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Carleson Measures and Toeplitz Operators for Weighted Bergman Spaces on the Unit Ball

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Abstract. We obtain some new characterizations on Carleson measures for weighted Bergman spaces on the unit ball involving product of functions. For these, we characterize bounded and compact Toeplitz operators between weighted Bergman spaces. The results are applied to characterize bounded and compact extended Cesàro operators and pointwise multiplication operators. The results are new even in the case of the unit disk.

1. Introduction

Let $\mathbb{C}^n$ denote the Euclidean space of complex dimension n. For any two points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we write $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, and $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : \ |z| < 1\}$ be the unit ball in $\mathbb{C}^n$. Let $H(\mathbb{B}_n)$ be the space of all holomorphic functions on the unit ball $\mathbb{B}_n$. Let dv be the normalized volume measure on $\mathbb{B}_n$ such that $v(\mathbb{B}_n) = 1$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, let $L^{p,\alpha} := L^p(\mathbb{B}_n, dv_\alpha)$ denote the weighted Lebesgue spaces that contain measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)\right)^{1/p} < \infty,$$

where $dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \, dv(z)$, and c_α is the normalized constant such that $v_\alpha(\mathbb{B}_n) = 1$. We also denote by $A^p_\alpha = L^p(\mathbb{B}_n, dv_\alpha) \cap H(\mathbb{B}_n)$ the weighted Bergman space on $\mathbb{B}_n$, with the same norm. If $\alpha = 0$, then we simply write them as $L^p(\mathbb{B}_n, dv)$ and A^p, respectively, and $\|f\|_p$ for the norm of f in these spaces.

Let μ be a positive Borel measure on $\mathbb{B}_n$. For $\lambda > 0$ and $\alpha > -1$, we say that μ is a (λ, α)-*Bergman–Carleson measure* if for any two positive numbers p and q with $q/p = \lambda$, there is a positive constant $C > 0$ such that

$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \le C\|f\|^q_{p,\alpha}$$

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for any $f \in A^p_\alpha$. We also denote

$$\|\mu\|_{\lambda,\alpha} = \sup_{f \in A^p_\alpha, \|f\|_{p,\alpha} \le 1} \int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z).$$

The concept of Carleson measures was first introduced by L. Carleson in order to study interpolating sequences and the corona problem [4; 5] for the algebra H^∞ of all bounded analytic functions on the unit disk. It quickly became a powerful tool for the study of function spaces and operators acting on them. The Bergman–Carleson measures were first studied by Hastings [10], and further pursued by Oleinik [18], Luecking [13; 14], Cima and Wogen [7], and many others.

In this paper we will give new characterizations for (λ, γ)-Bergman–Carleson measures and vanishing (λ, γ)-Bergman–Carleson measures (defined in Section 4) on the unit ball $\mathbb{B}_n$ by using products of functions in weighted Bergman spaces. In order to prove these results, we have to characterize bounded and compact Toeplitz operators between weighted Bergman spaces, which is of independent interest. Our results will be applied to study boundedness and compactness of extended Cesàro operators and pointwise multiplication operators from weighted Bergman spaces to a general family of function spaces.

THEOREM 1.1. *Let μ be a positive Borel measure on $\mathbb{B}_n$. For any integer $k \ge 1$ and $i = 1, 2, \dots, k$, let $0 < p_i, q_i < \infty$ and $-1 < \alpha_i < \infty$. Let*

$$\lambda = \sum_{i=1}^{k} \frac{q_i}{p_i}; \qquad \gamma = \frac{1}{\lambda} \sum_{i=1}^{k} \frac{\alpha_i q_i}{p_i}. \tag{1.1}$$

Then μ is a (λ, γ)-Bergman–Carleson measure if and only if there is a constant $C > 0$ such that for any $f_i \in A^{p_i}_{\alpha_i}$, $i = 1, 2, \dots, k$,

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{q_i} \, d\mu(z) \le C \prod_{i=1}^{k} \|f_i\|^{q_i}_{p_i,\alpha_i}. \tag{1.2}$$

A similar result for Hardy spaces on the unit disk was given by the second author in [26]. Due to lack of Riesz factorization theorem for weighted Bergman spaces, the proof of the theorem will be quite different and involved. For the proof of one implication in the case $0 < \lambda < 1$, a description of bounded Toeplitz operators between different Bergman spaces is needed. We state this result, which may be of independent interest, as a theorem. Given $\beta > -1$ and a positive Borel measure μ on $\mathbb{B}_n$, define the Toeplitz operator T^β_μ as follows:

$$T^\beta_\mu f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} \, d\mu(w), \quad z \in \mathbb{B}_n.$$

THEOREM 1.2. *Let $0 < p_1, p_2 < \infty$ and $-1 < \alpha_1, \alpha_2 < \infty$. Suppose that*

$$n + 1 + \beta > n \max\left(1, \frac{1}{p_i}\right) + \frac{1 + \alpha_i}{p_i}, \quad i = 1, 2.$$

Let

$$\lambda = 1 + \frac{1}{p_1} - \frac{1}{p_2}, \qquad \gamma = \frac{1}{\lambda}\left(\beta + \frac{\alpha_1}{p_1} - \frac{\alpha_2}{p_2}\right).$$

Let μ *be a positive Borel measure on* $\mathbb{B}_n$. *Then the following statements are equivalent*:

(i) T_μ^β *is bounded from* $A^{p_1}_{\alpha_1}$ *to* $A^{p_2}_{\alpha_2}$.

(ii) *The measure* μ *is a* (λ, γ)*-Bergman–Carleson measure.*

Moreover, we have

$$\|T_\mu^\beta\|_{A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}} \asymp \|\mu\|_{\lambda,\gamma}.$$

REMARK. In Theorem 1.2, the condition

$$n + 1 + \beta > n \max\left(1, \frac{1}{p_1}\right) + \frac{1 + \alpha_1}{p_1} \tag{1.3}$$

is used to prove that (i) implies (ii), whereas the condition

$$n + 1 + \beta > n \max\left(1, \frac{1}{p_2}\right) + \frac{1 + \alpha_2}{p_2} \tag{1.4}$$

is needed to prove that (ii) implies (i). Moreover, when $p_1 \geq 1$, condition (1.3) reduces to $(1 + \beta)p_1 > 1 + \alpha_1$, and by Theorem 2.11 of Zhu's book [28], this is equivalent to the fact that P_β is a bounded projection from $L^{p_1}(\mathbb{B}_n, dv_{\alpha_1})$ onto $A^{p_1}_{\alpha_1}$. Here, the projection P_β is defined as

$$P_\beta f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}}.$$

In a similar way, when $p_2 \geq 1$, condition (1.4) is equivalent to the fact that P_β is a bounded projection from $L^{p_2}(\mathbb{B}_n, dv_{\alpha_2})$ onto $A^{p_2}_{\alpha_2}$.

An account of the theory of Toeplitz operators acting on Bergman spaces can be found, for example, in [29, Chapter 7]. Theorems 1.1 and 1.2 are proved together: we first prove the sufficiency in Theorem 1.1, and this is applied in order to get the sufficiency in Theorem 1.2. After that, using standard test functions in the case $\lambda \geq 1$ and a method developed by Luecking [17] using Khinchine's inequality in the case $0 < \lambda < 1$, we get the necessity in Theorem 1.2. Finally, applying the result on Toeplitz operators and mixing several existing techniques in a non standard an maybe a new way, we prove the necessity in Theorem 1.1. Of particular importance in the proof is the technical Lemma 3.3, a result that can be of independent interest since it may have more applications to be discovered in the next years.

