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Hilbert schemes of finite abelian group orbits and Gröbner fans

Tomohito Morita

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Abstract. Let G be a finite abelian subgroup of $PGL(r-1,K) = \mathrm{Aut}(\mathbb{P}_K^{r-1})$. In this paper, we prove that the normalization of the G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ is described as a toric variety, which corresponds to the Gröbner fan for some homogeneous ideal I of $K[x_1,\dots,x_r]$.

Key words: Gröbner fan, G-Hilbert schemes, toric singularity.

1. Introduction

Let K be an algebraically closed field and G a finite subgroup of $GL(r,K) \subset \mathrm{Aut}(\mathbb{A}^r)$ of order prime to the characteristic of K. The G-Hilbert scheme (or the Hilbert scheme of G-orbits) $\mathrm{Hilb}^G(\mathbb{A}^r)$ is introduced by Ito and Nakamura [11]. The G-Hilbert scheme is the main component of the scheme parameterizing all G-invariant zero-dimensional subschemes Z of $\mathbb{A}^r$ of length $m := |G|$ for which $H^0(Z,\mathcal{O}_Z)$ is the regular representation of G.

Nakamura [12] proved that for a finite abelian group $G \subset GL(r,K)$, the normalization of $\mathrm{Hilb}^G(\mathbb{A}^r)$ is the toric variety corresponding to the fan defined by the G-graph, which is defined in [12], and he also proved that $\mathrm{Hilb}^G(\mathbb{A}^3)$ is a crepant smooth resolution of $\mathbb{A}^3/G$ if $G \subset SL(3,K)$. Furthermore, Ito [9] proved the following result.

Theorem 1.1 (Ito [9, Theorem 1.1]) *Let G be a finite small cyclic group in $GL(2,\mathbb{C})$ and I the G-invariant ideal of $\mathbb{C}[x,y]$ defined by the free G-orbit of $(1,1) \in (\mathbb{C}^*)^2$. Then the toric variety corresponding to the Gröbner fan for I is isomorphic to the minimal resolution of $\mathbb{C}^2/G$.*

The G-Hilbert scheme is isomorphic to the minimal resolution of the quotient singularity if $G \subset GL(2,K)$. Then it follows that $\mathrm{Hilb}^G(\mathbb{C}^2)$ is described by the Gröbner fan from this theorem.

In this paper, we consider the case when G is a finite abelian subgroup

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of the diagonal subgroup T' of $PGL(r-1, K) = \mathrm{Aut}(\mathbb{P}^{r-1})$, and the G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ of the projective space. We prove the following result.

Theorem 1.2 *Let G be a finite abelian group of the diagonal subgroup T' of $PGL(r-1, K)$ and I the G-invariant ideal of $K[x_1, \ldots, x_r]$ defined by a free G-orbit which is contained in $\{x_1 \cdots x_r \neq 0\} \subset \mathbb{P}^{r-1}$. Then the toric variety corresponding to the fan consisting of the images in the quotient lattice of the Gröbner cones for I is isomorphic to the normalization of the G-Hilbert scheme* $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$.

The corresponding results on $\mathrm{Hilb}^G(\mathbb{A}^{r-1})$ are easily deduced from it (Corollary 4.4). This gives an alternative proof and a generalization of Ito's result.

The proof of our theorem consists of three steps. Note that T' is isomorphic to an algebraic torus $(K^*)^{r-1} \subset \mathbb{P}^{r-1}$. Let $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ denote the Hilbert scheme of m points in $\mathbb{P}^{r-1}$, where $m = |G|$. For any homogeneous ideal I as in the theorem, I defines a point P of $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$, and we prove that the closure in $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ of T'-orbit of P is coincides with $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$. Next, we show that the normalization of $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ is described by the state polytope of I, defined in (3.3) (Proposition 3.2). Finally, we show that the normal fan of the state polytope coincides with the Gröbner fan for I by a theorem of Bayer and Morrison [1] (Theorem 3.4).

Gröbner fans are computable (for example by [6]). Hence we hope that the results in this paper are useful for the study on the G-Hilbert schemes, especially in higher dimensional cases.

This paper is organized as follows. In Section 2, we recall the basic notations and results for Gröbner basis and Gröbner fan. In Section 3, we ready some results for the proof of our main theorem. We explain that a toric variety corresponding to a special torus action is described by a Gröbner fan. In Section 4, we prove our main theorem. In Section 5, we give two and three dimensional examples. Two dimensional example gives the minimal resolution of the quotient singularity. But three dimensional example is singular even if an abelian group is a cyclic group.

After we wrote up this paper, the author found the paper [3] by Craw, Maclagan, and Thomas, where they study the moduli space of McKay quiver representations. Our work is related to their theory in the case of the

G-Hilbert schemes, but the method is different. T. Yasuda also obtained related results in [18], but his method is different from ours. Y. Ito communicated to the author that Y. Sekiya [15] independently obtained a similar result to ours.

2. Gröbner basis and Gröbner fan

We recall the basic notations for Gröbner basis and Gröbner fan. These notations are based on the book by Sturmfels [16].

Let K be any field and $K[X] = K[x_1, \ldots, x_r]$ the polynomial ring in r indeterminates. Let $\mathbb{N}$ be the set of non-negative integers. For $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$, let X^a denote the monomial $x_1^{a_1} \cdots x_r^{a_r}$. By this correspondence $a \mapsto X^a$, the lattice $\mathbb{N}^r$ is embedded in $K[X]$ as a multiplicative semigroup and this image coincides with the set of monomials in $K[X]$.

