}(r;k+n)^{q+1} \subset \left\langle f, x\frac{\partial f}{\partial x} \right\rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(r;k+n)\left\langle \frac{\partial f}{\partial y} \right\rangle + \mathfrak{M}(n)\left\langle \frac{\partial f}{\partial u} \right\rangle.$$

Therefore we have that

$$\mathcal{E}(r;k+n) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u} \right\rangle_{\mathcal{E}(n)} + \left\langle \frac{\partial E_{\tau_0}}{\partial t}|_{t=0} \right\rangle_{\mathbb{R}}.$$

Since $E_{\tau_0}(x,y,u,t) - f(x,y,u) \in \mathfrak{M}(m)\mathcal{E}(r;k+n+m)$, we have that

$$\begin{aligned}
&\mathcal{E}(r;k+n+m) \\
&\quad = \mathcal{E}(r;k+n) + \mathfrak{M}(m)\mathcal{E}(r;k+n+m) \\
&\quad = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u} \right\rangle_{\mathcal{E}(n)} + \left\langle \left.\frac{\partial E_{\tau_0}}{\partial t}\right|_{t=0} \right\rangle_{\mathbb{R}} \\
&\qquad + \mathfrak{M}(m)\mathcal{E}(r;k+n+m) \\
&\quad = \left\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x}, \frac{\partial E_{\tau_0}}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial E_{\tau_0}}{\partial u} \right\rangle_{\mathcal{E}(n+m)} + \left\langle \frac{\partial E_{\tau_0}}{\partial t} \right\rangle_{\mathcal{E}(m)} \\
&\qquad + \mathfrak{M}(m)\mathcal{E}(r;k+n+m).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
&\mathcal{E}(r;k+n+m+1) \\
&\quad = \mathcal{E}(r;k+n+m) + \mathfrak{M}(1)\mathcal{E}(r;k+n+m+1) \\
&\quad = \left\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x}, \frac{\partial E_{\tau_0}}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial E_{\tau_0}}{\partial u} \right\rangle_{\mathcal{E}(n+m)} + \left\langle \frac{\partial E_{\tau_0}}{\partial t} \right\rangle_{\mathcal{E}(m)} \\
&\qquad + \mathfrak{M}(m)\mathcal{E}(r;k+n+m) + \mathfrak{M}(1)\mathcal{E}(r;k+n+m+1) \\
&\quad = \left\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x}, \frac{\partial E_{\tau_0}}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m+1)} + \left\langle \frac{\partial E_{\tau_0}}{\partial u} \right\rangle_{\mathcal{E}(n+m+1)} \\
&\qquad + \left\langle \frac{\partial E_{\tau_0}}{\partial t} \right\rangle_{\mathcal{E}(m+1)} + \mathfrak{M}(m+1)\mathcal{E}(r;k+n+m+1).
\end{aligned}$$

By Malgrange preparation theorem, we have

$$\mathcal{E}(r;k+n+m+1)=\left\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x}, \frac{\partial E_{\tau_0}}{\partial y}\right\rangle_{\mathcal{E}(r;k+n+m+1)} + \left\langle \frac{\partial E_{\tau_0}}{\partial u}\right\rangle_{\mathcal{E}(n+m+1)} + \left\langle \frac{\partial E_{\tau_0}}{\partial t}\right\rangle_{\mathcal{E}(m+1)}. \qquad \square$$

Theorem 3.12 *Let $F(x,y,u,t)\in\mathfrak{M}(r;k+n+m)$ unfold $f(x,y,u)\in\mathfrak{M}(r;k+n)$ and $f_0(x,y)\in\mathfrak{M}(r;k)$. Suppose that f_0 is reticular $\mathcal{K}$-l-determined and $q\geq lm+l+m+1$. Then the following are equivalent.*
(1) *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal.*
(2) *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-stable.*
(3) *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-versal.*

Proof. Let $z=j^q f(0)$. (1)⇒(2). Let F be a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f. Let $\tilde{F}\in C^\infty(U,\mathbb{R})$ be a representative of F. Set $V=U\cap(\{0\}\times\mathbb{R}^{k+n+m})$. Define

$$N_{\tilde{F}}=\big\{\tilde{G}\in C^\infty(U,\mathbb{R}) \mid j^q_1\tilde{G}|_{x=0} \text{ is transversal to } O^q_{r\mathcal{P}\text{-}\mathcal{K}}(z) \text{ and } j^q_1\tilde{G}|_{x=0}(V)\cap O^q_{r\mathcal{P}\text{-}\mathcal{K}}(z)\neq\emptyset\big\}.$$