The product characterization of Carleson measures obtained in Theorem 1.1 can be applied to a number of questions that arise naturally in connection with function and operator theory in the ball. We are going to give some examples of that. First, an immediate consequence of Theorem 1.1 is the result stated in

Corollary 5.1: μ is a (λ, γ)-Bergman–Carleson measure if and only if for any $f \in A^p_\alpha$ and for some (any) $t > 0$,

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f(z)|^q \frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{s+t}}\, d\mu(z) \le C\|f\|^q_{p,\alpha}.$$

The expression of λ, γ in terms of the parameters p, q, and s and some conditions required on the parameters can be found in the statement of Corollary 5.1. This result is applied in order to study the boundedness and compactness of the extended Cesàro operators

$$J_g f(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t} \quad \text{and} \quad I_g f(z) = \int_0^1 Rf(tz) g(tz) \frac{dt}{t}$$

acting from the weighted Bergman spaces A^t_α to a general family of function spaces $F(p,q,s)$. Here Rf denotes the radial derivative of the function f. The operator J_g was first used by Ch. Pommerenke to characterize *BMOA* functions on the unit disk. It was first systematically studied by Aleman and Siskakis [2]. They proved that J_g is bounded on the Hardy space H^p on the unit disk if and only if $g \in BMOA$. Thereafter there have been many works on these operators. See [1; 3; 11; 19; 20; 21], and [23] for a few examples. The space $F(p,q,s)$ is defined as the space of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^{q+s} \frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{2s}}\, dv(z) < \infty,$$

where $0 < p < \infty$, $-n-1 < q < \infty$, $0 \le s < \infty$, and $q + s > -1$. The family of spaces $F(p,q,s)$ on the unit disk was introduced in [25]. It contains, as special cases, many classical function spaces, such as the analytic Besov spaces, weighted Bergman spaces, Dirichlet spaces, the Bloch space, and *BMOA* and Q_p spaces. See [25] for the details. For $F(p,q,s)$ on the unit ball, we refer to [24].

In Theorem 5.4 we give a complete description of the boundedness of $J_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$, and a similar description for the boundedness of I_g is obtained in Theorem 5.5. As a consequence, a characterization of the pointwise multipliers from A^t_α to $F(p, p\beta - n - 1, s)$ is obtained in Theorem 5.8. It looks possible to obtain these results directly (essentially by extracting the relevant parts in the proof of our Theorem 1.1), but the use of Theorem 1.1 gives a better understanding of what is going on, and it looks that more applications in the setting of operator theory can be discovered in the future.

The paper is organized as follows. In Section 2 we recall some notation and preliminary results, which will be used later. Section 3 is devoted to the proofs of our main results, Theorem 1.1 and Theorem 1.2. In Section 4 we give similar characterizations for vanishing (λ, γ)-Bergman–Carleson measures. In Section 5 we apply Theorem 1.1 to characterize bounded extended Cesàro operators and pointwise multiplication operators from weighted Bergman spaces into a general family of function spaces.

In the following, the notation $A \lesssim B$ means that there is a positive constant C such that $A \le CB$, and the notation $A \asymp B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

In this section we introduce some notation and recall some well-known results that will be used throughout the paper.

For any $a \in \mathbb{B}_n$ with $a \neq 0$, we denote by $\phi_a(z)$ the Möbius transformation on $\mathbb{B}_n$ that interchanges the points 0 and a. It is known that ϕ_a satisfies the following properties: $\phi_a \circ \phi_a(z) = z$, and

$$1 - |\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}, \quad a, z \in \mathbb{B}_n. \tag{2.1}$$

For $z, w \in \mathbb{B}_n$, the *pseudo-hyperbolic distance* between z and w is defined by

$$\rho(z, w) = |\phi_z(w)|,$$

and the *hyperbolic distance* on $\mathbb{B}_n$ between z and w induced by the Bergman metric is given by

$$\beta(z, w) = \tanh \rho(z, w) = \frac{1}{2} \log \frac{1+|\phi_z(w)|}{1-|\phi_z(w)|}.$$

For $z \in \mathbb{B}_n$ and $r > 0$, the *Bergman metric ball* at z is given by

$$D(z, r) = \{w \in \mathbb{B}_n : \ \beta(z, w) < r\}.$$

It is known that, for a fixed $r > 0$, the weighted volume

$$v_\alpha(D(z, r)) \asymp (1-|z|^2)^{n+1+\alpha}.$$

We refer to [28] for these facts.

We cite two results for Bergman–Carleson measures that justify the fact that a Bergman–Carleson measure depends only on α and the ratio $\lambda = q/p$. The first result was obtained by several authors and can be found, for example, in [27, Theorem 50] and the references there.

THEOREM A. *For a positive Borel measure μ on $\mathbb{B}_n$, $0 < p \le q < \infty$, and $-1 < \alpha < \infty$, the following statements are equivalent*:

(i) *There is a constant $C_1 > 0$ such that, for any $f \in A^p_\alpha$,*

$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \le C_1 \|f\|^q_{p,\alpha}.$$

(ii) *There is a constant $C_2 > 0$ such that, for any real number r with $0 < r < 1$ and any $z \in \mathbb{B}_n$,*

$$\mu(D(z, r)) \le C_2 (1-|z|^2)^{(n+1+\alpha)q/p}.$$

(iii) *There is a constant $C_3 > 0$ such that, for some (every) $t > 0$,*

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^t}{|1-\langle z, a\rangle|^{[(n+1+\alpha)q/p]+t}} \, d\mu(z) \le C_3.$$

Furthermore, the constants C_1, C_2, *and* C_3 *are all comparable to* $\|\mu\|_{\lambda,\alpha}$ *with* $\lambda = q/p$.

REMARK. Let $\lambda = q/p$. Then this result states that a positive Borel measure μ on $\mathbb{B}_n$ is a (λ, α)-Bergman–Carleson measure if and only if

$$\sup_{s\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{(n+1+\alpha)\lambda+t}}\, d\mu(z) < \infty$$

for some (every) $t > 0$.

For the case $0 < q < p < \infty$, we need a well-known result on decomposition of the unit ball $\mathbb{B}_n$. A sequence $\{a_k\}$ of points in $\mathbb{B}_n$ is called a *separated sequence* (in the Bergman metric) if there exists a positive constant $\delta > 0$ such that $\beta(z_i, z_j) > \delta$ for any $i \neq j$. The following result is Theorem 2.23 in [28].

LEMMA A. *There exists a positive integer* N *such that for any* $0 < r < 1$, *we can find a sequence* $\{a_k\}$ *in* $\mathbb{B}_n$ *with the following properties*:

(i) $\mathbb{B}_n = \bigcup_k D(a_k, r)$.
(ii) *The sets* $D(a_k, r/4)$ *are mutually disjoint.*
(iii) *Each point* $z \in \mathbb{B}_n$ *belongs to at most* N *of the sets* $D(a_k, 4r)$.

Any sequence $\{a_k\}$ satisfying the conditions of the lemma is called a *lattice* (or an r-*lattice* if one wants to stress the dependence on r) in the Bergman metric. Obviously, any r-lattice is separated. For convenience, we will denote by $D_k = D(a_k, r)$ and $\tilde{D}_k = D(a_k, 4r)$. Then Lemma A says that $\mathbb{B}_n = \bigcup_{k=1}^{\infty} D_k$ and there is a positive integer N such that every point z in $\mathbb{B}_n$ belongs to at most N of sets $\tilde{D}_k$.

The following result is essentially due to Luecking [16; 17] for the case $\alpha = 0$ (note that the discrete form (iii) is actually given in Luecking's proof). For $-1 < \alpha < \infty$, the result can be similarly proved as in [17]. The condition in part (iv) first appeared in [6] (see also [27, Theorem 54]), where it was used for the embedding of harmonic Bergman spaces into Lebesgue spaces.

THEOREM B. *For a positive Borel measure* μ *on* $\mathbb{B}_n$, $0 < q < p < \infty$, *and* $-1 < \alpha < \infty$, *the following statements are equivalent*:

(i) *There is a constant* $C_1 > 0$ *such that, for any* $f \in A^p_\alpha$,

$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \le C_1 \|f\|^q_{p,\alpha}.$$

(ii) *The function*

$$\widehat{\mu}_r(z) := \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha}}$$

is in $L^{p/(p-q),\alpha}$ *for any (some) fixed* $r \in (0,1)$.

(iii) *For any* r-*lattice* $\{a_k\}$ *and* D_k *as in Lemma* A, *the sequence*

$$\{\mu_k\} := \left\{ \frac{\mu(D_k)}{(1-|a_k|^2)^{(n+1+\alpha)(q/p)}} \right\}$$

belongs to $\ell^{p/(p-q)}$ *for any (some) fixed* $r \in (0,1)$.

(iv) *For any* $s > 0$, *the Berezin-type transform* $B_{s,\alpha}(\mu)$ *belongs to* $L^{p/(p-q),\alpha}$.