Definition 2.1 A total order $<$ on $\mathbb{N}^r$ is called a *term order* if $(0, \ldots, 0)$ is the unique minimal element, and $a < b$ implies $a + c < b + c$ for all $a, b, c \in \mathbb{N}^r$.

We denote by $\mathbb{R}_+$ the set of non-negative real numbers. Fix a weight vector $w = (w_1, \ldots, w_r) \in \mathbb{R}^r$ and a term order $<$. Let $\langle \ , \ \rangle$ denote the standard inner product of $\mathbb{R}^r$.

Definition 2.2 For a polynomial $f = \Sigma_{c_i \neq 0} c_i X^{a_i}$, the *initial term* $\mathrm{in}_w(f)$ of f with respect to w is the sum of all terms $c_i X^{a_i}$ such that the inner product $\langle w, a_i \rangle$ is maximal. For an ideal I of $K[X]$, the *initial ideal* $\mathrm{in}_w(I)$ is the ideal generated by the initial terms of all elements f in I:

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) | f \in I \rangle .$$

This ideal need not be a monomial ideal. However, $\mathrm{in}_w(I)$ is a monomial ideal if w is chosen sufficiently generic for a given I.

Definition 2.3 Let w be a weight vector in $\mathbb{R}_+^r$ and $<$ the fixed term order. The *weight term order* $<_w$ is the term order defined as follows:

$$X^a <_w X^b \Leftrightarrow \langle a, w \rangle < \langle b, w \rangle \,, \text{ or } \langle a, w \rangle = \langle b, w \rangle \text{ and } a < b.$$

Definition 2.4 For a polynomial f, the *initial term* $\mathrm{in}_{<_w}(f)$ of f with respect to $<_w$ is the maximal term of f with respect to $<_w$. For an ideal I of $K[X]$, the *initial ideal* $\mathrm{in}_{<_w}(I)$ is the ideal generated by the initial terms of all elements f in I:

$$\mathrm{in}_{<_w}(I) = \langle \mathrm{in}_{<_w}(f) | f \in I \rangle .$$

Definition 2.5 A finite subset $\mathcal{G} \subset I$ is a *Gröbner basis* for each w if $\mathrm{in}_{<_w}(I)$ is generated by $\{\mathrm{in}_{<_w}(g) | g \in \mathcal{G}\}$.

It is known that I is generated by the Gröbner basis of I.

Definition 2.6 The Gröbner basis $\mathcal{G}$ of I is *reduced* if for any two elements $g, h \in \mathcal{G}$, no term of h is divisible by $\mathrm{in}_{<_w}(g)$.

It is known that the reduced Gröbner basis of I is unique.

Definition 2.7 Two weight vectors $w, w' \in \mathbb{R}^r$ are called *I-equivalent* (or simply *equivalent*) if $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$.

Then we can consider the equivalence classes of weight vectors.

Proposition 2.8 ([16, Proposition 2.3]) *For any ideal I, and for any weight vector w, the equivalence class $c[w]$ of weight vectors is relatively open convex polyhedral cone. Moreover, if w is contained in $\mathbb{R}^r_+$ and chosen sufficiently generic, $c[w]$ is given by the reduced Gröbner basis $\mathcal{G}$ of I with respect to $<_w$ as follows*:

$$c[w] = \left\{ w' \in \mathbb{R}^r \,|\, \mathrm{in}_{w'}(g) = \mathrm{in}_w(g) \text{ for any } g \in \mathcal{G} \right\}.$$

Definition 2.9 The *Gröbner region* $GR(I)$ for I is the set of all $w \in \mathbb{R}^r$ such that $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$ for some $w' \in \mathbb{R}^r_+$. Clearly, $GR(I)$ contains $\mathbb{R}^r_+$.

Remark 2.10 In general, the $GR(I)$ does not coincide with $\mathbb{R}^r$. However, when the ideal I is homogeneous, it is known that $GR(I)$ coincides with $\mathbb{R}^r$.

Definition 2.11 The *Gröbner fan* $GF(I)$ for an ideal I is the set consisting of the faces of the closed cones of the form $\overline{c[w]}$ for some $w \in \mathbb{R}^r_+$. A closed cone σ is called *Gröbner cone* if $\sigma \in GF(I)$.

Proposition 2.12 ([16, Proposition 2.4]) *The Gröbner fan for I is a convex polyhedral fan.*

Remark 2.13 In general, the Gröbner fan for I does not consist of strongly convex cones.

3. Weight polytope and state polytope

In the first half of this section, we explain that the normalization of the closure of the torus orbit of a point in a projective space is the toric variety corresponding to some polytope, which is called the weight polytope (Proposition 3.2).