This is an open neighborhood of $\tilde{F}$ because the maps $\tilde{G}\mapsto j^q\tilde{G}\mapsto j^q_1\tilde{G}\mapsto j^q_1\tilde{G}|_{x=0}$ are given by compositions of continuous maps. Let $\tilde{G}\in N_{\tilde{F}}$ and take $(0,y_0,u_0,t_0)\in V$ such that $j^q_1\tilde{G}$ is transversal to $O^q_{r\mathcal{P}\text{-}\mathcal{K}}(z)$ at $(0,y_0,u_0,t_0)$. Let G be the germ of $\tilde{G}|_{\mathbb{H}^r\times\mathbb{R}^{k+n+m}}$ at $(0,y_0,u_0,t_0)$ and define $g\in\mathcal{E}(r;k+n)$ by $g(x,y,u)=G(x,y+y_0,u+u_0,t_0)$. Since $j^q g(0,0,0)=j^q_1\tilde{G}(0,y_0,u_0,t_0)\in O^q_{r\mathcal{P}\text{-}\mathcal{K}}(z)$, there exists $\phi\in\mathcal{B}_n(r;k+n)$ and a unit $\alpha\in\mathcal{E}(r;k+n)$ such that the germ $f'\in\mathcal{E}(r;k+n)$ defined by $f'(x,y,u)=\alpha(x,y,u)g\circ\phi(x,y,u)$ has the same q-jet of f. Since F is also reticular t-$\mathcal{P}$-$\mathcal{K}$-$(q-1)$-transversal and $q-1\geq lm+l+m$, we have by Lemma 3.9 that

$$\mathfrak{M}(r;k+n)^q\subset\left\langle f, x\frac{\partial f}{\partial x}\right\rangle_{\mathcal{E}(r;k+n)}+\mathfrak{M}(r;k+n)\left\langle\frac{\partial f}{\partial y}\right\rangle+\mathfrak{M}(n)\left\langle\frac{\partial f}{\partial u}\right\rangle.$$

This means by Lemma 3.10 that f is reticular $\mathcal{P}$-$\mathcal{K}$-q-determined. It follows that f' is reticular $\mathcal{P}$-$\mathcal{K}$-equivalent to f. So g is also reticular $\mathcal{P}$-$\mathcal{K}$-equivalent to f. Hence there exist $\phi'\in\mathcal{B}_n(r;k+n)$ and $\alpha'\in\mathcal{E}(r;k+n)$ such that g has the form $f(x,y,u)=\alpha'(x,y,u)g\circ\phi'(x,y,u)$ Define $G'\in\mathcal{E}(r;k+n+m)$

by $G'(x,y,u,t) = \alpha'(x,y,u)G(\phi'(x,y,u),t)$. Then G' is a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f. By Lemma 3.11 we have that F and G' are reticular t-$\mathcal{P}$-$\mathcal{K}$-f-equivalent. Therefore F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent.
(2)⇒(3). Let F be a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of f and let $\tilde{F} \in C^\infty(U,\mathbb{R})$ be a representative of F. By hypothesis and Theorem 3.2, there exist $\tilde{F}' \in C^\infty(U,\mathbb{R})$ and $(0,y_0,u_0,t_0) \in U$ such that $j_1^q \tilde{F}'|_{x=0}$ is transversal to $O^q_{r\mathcal{P}\text{-}\mathcal{K}}(z)$ and the germ $F' = \tilde{F}'|_{\mathbb{H}^r\times\mathbb{R}^{k+n+m}}$ at $(0,y_0,u_0,t_0)$ is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to F. By Proposition 3.5, we have that F is a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f.

Let an unfolding $G(x,y,u,s) \in \mathcal{E}(r;k+n+m_1)$ of f be given. Define $G'(x,y,u,t,s) \in \mathcal{E}(r;k+n+m+m_1)$ by $G'(x,y,u,t,s) = G(x,y,u,s) - f(x,y,u)+F(x,y,u,t)$. Then G' is a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f because F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal. Define $F''(x,y,u,t,s) \in \mathcal{E}(r;k+n+m+m_1)$ by $F''(x,y,u,t,s) = F(x,y,u,t)$. Then F'' is also a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding of f. By Lemma 3.11, we have that G' and F'' are reticular t-$\mathcal{P}$-$\mathcal{K}$-f-equivalent. Since G is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from G', and F'' is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from F, it follows that G is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from F. Therefore F is reticular t-$\mathcal{P}$-$\mathcal{K}$-versal.
(3)⇒(1). Let $F(x,y,u,t) \in \mathcal{E}(r;k+n+m_1)$ be a reticular t-$\mathcal{P}$-$\mathcal{K}$-versal unfolding of f. Take a reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal unfolding $G(x,y,u,s) \in \mathcal{E}(r;k+n+m_2)$ of f. By hypothesis, there exists a reticular t-$\mathcal{P}$-$\mathcal{K}$-f-morphism from G to F of the form:

$$G(x,y,u,s) = \alpha(x,y,u,s)F\big(x\phi_1(x,y,u,s),\phi_2(x,y,u,s),\phi_3(u,s),\phi_4(s)\big).$$

Since G is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal, we have

$$\mathcal{E}(r;k+n) = \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)} + \left\langle \frac{\partial G}{\partial s}|_{s=0}\right\rangle_{\mathbb{R}} + \mathfrak{M}(r;k+n)^{q+1}.$$

On the other hand, we have that

$$\left\langle \frac{\partial G}{\partial s}\bigg|_{s=0}\right\rangle_{\mathbb{R}} \subset \left\langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle_{\mathcal{E}(r;k+n)} + \left\langle \frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)} + \left\langle \frac{\partial F}{\partial t}\bigg|_{t=0}\right\rangle_{\mathbb{R}}$$