Furthermore, with $\lambda = q/p$, *we have*

$$\|\widehat{\mu}_r\|_{p/(p-q),\alpha} \asymp \|\{\mu_k\}\|_{\ell^{p/(p-q)}} \asymp \|B_{s,\alpha}(\mu)\|_{p/(p-q),\alpha} \asymp \|\mu\|_{\lambda,\alpha}.$$

Here, for a positive measure ν, the Berezin-type transform $B_{s,\alpha}(\nu)$ is

$$B_{s,\alpha}(\nu)(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{n+1+s+\alpha}}\, d\nu(w).$$

As a consequence of Theorem B, for $0 < \lambda < 1$, a positive Borel measure μ on $\mathbb{B}_n$ is a (λ,α)-Bergman–Carleson measure if and only if

$$\mu(D(z,r))(1-|z|^2)^{-n-1-\alpha} \in L^{1/(1-\lambda),\alpha}$$

or

$$\{\mu(D_k)(1-|a_k|^2)^{-(n+1+\alpha)\lambda}\} \in \ell^{1/(1-\lambda)}$$

for any (some) fixed $r \in (0,1)$.

The following integral estimate (see [28, Thm. 1.12]) has become indispensable in this area of analysis and will be used several times in this paper.

LEMMA B. *Suppose* $z \in \mathbb{B}_n$, $c > 0$ *and* $t > -1$. *The integral*

$$I_{c,t}(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\, dv(w)$$

is comparable to $(1-|z|^2)^{-c}$.

We also need a well known variant of the previous lemma.

LEMMA C. *Let* $\{z_k\}$ *be a separated sequence in* $\mathbb{B}_n$, *and let* $n < t < s$. *Then*

$$\sum_{k=1}^{\infty} \frac{(1-|z_k|^2)^t}{|1-\langle z,z_k\rangle|^s} \le C(1-|z|^2)^{t-s}, \quad z \in \mathbb{B}_n.$$

Lemma C can be deduced from Lemma B after noticing that, if a sequence $\{z_k\}$ is separated, then there is a constant $r > 0$ such that the Bergman metric balls $D(z_k,r)$ are pairwise disjoint. With all these preparations, now we are ready to prove the main results.

3. Proofs of Theorems 1.1 and 1.2

We first need the following lemma.

LEMMA 3.1. *Let* $-1 < \alpha < \infty$. *For* $i = 1,2,\dots,k$, *let* $0 < p_i, q_i < \infty$, *and let* $f_i \in A^{p_i/q_i}_{\alpha_i}$. *Let*

$$\lambda = \sum_{i=1}^{k} \frac{q_i}{p_i}; \qquad \gamma = \frac{1}{\lambda}\sum_{i=1}^{k} \frac{\alpha_i q_i}{p_i}.$$

Then $\prod_{i=1}^{k} f_i \in A_{\gamma}^{1/\lambda}$, *and*

$$\left\|\prod_{i=1}^{k} f_i\right\|_{1/\lambda,\gamma} \lesssim \prod_{i=1}^{k} \|f_i\|_{p_i/q_i,\alpha_i}.$$

Proof. Let $f_i \in A_{\alpha_i}^{p_i/q_i}$ $(i = 1, 2, \dots, k)$. Since $p_i\lambda/q_i > 1$ for any $i = 1, 2, \dots, k$, we can apply Hölder's inequality to obtain

$$\begin{aligned}
&\left\|\prod_{i=1}^{k} f_i\right\|_{1/\lambda,\gamma} \\
&\quad = \left(c_\gamma \int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{1/\lambda}(1-|z|^2)^{\gamma}\, dv(z)\right)^{\lambda} \\
&\quad \lesssim \prod_{i=1}^{k}\left(\int_{\mathbb{B}_n} |f_i(z)|^{(1/\lambda)(p_i\lambda/q_i)}(1-|z|^2)^{(q_i\alpha_i/(p_i\lambda))(p_i\lambda/q_i)}\, dv(z)\right)^{q_i/p_i} \\
&\quad = \prod_{i=1}^{k}\left(\int_{\mathbb{B}_n} |f_i(z)|^{p_i/q_i}(1-|z|^2)^{\alpha_i}\, dv(z)\right)^{q_i/p_i} \\
&\quad \lesssim \prod_{i=1}^{k} \|f_i\|_{p_i/q_i,\alpha_i}.
\end{aligned}$$

The result is proved. □

PROPOSITION 3.2. *Let* μ *be a positive Borel measure on* $\mathbb{B}_n$. *For any integer* $k \geq 1$ *and* $i = 1, 2, \dots, k$, *let* $0 < p_i, q_i < \infty$ *and* $-1 < \alpha_i < \infty$, *and let* λ *and* γ *be as in* (1.1). *If* μ *is a* (λ, γ)*-Bergman–Carleson measure*, *then* (1.2) *holds.*

Proof. If $k = 1$, then the result is just the definition. Let us now assume that $k \geq 2$. Let $h_i \in A_{\alpha_i}^{p_i/q_i}$, $i = 1, 2, \dots, k$. By Lemma 3.1, $\prod_{i=1}^{k} h_i \in A_{\gamma}^{1/\lambda}$, and

$$\left\|\prod_{i=1}^{k} h_i\right\|_{1/\lambda,\gamma} \lesssim \prod_{i=1}^{k} \|h_i\|_{p_i/q_i,\alpha_i}.$$

Since μ is a (λ, γ)-Bergman–Carleson measure,

$$\int_{\mathbb{B}_n}\left|\prod_{i=1}^{k} h_i(z)\right| d\mu(z) \leq C\left\|\prod_{i=1}^{k} h_i\right\|_{1/\lambda,\gamma} \leq C\prod_{i=1}^{k} \|h_i\|_{p_i/q_i,\alpha_i}. \tag{3.1}$$

Let

$$d\mu_1 = \left(\prod_{i=2}^{k} |h_i|\, d\mu\right) \Big/ \left(\prod_{i=2}^{k} \|h_i\|_{p_i/q_i,\alpha_i}\right).$$

Then (3.1) is equivalent to

$$\int_{\mathbb{B}_n} |h_1(z)|\, d\mu_1(z) \leq C\|h_1\|_{p_1/q_1,\alpha_1}.$$

Thus, μ_1 is a $(q_1/p_1, \alpha_1)$-Bergman–Carleson measure. Thus, for any $f_1 \in A^{p_1}_{\alpha_1}$,

$$\int_{\mathbb{B}_n} |f_1(z)|^{q_1}\, d\mu_1(z) \le C\|f_1\|^{q_1}_{p_1,\alpha_1},$$

which is the same as

$$\int_{\mathbb{B}_n} |f_1(z)|^{q_1} \prod_{i=2}^{k} |h_i(z)|\, d\mu(z) \le C\|f_1\|^{q_1}_{p_1,\alpha_1} \prod_{i=2}^{k} \|h_i\|_{p_i/q_i,\alpha_i}. \tag{3.2}$$

Let

$$d\mu_2 = \left(|f_1|^{q_1} \prod_{i=3}^{k} |h_i|\, d\mu\right) \Big/ \left(\|f_1\|^{q_1}_{p_1,\alpha_1} \prod_{i=3}^{k} \|h_i\|_{p_i/q_i,\alpha_i}\right).$$

Then (3.2) is the same as

$$\int_{\mathbb{B}_n} |h_2(z)|\, d\mu_2(z) \le C\|h_2\|_{p_2/q_2,\alpha_2}.$$

Thus, μ_2 is a $(q_2/p_2, \alpha_2)$-Bergman–Carleson measure. Thus, for any $f_2 \in A^{p_2}_{\alpha_2}$,

$$\int_{\mathbb{B}_n} |f_2(z)|^{q_2}\, d\mu_2(z) \le C\|f_2\|^{q_2}_{p_2,\alpha_2}$$

or

$$\int_{\mathbb{B}_n} |f_1(z)|^{q_1} |f_2(z)|^{q_2} \prod_{i=3}^{k} |h_i(z)|\, d\mu(z) \le C\|f_1\|^{q_1}_{p_1,\alpha_1} \|f_2\|^{q_2}_{p_2,\alpha_2} \prod_{i=3}^{k} \|h_i\|_{p_i/q_i,\alpha_i}.$$

Continuing this process we will eventually get (1.2). □

3.1. *Proof of Theorem* 1.2

3.1.1. (i) *Implies* (ii). We divide this part into two cases: $\lambda \ge 1$ and $0 < \lambda < 1$.