In the latter half of this section, we consider the Hilbert scheme $\mathrm{Hilb}_h(\mathbb{P}^{r-1})$ being embedded to the projective space $\mathbb{P}^n$ by the Plücker embedding, where $\mathrm{Hilb}_h(\mathbb{P}^{r-1})$ is the Hilbert scheme of $\mathbb{P}^{r-1}$ corresponding to a Hilbert polynomial $h(x) \in \mathbb{Z}[x]$. Then, the weight polytope corresponding to the normalization of the closure in $\mathrm{Hilb}_h(\mathbb{P}^{r-1})$ of the torus orbit of a Hilbert point I is called the state polytope of I. By a theorem of Bayer and Morrison [1], the normal fan of the state polytope coincides with the Gröbner fan for I (Theorem 3.4). In conclusion, the normalization of the closure of the torus orbit of a Hilbert point I is the toric variety corresponding to the Gröbner fan for I. We also prove that the normalization of the closure in $\mathrm{Hilb}_h(\mathbb{A}^{r-1})$ of the torus orbit of I in $\mathrm{Hilb}_h(\mathbb{A}^{r-1})$ is the toric variety corresponding to the Gröbner fan for the dehomogenization of I (Corollary 3.7).

Let V be an $n+1$ dimensional vector space over an algebraically closed field K and T an algebraic torus of dimension r. Let $\rho : T \to GL(V)$ be a rational linear representation of T such that $\rho(T)$ contains all scalar multiplications of $GL(V)$. The point of $\mathbb{P}(V) = (V\backslash\{0\})/k^*$ corresponding to $\mathbf{v} \in V\backslash\{0\}$ will be denoted by v. Here $GL(V)$ (and hence T via ρ) acts on $\mathbb{P}(V)$ in a natural way. For each $v \in \mathbb{P}(V)$, let $\overline{T \cdot v}$ be the closure in $\mathbb{P}(V)$ of the torus orbit of v. Then $\overline{T \cdot v}$ has the open dense orbit which is isomorphic to $T/\mathrm{Stab}(v)$.

In general, $\overline{T \cdot v}$ is not a normal variety. But its normalization is a toric variety which contains $T/\mathrm{Stab}(v)$ as an open dense orbit (cf. [13]). The corresponding fan of this toric variety is described by the weight polytope explained below.

Let M be the character group of T and N the dual lattice of M. It is known that V is the direct sum of its weight subspaces:

$$V = \bigoplus_{w\in M} V_w, \text{ where } V_w = \{\mathbf{v} \in V | \rho(t)\cdot\mathbf{v} = w(t)\cdot\mathbf{v} \text{ for all } t \in T\}.$$

Definition 3.1 Let $\mathbf{v} = \sum_{w\in M}\mathbf{v}_w$, where $\mathbf{v}_w \in V_w$. The convex hull $\mathrm{Wt}(\mathbf{v}) \subset M_{\mathbb{R}} = M \bigotimes_{\mathbb{Z}} \mathbb{R}$ of the set $\{w \in M | \mathbf{v}_w \neq 0\}$ will be called the *weight polytope* of $\mathbf{v}$.

Let v be a point of $\mathbb{P}(V)$. We denote by $\mathrm{Wt}(v)$ the weight polytope $\mathrm{Wt}(\mathbf{v})$ of a lift $\mathbf{v}$ of v. This definition doesn't depend on the choice of v.

When $T \to T\cdot v$ is injective, the corresponding fan to the normalization of $\overline{T\cdot v}$ is nothing but the normal fan of $\mathrm{Wt}(v)$ (see [13, Chapter 2.4]).

In general, we take a sublattice of M and construct the polytope corresponding to the toric variety as follows.

Let M' be the character group of $T/\mathrm{Stab}(v)$ and N' the dual lattice of M'. M' is identified with the sublattice of M which is generated by $w_1 - w_2$ for any $w_1, w_2 \in \{w \in M | \mathbf{v}_w \neq 0\}$, that is,

$$M' = \langle \{w_1 - w_2 \in M | \mathbf{v}_{w_1}, \mathbf{v}_{w_2} \neq 0\}\rangle .$$

The next proposition follows from [13, Theorem 2.22].

Proposition 3.2 (cf. [13, Theorem 2.22]) *The normalization of $\overline{T\cdot v}$ is isomorphic to the toric variety corresponding to the normal fan in $N'_{\mathbb{R}}$ of the polytope* $\mathrm{Wt}(v) - w \subset M'_{\mathbb{R}}$ *for any $w \in \mathrm{Wt}(v) \cap M$.*

Proof. Let X be the toric variety corresponding to the polytope $\mathrm{Wt}(v) - w$ and let the set of weights $\{w_1, \dots, w_s\} = \{w \in M | \mathbf{v}_w \neq 0\}$. Take a basis of V_w for each $w \in M$ which contains $\mathbf{v_w}$ if $\mathbf{v_w} \neq 0$. Gathering these bases, we get a base of V, and we may assume that the homogeneous coordinate of v with respect to this base is given by $(1 : \cdots : 1 : 0 : \cdots : 0)$. We have a morphism $\phi : X \to \overline{T\cdot v} \subset \mathbb{P}(V)$ as follows:

$$\phi : X \to \overline{T\cdot v} \subset \mathbb{P}(V)\ ;\ x \mapsto (\chi_{w_1}(x) : \cdots : \chi_{w_s}(x) : 0 : \cdots : 0),$$

where χ_w is the character corresponding to w. Since X is normal, the morphism ϕ factors through the normalization of $\overline{T\cdot v}$. The open torus orbit of X is isomorphic to the open torus orbit of the normalization of $\overline{T\cdot v}$ by the morphism ϕ. Moreover, the torus actions on X and on the normalization of $\overline{T\cdot v}$ are compatible with the morphism ϕ. Then the proposition follows from [14, theorem 4.1]. $\square$

Next, we consider the situation where the torus $T = (K^*)^r$ acts naturally on the vector space $V = N \bigotimes_{\mathbb{Z}} K \cong K^r$. Then we have a natural embedding $T \subset V$. Let S be the symmetric algebra of the dual space of V, and put $\mathbb{P} = \mathrm{Proj}(S)$. Then T acts on $S = K[x_1, \dots, x_r]$ in the following way:

$$T \times S \to S \; ; \; (t = (t_1, \dots, t_r), f(X)) \mapsto t \cdot f(X) := f\left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r}\right).$$

Further T acts on $\mathbb{P}$, too.