Therefore

$$\mathcal{E}(r;k+n)=\left\langle f,x\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\right\rangle_{\mathcal{E}(r;k+n)}+\left\langle\frac{\partial f}{\partial u}\right\rangle_{\mathcal{E}(n)}+\left\langle\left.\frac{\partial F}{\partial t}\right|_{t=0}\right\rangle_{\mathbb{R}}$$
$$+\mathfrak{M}(r;k+n)^{q+1}.$$

Hence F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal. □

Theorem 3.13 (Uniqueness of universal unfoldings) *Let $F(x,y,u,t)$, $G(x,y,u,t)\in\mathfrak{M}(r;k+n+m)$ be unfoldings of $f\in\mathfrak{M}(r;k+n)$. If F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-versal, then F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-f-equivalent.*

Proof. Since F is a reticular $\mathcal{P}$-$\mathcal{K}$-versal unfolding of $f_0=f|_{u=0}$ as $(n+m)$-dimensional unfolding. This means that f_0 is finitely determined. Choose an non-negative integer l such that (3) holds for f_0. Let $q\geq lm+l+m+1$. By Theorem 3.12, we have that F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal. By Lemma 3.11 we have that F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-f-equivalent. □

Theorem 3.14 *Let $F(x,y,u,t)\in\mathfrak{M}(r;k+n+m)$ be an unfolding of $f(x,y,u)\in\mathfrak{M}(r;k+n)$ and let f be an unfolding of $f_0(x,y)\in\mathfrak{M}(r;k)$. Then following are equivalent.*

1. *There exists a non-negative number l such that f_0 is reticular $\mathcal{K}$-l-determined and F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal for $q\geq lm+l+m+1$.*
2. *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-stable.*
3. *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-versal.*
4. *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally versal.*
5. *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.*
6. *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable.*

Proof. (2)⇒(5) F is also reticular $\mathcal{P}$-$\mathcal{K}$-stable unfolding of f_0 as $(n+m)$-dimensional unfolding. Therefore f_0 is reticular $\mathcal{K}$-finitely determined. Choose an non-negative integer l such that (3) holds for f_0. Let $q\geq lm+l+m+1$. By Theorem 3.12, we have that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal. Then the assertion (c) of Theorem 3.6 holds. Therefore F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.
(4)⇔(5) This is proved by Theorem 3.6.
(5)⇒(2) F is also reticular $\mathcal{P}$-$\mathcal{K}$-infinitesimally stable unfolding of f_0 as

$(n+m)$-dimensional unfolding. Therefore there exists a non-negative number l such that f_0 is reticular $\mathcal{K}$-l determined. By Theorem 3.12, we have that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-q-transversal for $q \geq lm + l + m + 1$. This means that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-stable by Theorem 3.12.
(1)⇔(2)⇔(3) This is proved in Theorem 3.12.
(5)⇒(6)

$$\begin{aligned}
&\mathcal{E}(r; k+n+m+1) \\
&\quad = \mathcal{E}(r; k+n+m) + \mathfrak{M}(1)\mathcal{E}(r; k+n+m+1) \\
&\quad = \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m)} + \left\langle \frac{\partial F}{\partial u} \right\rangle_{\mathcal{E}(n+m)} + \left\langle \frac{\partial F}{\partial t} \right\rangle_{\mathcal{E}(m)} \\
&\qquad + \mathfrak{M}(1)\mathcal{E}(r; k+n+m+1) \\
&\quad = \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m+1)} + \left\langle \frac{\partial F}{\partial u} \right\rangle_{\mathcal{E}(n+m+1)} + \left\langle \frac{\partial F}{\partial t} \right\rangle_{\mathcal{E}(m+1)} \\
&\qquad + \mathfrak{M}(m+1)\mathcal{E}(r; k+n+m+1).
\end{aligned}$$

By Malgrange preparation theorem, we have that

$$\begin{aligned}
\mathcal{E}(r; k+n+m+1) = &\left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(r;k+n+m+1)} \\
&+ \left\langle \frac{\partial F}{\partial u} \right\rangle_{\mathcal{E}(n+m+1)} + \left\langle \frac{\partial F}{\partial t} \right\rangle_{\mathcal{E}(m+1)}. \qquad (6)
\end{aligned}$$

This means that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable.
(6)⇒(5) Suppose that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable. Then (6) holds. Restrict this equation to $\mathbb{H}^r \times \mathbb{R}^{k+n+m}$. Then we have the equation (2). □

For $f \in \mathfrak{M}(r; k+n)$ if $a_1, \ldots, a_m \in \mathcal{E}(r; k+n)$ is a representative of a basis of $\mathcal{E}(r; k+n)/T_e(r\mathcal{P}\text{-}\mathcal{K})(f)$, then the function germ $f + a_1t_1 + \cdots + a_mt_m \in \mathfrak{M}(r; k+n+m)$ is a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of f.