Case 1: $\lambda \ge 1$. Fix $a \in \mathbb{B}_n$ and let $f_a(z) = (1 - \langle z, a\rangle)^{-(n+1+\beta)}$. Under the condition $(n+1+\beta)p_1 > n+1+\alpha_1$, it is easy to check using Lemma B that $f_a \in A^{p_1}_{\alpha_1}$ with

$$\|f_a\|^{p_1}_{p_1,\alpha_1} \lesssim (1 - |a|^2)^{(n+1+\alpha_1)-(n+1+\beta)p_1}.$$

Since

$$\begin{aligned} T^\beta_\mu f_a(z) &= \int_{\mathbb{B}_n} \frac{f_a(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}}\, d\mu(w) \\ &= \int_{\mathbb{B}_n} \frac{d\mu(w)}{(1 - \langle z, w\rangle)^{n+1+\beta} \cdot (1 - \langle w, a\rangle)^{n+1+\beta}}, \end{aligned}$$

we get

$$T^\beta_\mu f_z(z) = \int_{\mathbb{B}_n} \frac{d\mu(w)}{|1 - \langle z, w\rangle|^{2(n+1+\beta)}} \ge C \frac{\mu(D(z, r))}{(1 - |z|^2)^{2(n+1+\beta)}}.$$

On the other hand, by the pointwise estimate for functions in Bergman spaces (see [28, Thm. 2.1]) together with the boundedness of the Toeplitz operator T_μ^β, we get

$$\begin{aligned} T_\mu^\beta f_z(z) = |T_\mu^\beta f_z(z)| &\le \|T_\mu^\beta f_z\|_{p_2,\alpha_2}(1-|z|^2)^{-(n+1+\alpha_2)/p_2} \\ &\le \|T_\mu^\beta\| \cdot \|f_z\|_{p_1,\alpha_1}(1-|z|^2)^{-(n+1+\alpha_2)/p_2} \\ &\lesssim \|T_\mu^\beta\|(1-|z|^2)^{(n+1+\alpha_1)/p_1-(n+1+\alpha_2)/p_2-(n+1+\beta)}. \end{aligned}$$

Hence,

$$\begin{aligned} \mu(D(z,r)) &\lesssim \|T_\mu^\beta\|(1-|z|^2)^{(n+1+\beta)+(n+1+\alpha_1)/p_1-(n+1+\alpha_2)/p_2} \\ &= \|T_\mu^\beta\|(1-|z|^2)^{(n+1+\gamma)\lambda}. \end{aligned}$$

By Theorem A this means that μ is a (λ,γ)-Bergman–Carleson measure with

$$\|\mu\|_{\lambda,\gamma} \lesssim \|T_\mu^\beta\|.$$

Case 2: $0<\lambda<1$. Notice that the condition $0<\lambda<1$ is equivalent to $0<p_2<p_1<\infty$. Let $r_k(t)$ be a sequence of Rademacher functions (see [9, App. A]), and $\{a_k\}$ be any r-lattice on $\mathbb{B}_n$. Since

$$n+1+\beta > n\max\left(1,\frac{1}{p_1}\right)+\frac{1+\alpha_1}{p_1},$$

we know from Theorem 2.30 in [28] that, for any sequence of real numbers $\{\lambda_k\}\in\ell^{p_1}$, the function

$$f_t(z)=\sum_{k=1}^\infty \lambda_k r_k(t)\frac{(1-|a_k|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1}}{(1-\langle z,a_k\rangle)^{n+1+\beta}}$$

is in $A^{p_1}_{\alpha_1}$ with $\|f_t\|_{p_1,\alpha_1}\lesssim \|\{\lambda_k\}\|_{\ell^{p_1}}$ for almost every t in $(0,1)$. Denote by

$$f_k(z)=\frac{(1-|a_k|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1}}{(1-\langle z,a_k\rangle)^{n+1+\beta}}.$$

Since T_μ^β is bounded from $A^{p_1}_{\alpha_1}$ to $A^{p_2}_{\alpha_2}$, we get that for almost every t in $(0,1)$,

$$\begin{aligned} \|T_\mu^\beta f_t\|^{p_2}_{p_2,\alpha_2} &= \int_{\mathbb{B}_n}\left|\sum_{k=1}^\infty \lambda_k r_k(t) T_\mu^\beta f_k(z)\right|^{p_2} dv_{\alpha_2}(z) \\ &\lesssim \|T_\mu^\beta\|^{p_2}\cdot\|f_t\|^{p_2}_{p_1,\alpha_1} \lesssim \|T_\mu^\beta\|^{p_2}\left(\sum_{k=1}^\infty|\lambda_k|^{p_1}\right)^{p_2/p_1}. \end{aligned}$$

Integrating both sides with respect to t from 0 to 1, and using Fubini's theorem and Khinchine's inequality (see, e.g., [17]), we get

$$\int_{\mathbb{B}_n}\left(\sum_{k=1}^\infty |\lambda_k|^2|T_\mu^\beta f_k(z)|^2\right)^{p_2/2} dv_{\alpha_2}(z)\lesssim \|T_\mu^\beta\|^{p_2}\cdot\|\{\lambda_k\}\|^{p_2}_{\ell^{p_1}}. \tag{3.3}$$

Let $\{D_k\}$ be the associated sets to the lattice $\{a_k\}$ in Lemma A. Then

$$\begin{aligned}\sum_{k=1}^{\infty}|\lambda_k|^{p_2}\int_{\tilde{D}_k}|T_\mu^\beta f_k(z)|^{p_2}\,dv_{\alpha_2}(z)\\ =\int_{\mathbb{B}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^{p_2}|T_\mu^\beta f_k(z)|^{p_2}\chi_{\tilde{D}_k}(z)\right)dv_{\alpha_2}(z).\end{aligned}$$

If $p_2\geq 2$, then $2/p_2\leq 1$, and from the fact that ℓ^1 injects continuously into $\ell^{p_2/2}$ we have

$$\begin{aligned}&\sum_{k=1}^{\infty}|\lambda_k|^{p_2}\int_{\tilde{D}_k}|T_\mu^\beta f_k(z)|^{p_2}\,dv_{\alpha_2}(z)\\ &\leq\int_{\mathbb{B}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^{2}|T_\mu^\beta f_k(z)|^{2}\chi_{\tilde{D}_k}(z)\right)^{p_2/2}dv_{\alpha_2}(z)\\ &\leq\int_{\mathbb{B}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^{2}|T_\mu^\beta f_k(z)|^{2}\right)^{p_2/2}dv_{\alpha_2}(z).\end{aligned}$$

If $0<p_2<2$, then $2/p_2>1$. Thus, by Hölder's inequality we get

$$\begin{aligned}&\sum_{k=1}^{\infty}|\lambda_k|^{p_2}\int_{\tilde{D}_k}|T_\mu^\beta f_k(z)|^{p_2}\,dv_{\alpha_2}(z)\\ &\leq\int_{\mathbb{B}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^{2}|T_\mu^\beta f_k(z)|^{2}\chi_{\tilde{D}_k(z)}\right)^{p_2/2}\left(\sum_{k}\chi_{\tilde{D}_k}(z)\right)^{1-p_2/2}dv_{\alpha_2}(z)\\ &\leq N^{1-p_2/2}\int_{\mathbb{B}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^{2}|T_\mu^\beta f_k(z)|^{2}\right)^{p_2/2}dv_{\alpha_2}(z)\end{aligned}$$

since any point z belongs to at most N of the sets $\tilde{D}_k$. Combining the last two inequalities and applying (3.3), we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}|\lambda_k|^{p_2}\int_{\tilde{D}_k}|T_\mu^\beta f_k(z)|^{p_2}\,dv_{\alpha_2}(z)\\ &\leq\max\{1,N^{1-p_2/2}\}\int_{\mathbb{B}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^{2}|T_\mu^\beta f_k(z)|^{2}\right)^{p_2/2}dv_{\alpha_2}(z)\\ &\lesssim\|T_\mu^\beta\|^{p_2}\cdot\|\{\lambda_k\}\|_{\ell^{p_1}}^{p_2}.\end{aligned}$$