Let $h(x) \in \mathbb{Z}[x]$ be a polynomial and $\mathrm{Hilb}_h(\mathbb{P})$ the Hilbert scheme corresponding to h. We identify $I \in \mathrm{Hilb}_h(\mathbb{P})$ with the corresponding homogeneous ideal of S. Then d-graded part I_d of I is a subspace of S_d of codimension $h(d)$ for any large integer $d \gg 0$. Let $G(l - h(d), S_d)$ be the Grassmannian variety of subspaces of S_d of codimension $h(d)$, where l is $\dim_K S_d = \binom{r+d-1}{d}$. It is known that the mapping $\mathrm{Hilb}_h(\mathbb{P}) \to G(l - h(d), S_d) \; ; \; I \mapsto I_d$ is a closed embedding for any large integer $d \gg 0$. By the Plücker embedding, we have $\mathrm{Hilb}_h(\mathbb{P}) \subset G(l - h(d), S_d) \subset \mathbb{P}(\wedge^{l-h(d)} S_d) \cong \mathbb{P}^n$, where $n = \dim_K(\wedge^{l-h(d)} S_d) - 1$, and we have a natural action of T on S_d, and hence a natural action on $G(l - h(d), S_d)$, and also on $\mathbb{P}^n$.

Then we can apply the facts in the first half of this section to the torus orbit of a Hilbert point I.

Definition 3.3 Let I_d be the d-graded component of $I \in \mathrm{Hilb}_h(\mathbb{P})$. The weight polytope of the Hilbert point $I_d \in \mathbb{P}^n$ is called the *state polytope* $\mathrm{St}_d(I)$ of I in degree d.

The following is deduced from a theorem of Bayer and Morrison [1] (cf. Sturmfels [17]). For reader's convenience, we give here a slightly different proof.

Theorem 3.4 ([17, Theorem 2.1]) *Let $w \in \mathrm{St}_d(I) \cap M$. Then the normal fan in $N'_{\mathbb{R}}$ of the polytope $\mathrm{St}_d(I) - w \subset M'_{\mathbb{R}}$ coincides with the set of the images of the Gröbner cones for $J = \bigoplus_{i \geq d} I_i$ by the natural projection $N_{\mathbb{R}} \to N'_{\mathbb{R}}$.*

Proof. It is enough to prove that the pull back of the normal fan of $\mathrm{St}_d(I) - w$ coincides with the Gröbner fan for J. Therefore, we prove that the normal fan in $N_{\mathbb{R}}$ of the state polytope coincides with the Gröbner fan for J. First note that each Gröbner cone contains the vector $(1,\dots,1)$, and the kernel of the projection $N_{\mathbb{R}} \to N'_{\mathbb{R}}$ also contains the vector $(1,\dots,1)$. Then we only have to prove that the intersection of the normal fan in $N_{\mathbb{R}}$ of the state polytope and $\mathbb{R}^r_+ \subset N_{\mathbb{R}}$ coincides with the intersection of the Gröbner fan for J and $\mathbb{R}^r_+$.

First, we prove that the latter is a subdivision of the former. We recall that the coordinate of $\mathbb{P}^n$ is given by the Plücker coordinate. Let $f_1,\dots,f_k$ be a K-basis of $I_d \subset K[X]_d$, where $k = l - h(d)$. Then $f_1 \wedge \cdots \wedge f_k$ defines the Plücker coordinate of I_d. Let $\mathcal{S}$ be the set of weights defined by its all non-zero components. Then, by definition, the state polytope of I in degree d is the convex hull of $\mathcal{S}$. Let σ be a maximal cone of the Gröbner fan for J and $w \in \sigma \cap \mathbb{R}^r_+$ be a weight vector contained in the relative interior of σ. Let a set of monomials $X^{a_1},\dots,X^{a_k}$ be the basis of $\mathrm{in}_w(I)_d$. Then the weight $v \in M_{\mathbb{R}}$ corresponding to $X^{a_1} \wedge \cdots \wedge X^{a_k}$ is in $\mathcal{S}$. For $v'(\neq v) \in \mathcal{S}$, v' is defined by wedge product of $X^{a'_i}$, where $X^{a'_i}$ is a term of f_i, and at least one term does not coincide with X^{a_i}. Hence we have $\langle w, v\rangle > \langle w, v'\rangle$ for any $v' \neq v$. This implies that v is a vertex of the state polytope and w is contained in the normal cone of v. Hence σ is contained in the normal cone of v. Therefore, the intersection of the Gröbner fan for J and $\mathbb{R}^r_+$ is a subdivision of the intersection of the normal fan in $N_{\mathbb{R}}$ of the state polytope and $\mathbb{R}^r_+$.