4. A generic classification of unfoldings under the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence

Definition 4.1 We say that function germs $f_1(x,y) \in \mathfrak{M}(r_1;k_1)$ and $f_2(x,y) \in \mathfrak{M}(r_2;k_2)$ are *stably reticular $\mathcal{K}$-equivalent* if f_1 and f_2 are reticular $\mathcal{K}$-equivalent after additions of linear forms in x whose all coefficients are not zero and non-degenerate quadratic forms in the variables y. We also define *the stably reticular $\mathcal{P}$-$\mathcal{K}$-equivalence relation* and *the stably reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence relation* analogously.

Proposition 4.2 *Let $f_0 \in \mathfrak{M}(1;k)$. Then f_0 is stably reticular $\mathcal{K}$-equivalent to $y \in \mathfrak{M}(0;1)$ or there exists $f_0' \in \mathfrak{M}(r;k')^2$ ($r = 0$ or 1) such that f_0 and f_0' are stably reticular $\mathcal{K}$-equivalent.*

Proposition 4.3 (cf., [7, p. 126]) *Let $f_0(y) \in \mathfrak{M}(0;k)$ with $(r)\mathcal{K}$-cod$f_0 \leq 6$ be given. Then f_0 is stably (reticular) $\mathcal{K}$-equivalent to one of*

$$A_l : y^{l+1}(0 \leq l \leq 6),\ D_4^{\pm} : y_1^2 y_2 \pm y_2^3,\ D_5 : y_1^2 y_2 + y_2^4,$$
$$D_6^{\pm} : y_1^2 y_2 \pm y_2^5,\ E_6 : y_1^3 + y_2^4.$$

Let $f_0(x,y) \in \mathfrak{M}(1;k)$ with $r\mathcal{K}$-cod$f_0 \leq 4$ be given. Then f_0 is stably reticular $\mathcal{K}$-equivalent to one of

$$A_l : y^{l+1}(0 \leq l \leq 4),\ D_4^{\pm} : y_1^2 y_2 \pm y_2^3,\ B_l : x^l (1 \leq l \leq 4),$$
$$C_3^{\pm} : \pm xy + y^3,\ C_4 : xy + y^4,\ F_4 : x^2 + y^3.$$

Proposition 4.4 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ be a simple singularity, that is A_l, D_l, E_6, E_7, E_8 for $r = 0$, or B_l, C_l, F_4 for $r = 1$. Let Q_{f_0} be the local ring of f_0, that is $Q_{f_0} = \mathcal{E}(r;k)/\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle_{\mathcal{E}(r;k)}$. Then there exist monomials $\varphi_0, \varphi_1, \ldots, \varphi_n \in \mathfrak{M}(r;k)$ which consist a basis of Q_{f_0} such that*

1. $\mathfrak{M}(r;k) \cdot \varphi_0 \sim 0 \bmod Q_{f_0}$
2. *For any $i, j \in \{1, \ldots, n\} (i+j \geq n)$ there exists a non-zero real number a such that $\varphi_i \cdot \varphi_j \sim a\varphi_{i+j-n} \bmod Q_{f_0}$.*
3. *For any $i, j \in \{1, \ldots, n\} (i+j < n)$, $\varphi_i \cdot \varphi_j \sim 0 \bmod Q_{f_0}$,*

For example, if $f_0(x,y) = xy + y^4 (C_4)$ then we may choose that $\varphi_0 = y^3$, $\varphi_1 = y^2$, $\varphi_2 = y$, $\varphi_3 = 1$.

Proposition 4.5 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ be a simple singularity, that is A_l, D_l, E_6, E_7, E_8 for $r = 0$, or B_l, C_l, F_4 for $r = 1$. Choose monomials $\varphi_0(x,y), \ldots, \varphi_n(x,y)$ as the previous proposition. Then the function $F(x,y,u,t) = f_0(x,y) + \varphi_0(x,y)t + \sum_{i=1}^n \varphi_i(x,y)u_i$ is a reticular t-$\mathcal{P}$-$\mathcal{K}$-universal unfolding of $F|_{t=0}$.*

Proof. In this proof we write $\mathcal{E}(x,y,u,t)$ for $\mathcal{E}(r;k+n+1)$ and write $\mathcal{E}(u)$ for $\mathcal{E}(n)$ and write other notations analogously. Since $F - f_0 \in \mathfrak{M}(u,t)\mathcal{E}(x,y,u,t)$, we have that

$$x_i\frac{\partial F}{\partial x_i} - x_i\frac{\partial f_0}{\partial x_i}, \quad \frac{\partial F}{\partial y_j} - \frac{\partial f_0}{\partial y_j} \in \mathfrak{M}(u,t)\mathcal{E}(x,y,u,t).$$

It follows that

$$\left\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\right\rangle_{\mathfrak{M}(u,t)} \subset \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle_{\mathcal{E}(x,y,u,t)} + \mathfrak{M}(u,t)^2\mathcal{E}(x,y,u,t). \tag{7}$$

Therefore we have that

$$\begin{aligned}\left\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\right\rangle_{\mathfrak{M}(u,t)\mathcal{E}(x,y,u,t)} \\ \subset \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle_{\mathcal{E}(x,y,u,t)} + \mathfrak{M}(u,t)^2\mathcal{E}(x,y,u,t).\end{aligned} \tag{8}$$