Since, by subharmonicity (see [28, Lemma 2.24]) we have

$$|T_\mu^\beta f_k(a_k)|^{p_2}\lesssim\frac{1}{(1-|a_k|^2)^{n+1+\alpha_2}}\int_{\tilde{D}_k}|T_\mu^\beta f_k(z)|^{p_2}\,dv_{\alpha_2}(z),$$

we get

$$\sum_{k=1}^{\infty} |\lambda_k|^{p_2} (1-|a_k|^2)^{n+1+\alpha_2} |T_\mu^\beta f_k(a_k)|^{p_2} \lesssim \|T_\mu^\beta\|^{p_2} \cdot \|\{\lambda_k\}\|_{\ell^{p_1}}^{p_2}. \tag{3.4}$$

Now, notice that

$$T_\mu^\beta f_k(a_k) = (1-|a_k|^2)^{n+1+\beta-(n+1+\alpha_1)/p_1} \int_{\mathbb{B}_n} \frac{d\mu(w)}{|1-\langle w, a_k\rangle|^{2(n+1+\beta)}}.$$

Therefore,

$$\frac{\mu(D_k)}{(1-|a_k|^2)^{n+1+\beta+(n+1+\alpha_1)/p_1}} \lesssim T_\mu^\beta f_k(a_k),$$

and putting this into (3.4), we get

$$\sum_{k=1}^{\infty} |\lambda_k|^{p_2} \left(\frac{\mu(D_k)}{(1-|a_k|^2)^s} \right)^{p_2} \lesssim \|T_\mu^\beta\|^{p_2} \cdot \|\{\lambda_k\}\|_{\ell^{p_1}}^{p_2}$$

with

$$s = n+1+\beta + \frac{(n+1+\alpha_1)}{p_1} - \frac{(n+1+\alpha_2)}{p_2} = (n+1+\gamma)\lambda. \tag{3.5}$$

Since the conjugate exponent of (p_1/p_2) is $(p_1/p_2)' = p_1/(p_1-p_2)$ by duality, we know that

$$\{\nu_k\} := \left\{ \left(\frac{\mu(D_k)}{(1-|a_k|^2)^s} \right)^{p_2} \right\} \in \ell^{p_1/(p_1-p_2)}$$

with

$$\|\{\nu_k\}\|_{\ell^{p_1/(p_1-p_2)}} \lesssim \|T_\mu^\beta\|^{p_2}$$

or

$$\{\mu_k\} := \left\{ \frac{\mu(D_k)}{(1-|a_k|^2)^{(n+1+\gamma)\lambda}} \right\} \in \ell^{p_1 p_2/(p_1-p_2)} = \ell^{1/(1-\lambda)}$$

with

$$\|\{\mu_k\}\|_{\ell^{1/(1-\lambda)}} = \|\{\nu_k\}\|_{\ell^{p_1/(p_1-p_2)}}^{1/p_2} \lesssim \|T_\mu^\beta\|.$$

By Theorem B this means that μ is a (λ, γ)-Bergman–Carleson measure with

$$\|\mu\|_{\lambda,\gamma} \lesssim \|T_\mu^\beta\|.$$

3.1.2. (ii) *Implies* (i). Now suppose (ii) holds, that is, μ is a (λ, γ)-Bergman–Carleson measure. We show that this implies (i). We divide the proof into three cases.

Case 1: $p_2 > 1$. For this case, let p_2' and α_2' be two numbers satisfying

$$\frac{1}{p_2} + \frac{1}{p_2'} = 1; \qquad \frac{\alpha_2}{p_2} + \frac{\alpha_2'}{p_2'} = \beta. \tag{3.6}$$

Then

$$\alpha_2' = \left(\beta - \frac{\alpha_2}{p_2}\right) p_2' = \frac{\beta p_2 - \alpha_2}{p_2 - 1} > -1$$

since $\beta > (1+\alpha_2)/p_2 - 1$. By a duality result due to Luecking (see [15] or [28, Thm. 2.12]), we know that $(A^{p_2}_{\alpha_2})^* = A^{p_2'}_{\alpha_2'}$ under the integral pairing

$$\langle f, g\rangle_\beta = \int_{\mathbb{B}_n} f(z)\overline{g(z)}\, dv_\beta(z).$$

Let $f \in A^{p_1}_{\alpha_1}$ and $h \in A^{p_2'}_{\alpha_2'}$. An easy computation using Fubini's theorem and the reproducing formula for Bergman spaces shows

$$\langle h, T^\beta_\mu f\rangle_\beta = \int_{\mathbb{B}_n} h(z)\overline{f(z)}\, d\mu(z).$$

The conditions for λ and γ in the theorem are equivalent to

$$\lambda = \frac{1}{p_1} + \frac{1}{p_2'}, \qquad \gamma = \frac{1}{\lambda}\left(\frac{\alpha_1}{p_1} + \frac{\alpha_2'}{p_2'}\right).$$

Thus, by Proposition 3.2,

$$|\langle h, T^\beta_\mu f\rangle_\beta| \le \int_{\mathbb{B}_n} |h(z)||f(z)|\, d\mu(z) \lesssim \|\mu\|_{\lambda,\gamma} \cdot \|f\|_{p_1,\alpha_1} \cdot \|h\|_{p_2',\alpha_2'}.$$

Hence, T^β_μ is bounded from $A^{p_1}_{\alpha_1}$ to $A^{p_2}_{\alpha_2}$ with $\|T^\beta_\mu\| \lesssim \|\mu\|_{\lambda,\gamma}$.

Case 2: $p_2 = 1$. Let $f \in A^{p_1}_{\alpha_1}$. For this case, since $\beta > (1+\alpha_2)/1 - 1 = \alpha_2$, by Fubini's theorem and Lemma B we have

$$\begin{aligned}
\|T^\beta_\mu f\|_{1,\alpha_2} &\le \int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n} \frac{|f(w)|}{|1-\langle z, w\rangle|^{n+1+\beta}}\, d\mu(w)\right) dv_{\alpha_2}(z) \\
&= \int_{\mathbb{B}_n} |f(w)|\left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\alpha_2}}{|1-\langle z, w\rangle|^{n+1+\beta}}\, dv(z)\right) d\mu(w) \\
&\lesssim \int_{\mathbb{B}_n} |f(w)|(1-|w|^2)^{\alpha_2-\beta}\, d\mu(w).
\end{aligned} \tag{3.7}$$

Let ν be the measure defined by $d\nu(w) = (1-|w|^2)^{\alpha_2-\beta}\, d\mu(w)$. Since μ is a (λ, γ)-Bergman–Carleson measure, using Theorems A and B, we easily see that ν is a $(1/p_1, \alpha_1)$-Bergman–Carleson measure and, moreover, $\|\nu\|_{1/p_1,\alpha_1} \lesssim \|\mu\|_{\lambda,\gamma}$. Thus, for any $f \in A^{p_1}_{\alpha_1}$, we have

$$\int_{\mathbb{B}_n} |f(w)|\, d\nu(w) \lesssim \|\nu\|_{1/p_1,\alpha_1} \cdot \|f\|_{p_1,\alpha_1} \lesssim \|\mu\|_{\lambda,\gamma} \cdot \|f\|_{p_1,\alpha_1}.$$

Thus, by (3.7) it follows that

$$\|T^\beta_\mu f\|_{1,\alpha_2} \lesssim \|\mu\|_{\lambda,\gamma} \cdot \|f\|_{p_1,\alpha_1},$$

and so T^β_μ is bounded from $A^{p_1}_{\alpha_1}$ to $A^{p_2}_{\alpha_2}$ with $\|T^\beta_\mu\| \lesssim \|\mu\|_{\lambda,\gamma}$.