Let $\sigma' \neq \sigma$ be a maximal cone of the Gröbner fan for J and $w' \in \sigma' \cap \mathbb{R}^r_+$ a weight vector contained in the relative interior of σ'. Let a set of monomials $X^{b_1},\dots,X^{b_k}$ be the basis of $\mathrm{in}_{w'}(I)_d$. Then we have $X^{a_1} \wedge \cdots \wedge X^{a_k} \neq X^{b_1} \wedge \cdots \wedge X^{b_k}$. Hence the intersection of the normal fan in $N_{\mathbb{R}}$ of the state polytope and $\mathbb{R}^r_+$ coincides with the intersection of the Gröbner fan for J and $\mathbb{R}^r_+$. □

Remark 3.5 Note that the Gröbner fan for I does not coincide with the normal fan of the state polytope. There may exist two weights w, w' such that $\mathrm{in}_w(I)$ and $\mathrm{in}_{w'}(I)$ are different as ideals of graded ring S, while they define a same point of $\mathrm{Hilb}_h(\mathbb{P})$.

Example 3.6 Let $I \subset K[x,y,z,w]$ be the ideal generated by $x^5 - w^5$, $x^2 - yw$, $x^3 - zw^2$, $y^2 - xz$, $z^2 - xw$. Let $\omega = (31,11,2,1)$ and $\omega' = (11,31,2,1)$ be 2 weight vectors. Then initial ideals with respect to ω and ω' are $J = \langle x^2, y^2, xw, yz^2, xz^2, z^5, yw^3\rangle$ and $J' = \langle yw, x^3, y^2, xw, z^5, x^2y,$

$yz^2, x^2z^2, xz^4\rangle$. These initial ideals are different. But we have $J_d = J'_d$ for any $d \geq 5$. Therefore, these define the same point in the Hilbert scheme. See example 5.2.

In the rest of this section, we consider the case where the dimension of $\mathrm{Stab}(v)$ is one dimensional. Then $\mathrm{Stab}(v)$ is generated by $(1, \ldots, 1)K^*$ and a finite abelian group. Therefore, $N'_{\mathbb{R}}$ is identified with $N_{\mathbb{R}}/(1, \ldots, 1)$.

Corollary 3.7 *Let I' be an ideal of $K\left[\frac{x_1}{x_r}, \ldots, \frac{x_{r-1}}{x_r}\right]$ and $I \subset K[x_1, \ldots, x_r]$ the homogenization of I'. We assume that the stabilizer of I is one dimensional. We take the image of the part $e_1, \ldots, e_{r-1}$ of the standard basis of $N_{\mathbb{R}}$ as the basis of $N'_{\mathbb{R}}$. Then the Gröbner fan for I' consists of the images of the Gröbner cones for $J = \bigoplus_{i \geq d} I_i$ which are contained in the Gröbner region $GR(I')$ for I' in $N'_{\mathbb{R}}$.*

Proof. We only have to prove that the image $\overline{\sigma}$ of a maximal Gröbner cone σ for J coincides with some maximal Gröbner cone for I' if and only if the intersection of $\mathbb{R}_+^{r-1} \cong \sum_{i=1}^{r-1} \mathbb{R}_+ e_i \subset N'_{\mathbb{R}}$ and the relative interior of $\overline{\sigma}$ is not empty.

The necessity is clear from the definition. We prove the sufficiency. For any $d \gg 0$, we have $\phi : \bigoplus_{i=0}^{d} I'_i \cong I_d$. Moreover, ϕ maps the reduced Gröbner basis of I' which is contained in $\bigoplus_{i=0}^{d} I'_i$, to the reduced Gröbner basis of J contained in I_d. Then this follows immediately. □

Corollary 3.8 *Let I be a homogeneous ideal in* $\mathrm{Hilb}_h(\mathbb{P})$. *We assume that the stabilizer* $\mathrm{Stab}(I)$ *of I is one dimensional and that the intersection of the algebraic torus of $\mathbb{P}$ and the closed set Z defined by I is an open dense subset of Z. Let $I_i \subset K\left[\frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}\right]$ be the dehomogenization of I and Σ_i the Gröbner fan for I_i with respect to the basis $\{\overline{e_1}, \ldots, \check{\overline{e_i}}, \ldots, \overline{e_r}\}$ of $N'_{\mathbb{R}}$. Then the set of the images of the Gröbner cones for $J = \bigoplus_{i \geq d} I_i$ coincides with the fan obtained as the union $\cup_i \Sigma_i$ of Σ_i.*

Proof. This follows from Corollary 3.7 immediately. Note that the Gröbner region $GR(I_i)$ for I_i contains $\sum_{j \neq i} \mathbb{R}_+ e_j \cong \mathbb{R}_+^{r-1}$. □

4. Main theorem and its applications

First, we recall the notation and the results in the previous section. Let K be an algebraically closed field and T the algebraic torus of dimension

r which is identified with the diagonal subgroup of $GL(r,K)$. Let $\mathbb{P}^{r-1}$ be the $(r-1)$-dimensional projective space over K and $S \cong k[x_1,\dots,x_r]$ the homogeneous coordinate ring of $\mathbb{P}^{r-1}$, so $(x_1:\dots:x_r)$ is a homogeneous coordinate of $\mathbb{P}^{r-1}$. Then T acts on $\mathbb{P}^{r-1}$ by $(t_1,\dots,t_r)\cdot(p_1:\dots:p_r) = (t_1p_1:\dots:t_rp_r)$, and T acts on S by $(t_1,\dots,t_r)\cdot f(x_1,\dots,x_r) = f\big(\frac{x_1}{t_1},\dots,\frac{x_r}{t_r}\big)$. Then T acts on the Hilbert scheme $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ of m points of $\mathbb{P}^{r-1}$.