Let a function germ $G(x,y,u,t) \in \mathcal{E}(x,y,u,t)$ be given. It is enough to prove that

$$\begin{aligned}G \in \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle_{\mathcal{E}(x,y,u,t)} + \langle\varphi_1,\ldots,\varphi_n\rangle_{\mathcal{E}(u,t)} + \langle\varphi_0\rangle_{\mathcal{E}(t)} \\ + \mathfrak{M}(u,t)^2\mathcal{E}(x,y,u,t),\end{aligned}$$

because this means by Lemma 2.1 that

$$\mathcal{E}(x,y,u,t) = \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle_{\mathcal{E}(x,y,u,t)} + \langle\varphi_1,\ldots,\varphi_n\rangle_{\mathcal{E}(u,t)} + \langle\varphi_0\rangle_{\mathcal{E}(t)}.$$

Since F is a reticular $\mathcal{P}$-$\mathcal{K}$-infinitesimal stable unfolding of f_0 as $(n+1)$-dimensional unfolding, we have that

$$\mathcal{E}(x,y,u,t) = \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(x,y,u,t)} + \langle \varphi_0, \varphi_1, \ldots, \varphi_n \rangle_{\mathcal{E}(u,t)}.$$

It follows that there exist function germs $g_0(u,t), \ldots, g_n(u,t) \in \mathcal{E}(u,t)$ such that

$$G \sim g_0(u,t)\varphi_0(x,y) + \cdots + g_n(u,t)\varphi_n(x,y) \bmod \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(x,y,u,t)}.$$

Then g_0 has the form $g_0(u,t) = g_0(0,t) + \sum_{i=1}^{n} a_i u_i + h(u,t)$, where $a_i \in \mathbb{R}$ and $h \in \mathfrak{M}(u,t)^2$. Since F is *quasi-homogeneous function germ* (see [1, p. 192] for the definition), and f_0 is simple singularity, there exist non-zero real numbers b_x, b_y, b_t, b_{u_i} such that F has the form:

$$F = b_x x\frac{\partial F}{\partial x} + b_y y\frac{\partial F}{\partial y} + b_t t\varphi_0 + b_{u_1} u_1\varphi_1 + \cdots + b_{u_n} u_n\varphi_n.$$

Then there exist non-zero real numbers b_i' such that

$$\varphi_{n-1}F \sim b_x\varphi_{n-1}x\frac{\partial F}{\partial x} + b_y\varphi_{n-1}y\frac{\partial F}{\partial y} + b_t t\varphi_0\varphi_{n-1} + b_1' u_1\varphi_0 + \cdots + b_n' u_n\varphi_{n-1}$$

mod $\langle f_0, x\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}\rangle_{\mathfrak{M}(u,t)\mathcal{E}(x,y,u,t)}$. Therefore we have by (8) that

$$0 \sim \varphi_{n-1}F \sim b_t t\varphi_0\varphi_{n-1} + b_1' u_1\varphi_0 + \cdots + b_n' u_n\varphi_{n-1}$$

mod the right hand side of (8). Since $\mathfrak{M}(x,y)\varphi_0 \sim 0 \bmod Q_{f_0}$, we have that

$$0 \sim \varphi_{n-1}F \sim b_1' u_1\varphi_0 + \cdots + b_n' u_{n-1}\varphi_{n-1}$$

mod the right hand side of (8). This means that

$$u_1\varphi_0 \in \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(x,y,u,t)} + \langle \varphi_1, \ldots, \varphi_n \rangle_{\mathfrak{M}(u,t)} + \mathfrak{M}(u,t)^2\mathcal{E}(x,y,u,t). \tag{9}$$

By considering $\varphi_{n-2}F, \dots, \varphi_0 F$ instead of $\varphi_{n-1}F$, we have that $u_2\varphi_0, \dots, u_n\varphi_0$, are included in the right hand side of (9). This means that $g_0(u,t)\varphi_0 \sim g_0(0,t)\varphi_0$ mod the right hand side of (9). Therefore we have that

$$G \in \left\langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle_{\mathcal{E}(x,y,u,t)} + \langle \varphi_1, \dots, \varphi_n \rangle_{\mathcal{E}(u,t)} + \langle \varphi_0 \rangle_{\mathcal{E}(t)} + \mathfrak{M}(u,t)^2 \mathcal{E}(x,y,u,t). \qquad \square$$

Lemma 4.6 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ be a simple singularity and $F(x,y,u,t) \in \mathfrak{M}(r;k+n+1)$ be a reticular $\mathcal{P}$-$\mathcal{K}$-universal unfoldings of f_0. If F is a reticular t-$\mathcal{P}$-$\mathcal{K}$-universal unfoldings of $f = F|_{t=0}$ and $r\mathcal{K}$-cod$f = 1$, then F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to the function germ of the form in Proposition* 4.5.