Case 3: $0 < p_2 < 1$. Let $\{a_k\}$ be an r-lattice of $\mathbb{B}_n$ in the Bergman metric, and $\{D_k\}$ be the corresponding sets as in Lemma A. Then we know that

$\mathbb{B}_n = \bigcup_{k=1}^{\infty} D_k$ and there is a positive integer N such that each point in $\mathbb{B}_n$ belongs to at most N of the sets $\tilde{D}_k$. Then

$$|T_\mu^\beta f(z)| \lesssim \sum_{k=1}^{\infty} \int_{D_k} \frac{|f(w)|}{|1-\langle z,w\rangle|^{n+1+\beta}}\, d\mu(w)$$
$$\lesssim \sum_{k=1}^{\infty} \frac{1}{|1-\langle z,a_k\rangle|^{n+1+\beta}} \int_{D_k} |f(w)|\, d\mu(w).$$

Now, for $w \in D_k$, we have

$$|f(w)|^{p_1} \lesssim \frac{1}{(1-|a_k|^2)^{n+1+\alpha_1}} \int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z).$$

From this we get

$$\int_{D_k} |f(w)|\, d\mu(w) \lesssim \frac{1}{(1-|a_k|^2)^{(n+1+\alpha_1)/p_1}} \left(\int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z) \right)^{1/p_1} \mu(D_k).$$

Since $0 < p_2 < 1$, this implies

$$|T_\mu^\beta f(z)|^{p_2} \lesssim \sum_{k=1}^{\infty} \frac{1}{|1-\langle z,a_k\rangle|^{(n+1+\beta)p_2}} \frac{\mu(D_k)^{p_2}}{(1-|a_k|^2)^{(n+1+\alpha_1)(p_2/p_1)}}$$
$$\times \left(\int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z) \right)^{p_2/p_1}.$$

Therefore, since $(n+1+\beta)p_2 > n+1+\alpha_2$, we can apply Lemma B to obtain

$$\|T_\mu^\beta f\|_{p_2,\alpha_2}^{p_2} \lesssim \sum_{k=1}^{\infty} \frac{\mu(D_k)^{p_2}}{(1-|a_k|^2)^{(n+1+\alpha_1)(p_2/p_1)}} \left(\int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z) \right)^{p_2/p_1}$$
$$\times \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\alpha_2}\, dv(z)}{|1-\langle z,a_k\rangle|^{(n+1+\beta)p_2}}$$
$$\lesssim \sum_{k=1}^{\infty} \frac{\mu(D_k)^{p_2}}{(1-|a_k|^2)^{(n+1+\alpha_1)(p_2/p_1)}} \left(\int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z) \right)^{p_2/p_1}$$
$$\times (1-|a_k|^2)^{n+1+\alpha_2-(n+1+\beta)p_2}. \tag{3.8}$$

First, assume that $\lambda \geq 1$. Since μ is a (λ,γ)-Bergman–Carleson measure, by Theorem A we get

$$\mu(D_k) \lesssim \|\mu\|_{\lambda,\gamma} (1-|a_k|^2)^{(n+1+\gamma)\lambda}.$$

Bearing in mind (3.5), this, together with (3.8) and the fact that $p_2 \geq p_1$ (due to the assumption $\lambda \geq 1$), yields

$$\|T_\mu^\beta f\|_{p_2,\alpha_2}^{p_2} \lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \sum_{k=1}^{\infty} \left(\int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z) \right)^{p_2/p_1}$$
$$\lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \left(\sum_{k=1}^{\infty} \int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z) \right)^{p_2/p_1} \lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \cdot \|f\|_{p_1,\alpha_1}^{p_2}.$$

Hence, T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$ with $\|T_\mu^\beta\| \lesssim \|\mu\|_{\lambda,\gamma}$. Next, assume that $0 < \lambda < 1$. Then $p_1 > p_2$, and using Hölder's inequality in (3.8), we get

$$\begin{aligned}\|T_\mu^\beta f\|_{p_2,\alpha_2}^{p_2} &\lesssim \sum_{k=1}^{\infty} \frac{\mu(D_k)^{p_2}}{(1-|a_k|^2)^{(n+1+\gamma)\lambda p_2}} \left(\int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z)\right)^{p_2/p_1} \\ &\le \left\{\sum_{k=1}^{\infty} \left[\frac{\mu(D_k)^{p_2}}{(1-|a_k|^2)^{(n+1+\gamma)\lambda p_2}}\right]^{p_1/(p_1-p_2)}\right\}^{1-p_2/p_1} \\ &\quad \times \left(\sum_{k=1}^{\infty} \int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z)\right)^{p_2/p_1}.\end{aligned}$$

Since μ is a (λ,γ)-Bergman–Carleson measure, by Theorem B we get that

$$\begin{aligned}\sum_{k=1}^{\infty} \left[\frac{\mu(D_k)^{p_2}}{(1-|a_k|^2)^{(n+1+\gamma)\lambda p_2}}\right]^{p_1/(p_1-p_2)} &= \sum_{k=1}^{\infty} \left[\frac{\mu(D_k)}{(1-|a_k|^2)^{(n+1+\gamma)\lambda}}\right]^{1/(1-\lambda)} \\ &\lesssim \|\mu\|_{\lambda,\gamma}^{1/(1-\lambda)} = \|\mu\|_{\lambda,\gamma}^{p_1p_2/(p_1-p_2)},\end{aligned}$$

and so

$$\|T_\mu^\beta f\|_{p_2,\alpha_2}^{p_2} \lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \left(\sum_{k=1}^{\infty} \int_{\tilde{D}_k} |f(z)|^{p_1}\, dv_{\alpha_1}(z)\right)^{p_2/p_1} \lesssim \|\mu\|_{\lambda,\gamma}^{p_2} \cdot \|f\|_{p_1,\alpha_1}^{p_2}.$$

Hence, T_μ^β is bounded from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$ with $\|T_\mu^\beta\| \lesssim \|\mu\|_{\lambda,\gamma}$. The proof is complete.

3.2. A Key Lemma

Now, we are going to use the result just proved on Toeplitz operators to obtain the following technical result that will be the key for the proof of the remaining part in Theorem 1.1.

LEMMA 3.3. *Let μ be a positive Borel measure on the unit ball $\mathbb{B}_n$. For $s, r > 0$ and $\alpha_1 > -1$, let*

$$S_{\mu,\alpha_1}^r f(z) = (1-|z|^2)^s \int_{\mathbb{B}_n} \frac{|f(w)|^r\, d\mu(w)}{|1-\langle z,w\rangle|^{n+1+s+\alpha_1}}.$$

For $q > 1$, $p > 0$, and $\alpha_2 > -1$, let

$$\lambda = 1 + \frac{r}{p} - \frac{1}{q} \quad \textit{and} \quad \gamma = \frac{1}{\lambda}\left(\alpha_1 + \frac{\alpha_2 r}{p} - \frac{\alpha_1}{q}\right). \tag{3.9}$$

Assume that

$$n + s > n \max\left(1, \frac{1}{p}\right) + \frac{1+\alpha_2}{p}. \tag{3.10}$$

The following conditions are equivalent:

(a) *μ is a (λ,γ)-Bergman–Carleson measure.*

(b) *There is a positive constant K such that $\|S_{\mu,\alpha_1}^r f\|_{q,\alpha_1} \le K\|f\|_{p,\alpha_2}^r$ for $f \in A_{\alpha_2}^p$.*

Moreover, we have $\|\mu\|_{\lambda,\gamma} \asymp K$.