Let M be the character group of the torus T and N the dual lattice of M. Since T acts on S, the Gröbner fan for an ideal $I \subset S$ is defined in $N_{\mathbb{R}}$. Let v be a point of $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ and I the homogeneous ideal of S corresponding to v. Then, by Proposition 3.2, the normalization of the closure in $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ of the torus orbit of v is the toric variety corresponding to the state polytope defined in (3.3) for $d \gg 0$. Note that the state polytope is a translate of a subset of $M'_{\mathbb{R}}$, where M' is the character group of $T/\mathrm{Stab}(v)$. Let N' be the dual lattice of M'. We proved that the normal fan in $N'_{\mathbb{R}}$ of the state polytope coincides with the set of images of the Gröbner cones for I by the natural projection $N_{\mathbb{R}} \to N'_{\mathbb{R}}$ (cf. Theorem 3.4). Then the set of images of the Gröbner cones for I becomes a fan which consists of strongly convex cones.

Next, following Nakamura [12], we introduce the G-Hilbert schemes $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$. Let G be a finite abelian subgroup of $GL(r,K)$ of order prime to the characteristic of K. Let $\phi : GL(r,K) \to PGL(r-1,K)$ be the natural projection. Let m be the order of G. We assume that $G \cong \phi(G)$ and that G is a subgroup of the diagonal subgroup of $PGL(r-1,K)$. Then we can identify $T/(1,\dots,1)K^*$ with an algebraic torus T' of $\mathbb{P}^{r-1}$ and the actions of G and T are commutative. Since G and T also act on S, the groups G and T act on the Hilbert scheme $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$.

Let $\mathrm{Ch}^m(\mathbb{P}^{r-1})$ be the Chow variety of m points in $\mathbb{P}^{r-1}$. We have a natural morphism $\phi : \mathrm{Hilb}^m(\mathbb{P}^{r-1}) \to \mathrm{Ch}^m(\mathbb{P}^{r-1})$, which is called the Hilbert-Chow morphism. Here G acts on $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ and on $\mathrm{Ch}^m(\mathbb{P}^{r-1})$, and ϕ is G-equivariant. Therefore we have a natural morphism between their G-fixed point sets. It is known that the G-fixed point set of $\mathrm{Ch}^m(\mathbb{P}^{r-1})$ contains U_r/G as a locally closed subset, where $U_r = \{x_r \neq 0\} \subset \mathbb{P}^{r-1}$, and its closure is an irreducible component of the G-fixed point set of $\mathrm{Ch}^m(\mathbb{P}^{r-1})$.

Definition 4.1 The G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ is defined to be the unique irreducible component of the G-fixed point set of $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$ which dominates the closure of U_r/G by the map ϕ.

Theorem 4.2 *Let I be the G-invariant ideal of S whose zero set is contained in $T' \subset \mathbb{P}^{r-1}$. Then the toric variety corresponding to the fan consisting of the images in $N'_{\mathbb{R}}$ of the Gröbner cones for I is isomorphic to the normalization of the G-Hilbert scheme* $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$.

Proof. By Proposition 3.2 and Theorem 3.4, it is enough to prove that $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ is the closure of the torus orbit of I in $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$.

We can consider that the torus T' is the diagonal subgroup of PGL $(r-1, K)$. If there exist $g, g' \in G$ and $t \in T/(1,\ldots,1) \cong T'$ such that $gt = g't$, then we have $g = g'$ in T'. Here G acts freely on the open torus of $\mathbb{P}^{r-1}$.

We prove that the torus orbit of I coincides with the torus contained in $\mathrm{Hilb}^G(\mathbb{P}^{n-1})$.

Let I' be a point of $T \cdot I$, then there exist $t \in T$ such that $I' = t \cdot I$. Since $G \subset T/(1,\ldots,1)$, I' is also a G-fixed point of $\mathrm{Hilb}^m(\mathbb{P}^{r-1})$.

Conversely, let J be an ideal of S defined by a free G-orbit whose zero set is contained in T'. Then J determines distinct m points $p_1, \ldots, p_m$ of T' and I determines distinct m points $q_1, \ldots, q_m$ of T'. Take a $t \in T$ satisfying $t \cdot p_1 = q_1$. Then we have

$$\begin{aligned}\{q_1, \ldots, q_m\} &= \{g \cdot q_1 | g \in G\} = \{g \cdot t \cdot p_1 | g \in G\} \\ &= \{t \cdot g \cdot p_1 | g \in G\} = t \cdot \{p_1, \ldots, p_m\}.\end{aligned}$$

Therefore J is contained in $T \cdot I$.

Since $T \cdot I$ is an open subset of $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ and $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ is irreducible, the closure of $T \cdot I$ coincides with $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$. □

Remark 4.3 From this theorem, we can obtain a flat family of subschemes parameterized by the toric variety corresponding to the Gröbner fan for I. One of the referees suggested the possibility of an alternative proof of the main theorem by constructing this family directly, via the idea in 15.8 of [4]. This should be investigated.

Corollary 4.4 *Let I be the G-invariant ideal of S whose zero set is contained in T' and $I' \subset K\left[\frac{x_1}{x_r}, \ldots, \frac{x_{r-1}}{x_r}\right]$ the dehomogenization of I. Then the toric variety corresponding to the Gröbner fan $GF(I')$ for I' is isomorphic to the normalization of G-Hilbert scheme* $\mathrm{Hilb}^G(\mathbb{A}^{r-1})$.