Proof. We may assume that f_0 has the normal from. Then F is reticular $\mathcal{P}$-$\mathcal{K}$-equivalent to $F_0 = f_0(x,y) + t\varphi_0(x,y) + u_1\varphi_1(x,y) \cdots + u_n\varphi_n(x,y)$. Therefore there exists a reticular $\mathcal{P}$-$\mathcal{K}$-isomorphism (α, Φ) from F_0 to F. We write $\Phi = (x\phi_1, \phi_2, \phi_3, \phi_4)$. We set $f^0 \in \mathfrak{M}(r;k)$ by $f^0 = F_0|_{t=0}$, that is $f^0 = f_0(x,y) + u_1\varphi_1(x,y) \cdots + u_n\varphi_n(x,y)$. Since $r\mathcal{K}$-cod$f = 1$, it follows that the map germ $u \mapsto \phi_3(u,0)$ is invertible. Therefore we may reduce F to the form: $F(x,y,u,t) = f_0(x,y) + a(u,t)\varphi_0(x,y) + u_1\varphi_1(x,y) \cdots + u_n\varphi_n(x,y)$ for some $a \in \mathfrak{M}(n+1)$ with $\frac{\partial a}{\partial t}(0) \neq 0$. By an analogous method of Proposition 4.5, we have that

$$\mathfrak{M}(u)\varphi_0 \in \left\langle f^0, x\frac{\partial f^0}{\partial x} \right\rangle_{\mathcal{E}(x,y,u)} + \mathfrak{M}(x,y,u)\left\langle \frac{\partial f^0}{\partial y} \right\rangle + \mathfrak{M}(u)\langle \varphi_1, \dots, \varphi_n \rangle.$$

We fix $\tau_0 \in [0,1]$ and define $E_{\tau_0}(x,y,u,\tau) \in \mathfrak{M}(r;k+n+1)$ by $E_{\tau_0}(x,y,u,\tau) = f_0(x,y) + (\tau_0+\tau)a(u,0)\varphi_0(x,y) + u_1\varphi_1(x,y) \cdots + u_n\varphi_n(x,y)$. Since $E_{\tau_0} - f^0 = (\tau_0+\tau)a(u,0)\varphi_0$, it follows that

$$\frac{\partial E_{\tau_0}}{\partial \tau} \in \left\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x} \right\rangle_{\mathcal{E}(x,y,u,\tau)} + \mathfrak{M}(x,y,u,\tau)\left\langle \frac{\partial E_{\tau_0}}{\partial y} \right\rangle + \mathfrak{M}(u,\tau)\left\langle \frac{\partial E_{\tau_0}}{\partial u} \right\rangle.$$

By an analogous method of [10, p. 26 Lemma 1.27], we have that $F|_{t=0}$ and f^0 are reticular $\mathcal{P}$-$\mathcal{K}$-equivalent. By Theorem 3.13, it follows that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to F_0. $\square$

Now we classify reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfoldings in $\mathfrak{M}(r; k+n+1)$ with respect to stably reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence for the case $r = 0, n \leq 5$ and $r = 1, n \leq 3$. We prove only the case $r = 1, n \leq 3$.

Let a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding $F(x, y, u, t) \in \mathfrak{M}(1; k+n+1)$ with $n \leq 3$ be given. We set $f = F|_{t=0}$ and $f_0 = f|_{u=0}$. Since F is a reticular $\mathcal{P}$-$\mathcal{K}$-stable unfolding of f_0 as $(n+1)$-dimensional unfolding, it follows that f_0 is stably reticular $\mathcal{K}$-equivalent to one of the types in Proposition 4.3. So we may assume that f_0 has the normal form in $\mathfrak{M}(1; 1)$. We denote X the type of f_0. Then the local ring Q_{f_0} has basis $\varphi_0, \ldots, \varphi_{l-1}$ $(l \leq n+1)$ and φ_0 has the maximal degree. The function germ $F_0(x, y, u, t) = f_0 + t\varphi_0 + u_1\varphi_1 + \cdots u_{l-1}\varphi_{l-1} \in \mathfrak{M}(1; 1 + (l-1) + 1)$ is a reticular t-$\mathcal{P}$-$\mathcal{K}$-universal unfolding of f_0 by Proposition 4.5. Since F is a reticular $\mathcal{P}$-$\mathcal{K}$-stable unfolding of f_0, there exists a diffeomorphism germ ϕ on $(\mathbb{R}^{n+1}, 0)$ such that $F_1 \in \mathfrak{M}(r; k + (l-1) + 1)$ given by $F_1(x, y, u, t) = F(x, y, \phi(u_1, \ldots, u_{l-1}, t, 0, \ldots, 0))$ is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to F_0. So we may reduce F_1 to F_0. Therefore F has the form

$$F(x, y, u, t) = f_0(x, y) + a_0(u, t)\varphi_0(x, y) + \cdots + a_{l-1}(u, t)\varphi_{l-1}(x, y),$$

where the map germ $(u_1, \ldots, u_n, t) \mapsto (a_0(u, t), \ldots, a_{l-1}(u, t))$ is a submersion.

In the case that the map germ $(u_1, \ldots, u_n) \mapsto (a_0(u, 0), \ldots, a_{l-1}(u, 0))$ is also a submersion, then F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to ${}^0X$.