Proof. Suppose first that μ is a (λ,γ)-Bergman–Carleson measure. Consider a lattice $\{a_j\}$ and its associated sets $\{D_j\}$. Since $|1-\langle z,w\rangle|$ is comparable with $|1-\langle z,a_j\rangle|$ for w in D_j, we have

$$\begin{aligned}|S^r_{\mu,\alpha_1} f(z)| &\lesssim (1-|z|^2)^s \sum_{j=1}^{\infty} \int_{D_j} \frac{|f(w)|^r\, d\mu(w)}{|1-\langle z,w\rangle|^{n+1+s+\alpha_1}} \\ &\asymp (1-|z|^2)^s \sum_{j=1}^{\infty} \frac{1}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \int_{D_j} |f(w)|^r\, d\mu(w).\end{aligned}$$

Using the notation

$$|\widehat{f}(a_j)| := \left(\frac{1}{(1-|a_j|^2)^{n+1+\alpha_2}} \int_{\tilde{D}_j} |f(\zeta)|^p\, dv_{\alpha_2}(\zeta) \right)^{1/p},$$

we have

$$|f(w)|^r \lesssim |\widehat{f}(a_j)|^r, \quad w \in D_j.$$

This gives

$$|S^r_{\mu,\alpha_1} f(z)|^q \lesssim (1-|z|^2)^{sq} \left(\sum_{j=1}^{\infty} \frac{|\widehat{f}(a_j)|^r \mu(D_j)}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \right)^q.$$

Now, pick $\varepsilon > 0$ so that $\alpha_1 - \varepsilon \max(q,q') > -1$ with q' being the conjugate exponent of q, that is, $1/q + 1/q' = 1$. By Hölder's inequality with exponent $q > 1$ we get

$$\begin{aligned}&\left(\sum_{j=1}^{\infty} \frac{|\widehat{f}(a_j)|^r \mu(D_j)}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \right)^q \\ &\quad\le \left(\sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^{n+1+\alpha_1-\varepsilon q'}}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \right)^{q-1} \\ &\qquad\times \left(\sum_{j=1}^{\infty} \frac{|\widehat{f}(a_j)|^{rq} \mu(D_j)^q (1-|a_j|^2)^{(n+1+\alpha_1)(1-q)+\varepsilon q}}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \right).\end{aligned}$$

Since the sequence $\{a_j\}$ is separated and $n+1+\alpha_1-\varepsilon q' > n$, using Lemma C, we have

$$\sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^{n+1+\alpha_1-\varepsilon q'}}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \lesssim (1-|z|^2)^{-s-\varepsilon q'},$$

and therefore

$$\begin{aligned}&|S^r_{\mu,\alpha_1} f(z)|^q \\ &\quad\lesssim (1-|z|^2)^{s-\varepsilon q} \left(\sum_{j=1}^{\infty} \frac{|\widehat{f}(a_j)|^{rq} \mu(D_j)^q (1-|a_j|^2)^{(n+1+\alpha_1)(1-q)+\varepsilon q}}{|1-\langle z,a_j\rangle|^{n+1+s+\alpha_1}} \right).\end{aligned}$$

This, together with the typical integral estimate in Lemma B, gives

$$\begin{aligned}\|S^r_{\mu,\alpha_1} f\|^q_{q,\alpha_1} &\lesssim \sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{rq} \mu(D_j)^q (1-|a_j|^2)^{(n+1+\alpha_1)(1-q)+\varepsilon q} \\ &\qquad \times \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{s+\alpha_1-\varepsilon q}}{|1-\langle z, a_j\rangle|^{n+1+s+\alpha_1}}\, dv(z) \\ &\lesssim \sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{rq} \mu(D_j)^q (1-|a_j|^2)^{(n+1+\alpha_1)(1-q)}.\end{aligned}$$

If $\lambda \geq 1$, then $\mu(D_j) \lesssim \|\mu\|_{\lambda,\gamma}(1-|a_j|^2)^{(n+1+\gamma)\lambda}$ due to Theorem A. Moreover, the condition $\lambda \geq 1$ also implies $p/(rq) \leq 1$, and therefore we have

$$\begin{aligned}\|S^r_{\mu,\alpha_1} f\|^q_{q,\alpha_1} &\lesssim \|\mu\|^q_{\lambda,\gamma} \sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{rq} (1-|a_j|^2)^{(n+1+\alpha_2)rq/p} \\ &\leq \|\mu\|^q_{\lambda,\gamma} \left(\sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{p} (1-|a_j|^2)^{n+1+\alpha_2} \right)^{rq/p}.\end{aligned}$$

If $0 < \lambda < 1$, then we use Hölder's inequality with exponent $p/(rq) > 1$. Observe that the conjugate exponent of $p/(rq)$ is

$$\frac{p/(rq)}{p/(rq)-1} = \frac{p}{p-rq} = \frac{1}{q(1-\lambda)}.$$

We obtain, after an application of Theorem B,

$$\begin{aligned}\|S^r_{\mu,\alpha_1} f\|^q_{q,\alpha_1} &\lesssim \left(\sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{p} (1-|a_j|^2)^{n+1+\alpha_2} \right)^{rq/p} \\ &\qquad \times \left(\sum_{j=1}^{\infty} \left(\frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}} \right)^{1/(1-\lambda)} \right)^{q(1-\lambda)} \\ &\lesssim \|\mu\|^q_{\lambda,\gamma} \left(\sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{p} (1-|a_j|^2)^{n+1+\alpha_2} \right)^{rq/p}.\end{aligned}$$

Finally, in both cases, we obtain the inequality in part (b) after noticing that

$$\sum_{j=1}^{\infty} |\widehat{f}(a_j)|^{p} (1-|a_j|^2)^{n+1+\alpha_2} \lesssim \|f\|^p_{p,\alpha_2}.$$

Conversely, assume that (b) holds. We want to show that μ is a (λ,γ)-Bergman–Carleson measure. We split the proof in two cases.

If $\lambda \geq 1$, then, for each $a \in \mathbb{B}_n$, consider the functions

$$f_a(z) = (1-\langle z, a\rangle)^{-\sigma}$$

with σ big enough, that is, with $p\sigma > n+1+\alpha_2$. By Lemma B we have

$$\|f_a\|_{p,\alpha_2}^p = \int_{\mathbb{B}_n} \frac{dv_{\alpha_2}(z)}{|1-\langle z,a\rangle|^{p\sigma}} \lesssim (1-|a|^2)^{n+1+\alpha_2-p\sigma}.$$

Also, for any $\tau > 0$, we get

$$\begin{aligned}\frac{(1-|z|^2)^s |f_a(a)|^r}{|1-\langle z,a\rangle|^{n+1+s+\alpha_1}} \mu(D(a,\tau)) &\lesssim (1-|z|^2)^s \int_{D(a,\tau)} \frac{|f_a(w)|^r \, d\mu(w)}{|1-\langle z,w\rangle|^{n+1+s+\alpha_1}} \\ &\le S^r_{\mu,\alpha_1} f_a(z).\end{aligned}$$

Moreover, since $v_{\alpha_1}(D(a,\tau)) \asymp (1-|a|^2)^{n+1+\alpha_1}$, we have

$$\begin{aligned}(1-|a|^2)^{(n+1+\alpha_1)(1-q)} &\lesssim \int_{D(a,\tau)} \left(\frac{(1-|z|^2)^s}{|1-\langle z,a\rangle|^{n+1+s+\alpha_1}}\right)^q dv_{\alpha_1}(z) \\ &\le \int_{\mathbb{B}_n} \left(\frac{(1-|z|^2)^s}{|1-\langle z,a\rangle|^{n+1+s+\alpha_1}}\right)^q dv_{\alpha_1}(z).\end{aligned}$$

Hence,

$$\begin{aligned}(1-|a|^2)^{(n+1+\alpha_1)(1-q)} |f_a(a)|^{rq} \mu(D(a,\tau))^q &\lesssim \int_{\mathbb{B}_n} S^r_{\mu,\alpha_1} f_a(z)^q \, dv_{\alpha_1}(z) \\ &\le K^q \|f_a\|_{p,\alpha_2}^{rq} \\ &\lesssim K^q ((1-|a|^2)^{n+1+\alpha_2-p\sigma})^{rq/p}.\end{aligned}$$

This gives

$$\begin{aligned}\mu(D(a,\tau)) &\lesssim K(1-|a|^2)^{(n+1+\alpha_1)(q-1)/q} (1-|a|^2)^{(n+1+\alpha_2)r/p} \\ &= K(1-|a|^2)^{(n+1+\gamma)\lambda}.\end{aligned}$$

By Theorem A it follows that μ is a (λ,γ)-Bergman–Carleson measure with $\|\mu\|_{\lambda,\gamma} \lesssim K$.

For $0<\lambda<1$, we split the proof in several cases.

Case $r=1$: In that case, it is easy to see that the condition implies that the Toeplitz operator $T^\beta_\mu : A^p_{\alpha_2} \to A^q_\sigma$ is bounded with

$$\beta = s+\alpha_1 \quad \text{and} \quad \sigma = \alpha_1 + sq.$$

Therefore, part (a) is an immediate consequence of Theorem 1.2 after checking that the parameter β satisfies the conditions of that theorem, that is, we need to check (1.3) and (1.4). Observe that, since $\beta > s-1$, condition (3.10) ensures that β satisfies (1.3). On the other hand, since $q>1$, (1.4) becomes

$$1+\beta > \frac{1+\sigma}{q} \quad \Leftrightarrow \quad 1+\alpha_1 > \frac{1+\alpha_1}{q} \quad \Leftrightarrow \quad q>1.$$

This finishes the proof of this case.