Remark 4.5 In this case, the Gröbner fan for I' consists of strongly convex cones.

Proof. We denote by $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbb{P}^{r-1})$ (resp. $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbb{A}^{r-1})$) the normalization of $\mathrm{Hilb}^G(\mathbb{P}^{r-1})$ (resp. $\mathrm{Hilb}^G(\mathbb{A}^{r-1})$). A point $p \in \mathrm{Hilb}^G_{\mathrm{norm}}(\mathbb{P}^{r-1})$ is contained in $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbb{A}^{r-1})$ if and only if p is contained in the torus orbit corresponding to a Gröbner cone σ of $GF(I')$. Then this corollary follows from Corollary 3.7 and Theorem 4.2 immediately. $\square$

Remark 4.6 Another dehomogenization ideal $I'_i \subset K\left[\frac{x_1}{x_i}, \dots, \frac{x_{r-1}}{x_i}\right]$ of I corresponds to $\mathrm{Hilb}^G_{\mathrm{norm}}(U_i)$, where $U_i = \{x_i \neq 0\} \cong \mathbb{A}^{r-1}$. Then $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbb{P}^{r-1})$ can be obtained by the gluing of $\mathrm{Hilb}^G_{\mathrm{norm}}(U_i)$ by Corollary 3.8.

Corollary 4.7 (Ito [9, Theorem 1.1]) *Let G be a finite small cyclic group in $GL(2, \mathbb{C})$ and I the G-invariant ideal of $\mathbb{C}[x, y]$ whose zero set is contained in $(\mathbb{C}^\times)^2$. Then the toric variety corresponding to the Gröbner fan for I is isomorphic to the minimal resolution of $\mathbb{C}^2/G$.*

Proof. Corollary 4.4 gives that the toric variety corresponding to the Gröbner fan for I is isomorphic to the normalization of G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{A}^2)$. Ishii [8, theorem 3.1] shows that the G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$ when G is a finite small subgroup of $GL(2, \mathbb{C})$. Hence the proof completes. $\square$

5. Example

Example 5.1 Let G be a cyclic group which is generated by the matrix

$$\begin{pmatrix} e & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where e is a primitive fifth root of the unity. Let I be the ideal of $K[x, y, z]$ which is generated by $x^3 - yz^2$, $x^2y - z^3$, $y^2 - xz$. The set of zeros of I is $\{(1:1:1), (e:e^3:1), (e^2:e^1:1), (e^3:e^4:1), (e^4:e^2:1)\}$. Then I is a G-invariant ideal whose zero set is contained in $(K^*)^2$.

We denote by $\mathcal{G}_w$ the reduced Gröbner basis with respect to w and $c[w]$ the Gröbner cone with respect to w. The Gröbner fan for I is defined by 11 maximal cones:

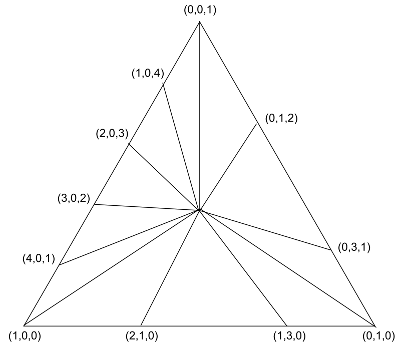

Figure 1. The intersection of the Gröbner fan for I and $\{x+y+z=1\}$

$$
\begin{aligned}
\mathcal{G}_{w_1} &= \{x^3-yz^2,\ x^2y-z^3,\ xz-y^2,\ xy^3-z^4,\ y^5-z^5\}\\
\mathcal{G}_{w_2} &= \{x^3-yz^2,\ x^2y-z^3,\ y^2-xz\}\\
\mathcal{G}_{w_3} &= \{yz^2-x^3,\ x^2y-z^3,\ x^5-z^5\}\\
\mathcal{G}_{w_4} &= \{yz^2-x^3,\ x^2y-z^3,\ y^2-xz,\ z^5-x^5\}\\
\mathcal{G}_{w_5} &= \{yz^2-x^3,\ z^3-x^2y,\ y^2-xz\}\\
\mathcal{G}_{w_6} &= \{yz^2-x^3,\ z^3-x^2y,\ y^3z-x^4,\ y^5-x^5\}\\
\mathcal{G}_{w_7} &= \{yz^2-x^3,\ z^3-x^2y,\ xz-y^2,\ y^3z-x^4,\ x^5-y^5\}\\
\mathcal{G}_{w_8} &= \{yz^2-x^3,\ z^3-x^2y,\ xz-y^2,\ x^4-y^3z\}\\
\mathcal{G}_{w_9} &= \{x^3-yz^2,\ z^3-x^2y,\ xz-y^2\}\\
\mathcal{G}_{w_{10}} &= \{x^3-yz^2,\ x^2y-z^3,\ xz-y^2,\ z^4-xy^3\}\\
\mathcal{G}_{w_{11}} &= \{x^3-yz^2,\ x^2y-z^3,\ xz-y^2,\ xy^3-z^4,\ z^5-y^5\}
\end{aligned}
$$