In the case that the map germ $(u_1, \ldots, u_n) \mapsto (a_0(u, 0), \ldots, a_{l-1}(u, 0))$ is not a submersion. Then $r\mathcal{K}$-cod$F|_{t=0} = 1$. It follows that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to F_0 by Lemma 4.6. Therefore F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to the function germ:

$$f_0 + (t + a_0)\varphi_0 + (u_1 + a_1)\varphi_1 + \cdots (u_{l-1} + a_{l-1})\varphi_{l-1},$$

where $a_i \in \mathfrak{M}(u_l, \ldots, u_n)\mathcal{E}(u)$ for $i = 1, \ldots, l-1$. Hence F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to the function germ:

$$f_0 + (t + a_0)\varphi_0 + u_1\varphi_1 + \cdots u_{l-1}\varphi_{l-1}.$$

Let $l - 1 = n$. Since $a_0 = 0$, it follows that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to ${}^1X$.

Let $l-1<n$. Then $\frac{\partial a_0}{\partial u_i}(0)=0$ for all $i=l,\dots,n$. If $\left(\frac{\partial^2 a_0}{\partial u_i \partial u_j}(0)\right)_{i,j=l,\dots,n}$ is degenerate then $r\mathcal{K}\text{-cod}F|_{t=0}>1$. It follows that F is not reticular t-$\mathcal{P}$-$\mathcal{K}$-stable. Therefore $\left(\frac{\partial^2 a_0}{\partial u_i \partial u_j}(0)\right)_{i,j=l,\dots,n}$ is non-degenerate. Since $a_0|_{u_1=\cdots=u_{l-1}=0}$ is a Morse function on $u_l,\dots,u_n$, We have that F is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to ${}^1X$.

Theorem 4.7 *Let $r=0$, $n\leq 5$ or $r=1$, $n\leq 3$ and U be a neighborhood of 0 in $\mathbb{H}^r\times\mathbb{R}^{k+n+1}$. Then there exists a residual set $O\subset C^\infty(U,\mathbb{R})$ such that the following condition holds: For any $\tilde{F}\in O$ and $(0,y_0,u_0,t_0)\in U$, the function germ $F(x,y,u,t)\in\mathfrak{M}(r;k+n+1)$ given by $F(x,y,u,t)=\tilde{F}(x,y+y_0,u+u_0,t+t_0)-\tilde{F}(0,y_0,u_0,t_0)$ is a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of $F|_{t=0}$.*

In the case $r=0,n\leq 5$, F is stably reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to one of the following type:

$({}^0A_l)$ $y_1^{l+1}+\sum_{i=1}^{l-1}u_iy_1^i+u_l$ $(0\leq l\leq 5)$,
$({}^0D_4^\pm)$ $y_1^2y_2\pm y_2^3+u_1y_2^2+u_2y_2+u_3y_1+u_4$,
$({}^0D_5)$ $y_1^2y_2+y_2^4+u_1y_2^3+u_2y_2^2+u_3y_2+u_4y_1+u_5$,
$({}^1A_l)$ $y_1^{l+1}+(t\pm u_{l-1}^2\pm\cdots\pm u_n^2)y_1^{l-1}+\sum_{i=1}^{l-2}u_iy_1^i+u_l$ $(2\leq l\leq 6)$,
$({}^1D_4^\pm)$ $y_1^2y_2\pm y_2^3+ty_2^2+u_1y_2+u_2y_1+u_3$, $y_1^2y_2\pm y_2^3+(t\pm u_4^2)y_2^2+u_1y_2+u_2y_1+u_3$,
$({}^1D_5)$ $y_1^2y_2+y_2^4+ty_2^3+u_1y_2^2+u_2y_2+u_3y_1+u_4$, $y_1^2y_2+y_2^4+(t\pm u_5^2)y_2^3+u_1y_2^2+u_2y_2+u_3y_1+u_4$,
$({}^1D_6^\pm)$ $y_1^2y_2\pm y_2^5+ty_2^6+u_1y_2^3+u_2y_2^2+u_3y_2+u_4y_1+u_5$,
$({}^1E_6)$ $y_1^3+y_2^4+ty_1y_2^2+u_1y_1y_2+u_2y_2^2+u_3y_1+u_4y_2+u_5$.

In the case $r=1,n\leq 3$, F is stably reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to one of the following type:

$({}^0A_l)$ $(0\leq l\leq 3)$,
$({}^0B_1)$ $x+u$,
$({}^0B_2)$ $x^2+u_1x+u_2$,
$({}^0B_3)$ $x^3+u_1x^2+u_2x+u_3$,
$({}^0C_3^\pm)$ $\pm xy+y^3+u_1y^2+u_2y+u_3$,
$({}^1A_l)$ $(2\leq l\leq 4)$, $({}^1D_4^\pm)$,
$({}^1B_1)$ $x+t$,
$({}^1B_2)$ x^2+tx+u_1, $x^2+(t\pm u_2^2)x+u_1$, $x^2+(t\pm u_2^2\pm u_3^2)x+u_1$,
$({}^1B_3)$ $x^3+tx^2+u_1x+u_2$, $x^3+(t\pm u_3^2)x^2+u_1x+u_2$,
$({}^1B_4)$ $x^4+tx^3+u_1x^2+u_2x+u_3$,

$({}^1C_3^{\pm})$ $\pm xy+y^3+ty^2+u_1y+u_2,\ \pm xy+y^3+(t\pm u_3^2)y^2+u_1y+u_2,$
$({}^1C_4)$ $xy+y^4+ty^3+u_1y^2+u_2y+u_3,$
$({}^1F_4)$ $x^2+y^3+txy+u_1x+u_2y+u_3.$

We remark that a class ${}^1X$ is not one equivalent class, since non-degenerate quadratic forms $+u^2$ and $-u^2$ may define different classes.

Proof. We prove only the case $r=1, n\leq 3$. All function germ in $\mathfrak{M}(1;k)$ with the reticular $\mathcal{K}$-codimension ≤ 3 are stably reticular $\mathcal{K}$-equivalent to one of the types in Proposition 4.3. We define the stably reticular $\mathcal{P}$-$\mathcal{K}$-equivalence classes by

$({}^0A_l)$ $y_1^{l+1}+\sum_{i=1}^{l-1}u_iy_1^i+u_l$ $(0\leq l\leq 3),$
$({}^0B_1)$ $x+u,$
$({}^0B_2)$ $x^2+u_1x+u_2,$
$({}^0B_3)$ $x^3+u_1x^2+u_2x+u_3,$
$({}^0C_3^{\pm})$ $\pm xy+y^3+u_1y^2+u_2y+u_3,$
$({}^1A_l)$ $y_1^{l+1}+(\pm u_{l-1}^2\pm\cdots\pm u_n^2)y_1^{l-1}+\sum_{i=1}^{l-2}u_iy_1^i+u_l$ $(2\leq l\leq 4),$
$({}^1D_4^{\pm})$ $y_1^2y_2\pm y_2^3+u_1y_2+u_2y_1+u_3,\ y_1^2y_2\pm y_2^3\pm u_4^2y_2^2+u_1y_2+u_2y_1+u_3,$
$({}^0B_1)$ $x,$
$({}^1B_2)$ $x^2+u_1,\ x^2\pm u_2^2x+u_1,\ x^2+(\pm u_2^2\pm u_3^2)x+u_1,$
$({}^1B_3)$ $x^3+u_1x+u_2,\ x^3\pm u_3^2x^2+u_1x+u_2,$
$({}^1B_4)$ $x^4+u_1x^2+u_2x+z,$
$({}^1C_3^{\pm})$ $\pm xy+y^3+u_1y+u_2,\ \pm xy+y^3\pm u_3^2y^2+u_1y+u_2,$
$({}^1C_4)$ $xy+y^4+u_1y^2+u_2y+u_3,$
$({}^1F_4)$ $x^2+y^3+u_1x+u_2y+u_3.$

We define that

$$O'=\left\{F\in C^\infty(U,\mathbb{R})\mid j_1^lF|_{x=0}\text{ is transversal to }[X]\text{ for all above }X\right\}$$

Then O' is a residual set in $C^\infty(U,\mathbb{R})$.

We set

$$Y=\left\{j^lf(0)\in J^l(r+k+n)\mid r\mathcal{P}\text{-}\mathcal{K}\mathrm{cod}f>1.\right\}$$

Then Y is an algebraic set in $J^l(r+k+n)$. We also set

$$O''=\left\{F\in C^\infty(U,\mathbb{R})\mid j_1^lF|_{x=0}\text{ is transversal to }Y\right\}.$$

Then Y has codimension $> k+n+1$ because all function germ $f \in \mathfrak{M}(1; k+n)$ with $j^l f(0) \in Y$ is adjacent to one of the above list which are simple. Then we have that

$$O'' = \{F \in C^\infty(U, \mathbb{R}) \mid j_1^l F(U \cap \{x = 0\}) \cap Y = \emptyset\}.$$

We set $O = O' \cap O''$. Then O has the required condition. □

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References

[1] Arnold V.I., Gusein-Zade S.M., Varchenko A.N., *Singularities of differential maps I*, Birkhäuser, 1985.

[2] Th. bröcker, *Differentiable Germs and Catastrophes*. London Math. Soc. Lecture Note Ser., **17**, Cambridge Univ. Press, 1975.

[3] Izumiya S., *Perestroikas of Optical Wave fronts and Graphlike Legendrian Unfoldings*. J. Diff. Geom. **38** (1993), 485–500.

[4] Jänich K., *Caustics and catastrophes*, Math. Ann. **209** (1974), 161–180.

[5] Tsukada T., *Reticular Lagrangian, legendrian Singularities and their applications*. PhD Thesis, Hokkaido University, 1999.

[6] Tsukada T., *Reticular Lagrangian Singularities*. Asian J. Math. **1** (1997), 572–622.

[7] Tsukada T., *Reticular Legendrian Singularities*. Asian J. Math. **5** (2001), 109–127.

[8] Tsukada T., *Bifurcations of Wavefronts on an r-corner I: Generating families*. in preparation.

[9] Tsukada T., *Bifurcations of Wavefronts on an r-corner II: Stabilities and a generic classification*. in preparation.

[10] Wassermann G., *Stability of unfolding*. Lecture note in mathematics, **393**.

[11] Wassermann G., *Stability of unfolding in space and time*. Acta Math. **135** (1975), 57–128.

[12] Zakalyukin V.M., *Reconstruction of fronts and caustics depending on a parameter and versality of mappings*. J. of Soviet Math. **27** (1984), 2713–2735.

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