In this case, these initial ideal define different points of $\mathrm{Hilb}^5(\mathbb{P}^2)$. Then, this fan coincides with the Gröbner fan for $J=\bigoplus_{i\geq d} I_i$.

$$
\begin{aligned}
c[w_1] &= \{(x,y,z)\in\mathbb{R}^3 | y-z\geq 0,\ x-2y+z\geq 0\}\\
c[w_2] &= \{(x,y,z)\in\mathbb{R}^3 | x-2y+z\leq 0,\ 3x-y-2z\geq 0\}\\
c[w_3] &= \{(x,y,z)\in\mathbb{R}^3 | 3x-y-2z\leq 0,\ x-z\geq 0\}\\
c[w_4] &= \{(x,y,z)\in\mathbb{R}^3 | x-z\leq 0,\ 2x+y-3z\geq 0\}
\end{aligned}
$$

$$
\begin{aligned}
c[w_5] &= \{(x,y,z) \in \mathbb{R}^3 | 2x+y-3z \leq 0,\ x-2y+z \leq 0\} \\
c[w_6] &= \{(x,y,z) \in \mathbb{R}^3 | x-2y+z \geq 0,\ x-y \leq 0\} \\
c[w_7] &= \{(x,y,z) \in \mathbb{R}^3 | x-y \geq 0,\ 4x-3y-z \leq 0\} \\
c[w_8] &= \{(x,y,z) \in \mathbb{R}^3 | 4x-3y-z \geq 0,\ 3x-y-2y \leq 0\} \\
c[w_9] &= \{(x,y,z) \in \mathbb{R}^3 | 3x-y-2z \geq 0,\ 2x+y-3z \leq 0\} \\
c[w_{10}] &= \{(x,y,z) \in \mathbb{R}^3 | 2x+y-3z \geq 0,\ x+3y-4z \leq 0\} \\
c[w_{11}] &= \{(x,y,z) \in \mathbb{R}^3 | x+3y-4z \geq 0,\ y-z \leq 0\}
\end{aligned}
$$

Let N be the lattice corresponding to the torus T. The stabilizer of I is generated by $(1,1,1) \cdot K^*$ and $(e, e^3, 1)$. Then the lattice N' corresponding to $T/\mathrm{Stab}(I)$ is identified with $\mathbb{Z} \oplus \mathbb{Z} \oplus (\frac{1}{5}, \frac{3}{5})\mathbb{Z}$ and the lattice homomorphism $N \to N'$ is described as follows:

$$N \cong \mathbb{Z}^3 \to N' \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \left(\frac{1}{5}, \frac{3}{5}\right)\mathbb{Z} : (a,b,c) \mapsto (a-c, b-c).$$

Then the fan corresponding to $\mathrm{Hilb}^G(\mathbb{P}^2)$ is generated by the images of these cones.

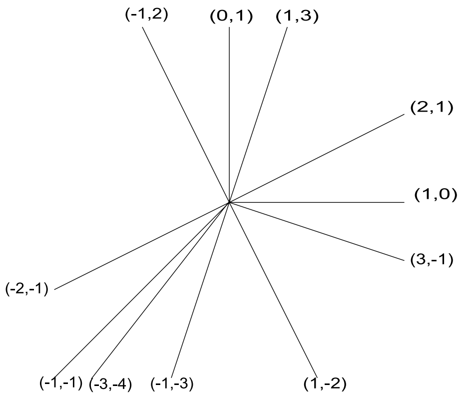

Figure 2. The image of the Gröbner fan for I

This fan corresponds to a nonsingular toric variety, and the subfan which is contained in $\mathbb{R}^2_+$ corresponds to the minimal resolution of $\{(x:y:z) \in \mathbb{P}^2 | z \neq 0\}/G \cong K^2/G$.

Example 5.2 Let G be a cyclic group which is generated by the matrix

$$\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where e is a primitive fifth root of the unity. Let I be the ideal of $K[x, y, z, w]$ which is generated by $x^5 - w^5$, $x^2 - yw$, $x^3 - zw^2$, $y^2 - xz$, $z^2 - xw$. The set of zeros of I is $\{(1 : 1 : 1 : 1), (e : e^2 : e^3 : 1), (e^2 : e^4 : e : 1), (e^3 : e : e^4 : 1), (e^4 : e^3 : e^2 : 1)\}$. Then I is a G-invariant ideal whose zero set is contained in $(K^*)^3$. Here G acts on $\{w \neq 0\} \cong \mathbb{A}^3$. Let $I' \subset K[\frac{x}{w}, \frac{y}{w}, \frac{z}{w}]$ be the dehomogenization of I. The Gröbner fan for I' has 8 edges, 17 facets, and 10 maximal cones. 9 maximal cones are simplicial and nonsingular, but 1 maximal cone is not simplicial. Then $\mathrm{Hilb}^G(\mathbb{A}^3)$ is singular.

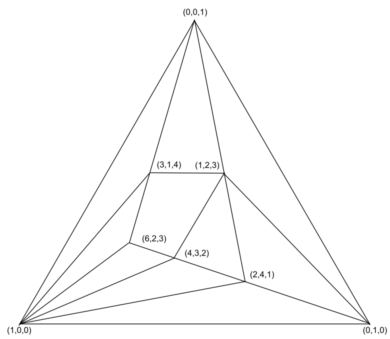

Figure 3. The intersection of the Gröbner fan for I' and $\{x + y + z = 1\}$

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Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan
E-mail: morita.t.ae@m.titech.ac.jp