Case $r>1$: We want to show that μ is a (λ,γ)-Bergman–Carleson measure, or equivalently, that $B_{s,\gamma}(\mu)$ belongs to $L^{1/(1-\lambda),\gamma}$. By Theorem B and the result on Toeplitz operators (Theorem 1.2) we have

$$\|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma} \le C\|T^\beta_\mu\|_{A^p_{\alpha_2}\to A^t_\sigma} \tag{3.11}$$

with

$$\beta = s + \frac{\alpha_1}{r} + \frac{\gamma(r-1)}{r},$$
$$t = \frac{1}{1-\lambda+1/p}$$

and σ is determined by the relation

$$\gamma = \frac{1}{\lambda}\left(\beta + \frac{\alpha_2}{p} - \frac{\sigma}{t}\right).$$

Again, it must be checked that the parameter β satisfies the condition indicated in the second line of the statement of Theorem 1.2. Since $\beta > s-1$, condition (1.3) is satisfied due to (3.10). On the other hand, the corresponding condition (1.4) becomes

$$n+1+\beta > n\max\left(1, \frac{1}{t}\right) + \frac{1+\sigma}{t}.$$

It is easy to see that $t > 1$, so that we must check the condition

$$1+\beta > \frac{1+\sigma}{t}$$

or, equivalently,

$$1+\beta > \frac{1}{t} + \beta + \frac{\alpha_2}{p} - \gamma\lambda.$$

Taking into account the expression for $\lambda\gamma$ given in (3.9), we see that this condition is equivalent to

$$1 > \frac{1}{t} + \frac{\alpha_2}{p}(1-r) - \frac{\alpha_1}{q}(q-1) = \frac{(1+\alpha_2)}{p}(1-r) + \frac{(1+\alpha_1)}{q} - \alpha_1.$$

Since $r > 1$, this holds if

$$1+\alpha_1 > \frac{1+\alpha_1}{q},$$

and this is clearly satisfied because $q > 1$ and $(1+\alpha_1) > 0$.

Next, we continue with the proof. Assume first that μ has compact support on $\mathbb{B}_n$. By Hölder's inequality,

$$|T_\mu^\beta f(z)|^t \le \left(\int_{\mathbb{B}_n} \frac{|f(w)|^r\, d\mu(w)}{|1-\langle z, w\rangle|^{n+1+s+\alpha_1}}\right)^{t/r} \left(\int_{\mathbb{B}_n} \frac{d\mu(w)}{|1-\langle z, w\rangle|^{n+1+s+\gamma}}\right)^{t/r'},$$

where $r' = \frac{r}{r-1}$ is the conjugate exponent of r. This yields

$$\|T_\mu^\beta f\|_{t,\sigma}^t \le \int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^s |f(w)|^r\, d\mu(w)}{|1-\langle z, w\rangle|^{n+1+s+\alpha_1}}\right)^{t/r} (B_{s,\gamma}\mu(z))^{t/r'}\, dv_{\sigma-st}(z).$$

Now, since $r'/((1-\lambda)t) > 1$ (because $(1-\lambda)t < 1$), we can apply Hölder's inequality again to obtain

$$\|T_\mu^\beta f\|_{t,\sigma}^t \le \|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma}^{t/r'}$$
$$\times \left[\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^s |f(w)|^r\, d\mu(w)}{|1-\langle z, w\rangle|^{n+1+s+\alpha_1}}\right)^q dv_{\gamma+(\sigma-st-\gamma)\eta}(z)\right]^{1/\eta}.$$

Observe that

$$\eta := \left(\frac{r'}{(1-\lambda)t}\right)' = \frac{r'/((1-\lambda)t)}{r'/((1-\lambda)t)-1} = \frac{r'}{r'-(1-\lambda)t} = \frac{r}{r-(1-\lambda)t(r-1)}$$

and therefore

$$\begin{aligned}\frac{t\eta}{r} &= \frac{t}{r-(1-\lambda)t(r-1)} = \frac{1}{r/t-(1-\lambda)(r-1)} \\ &= \frac{1}{r(1-\lambda+1/p)-(1-\lambda)(r-1)} = \frac{1}{r/p+1-\lambda} = q.\end{aligned}$$

After some long and tedious but elementary computations, it is possible to check that

$$\gamma + (\sigma - st - \gamma)\eta = \alpha_1. \tag{3.12}$$

Indeed, since $1-\eta = \frac{-(1-\lambda)t(r-1)}{r-(1-\lambda)t(r-1)}$, identity (3.12) is equivalent to

$$-\gamma(1-\lambda)t(r-1) + \sigma r - str = \alpha_1[r-(1-\lambda)t(r-1)].$$

Using that $\sigma = \beta t + \alpha_2 t/p - \gamma\lambda t$ and the expression of β, after some simplifications, we see that the previous identity is equivalent to

$$-\gamma\lambda t + \alpha_2 t\frac{r}{p} = \alpha_1 r - \alpha_1 tr(1-\lambda) - \alpha_1\lambda t.$$

Now, using the expressions of λ, γ given in (3.9), we must check that

$$\frac{\alpha_1 t}{p} = \alpha_1 - \alpha_1 t(1-\lambda).$$

This is obvious if $\alpha_1 = 0$. If $\alpha_1 \neq 0$, this is equivalent to

$$t = \frac{1}{1-\lambda+1/p},$$

and this is our choice of t. Hence, (3.12) holds.

Then, by our condition (b) we obtain

$$\begin{aligned}\|T_\mu^\beta f\|_{t,\sigma}^t &\le \|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma}^{t/r'} \cdot \|S_{\mu,\alpha_1}^r f\|_{q,\alpha_1}^{q/\eta} \\ &\le K^{q/\eta} \cdot \|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma}^{t/r'} \cdot \|f\|_{p,\alpha_2}^t.\end{aligned}$$

This, together with (3.11), gives

$$\|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma} \lesssim \|T_\mu^\beta\| \lesssim K^{q/\eta t} \cdot \|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma}^{1/r'},$$

and since $q/\eta t = 1/r$, this implies

$$\|B_{s,\gamma}(\mu)\|_{1/(1-\lambda),\gamma} \lesssim K,$$

proving the result when μ has compact support on $\mathbb{B}_n$. The result for arbitrary μ follows from this by an easy limit argument.

Case $r < 1$: Fix a number $m > 1$ and consider the measure $\widetilde{\mu}$ given by

$$d\widetilde{\mu}(z) = (1-|z|^2)^A \, d\mu(z)$$

with

$$A = (m-r)\frac{(n+1+\alpha_2)}{p}.$$

Let

$$\gamma^* = \gamma + \frac{A}{\lambda},$$
$$\beta = s + \frac{\alpha_1}{m} + \frac{\gamma^*(m-1)}{m},$$
$$t = \frac{1}{1-\lambda+1/p},$$

and let σ be determined by the relation

$$\gamma^* = \frac{1}{\lambda}\left(\beta + \frac{\alpha_2}{p} - \frac{\sigma}{t}\right).$$

As done in the previous cases, it can be checked that the parameter β satisfies the condition indicated in the second line of the statement of Theorem 1.2.

Again, assume first that μ has compact support on $\mathbb{B}_n$. Obviously, then the measure $\widetilde{\mu}$ also has compact support. By Theorem 1.2 applied to the Toeplitz operator $T^{\beta}_{\widetilde{\mu}} : A^p_{\alpha_2} \to A^t_{\sigma}$ we have

$$\|B_{s,\gamma^*}(\widetilde{\mu})\|_{1/(1-\lambda),\gamma^*} \leq C\|T^{\beta}_{\widetilde{\mu}}\|_{A^p_{\alpha_2}\to A^t_{\sigma}}. \tag{3.13}$$

Arguing as in the previous case, we get

$$\|T^{\beta}_{\widetilde{\mu}} f\|^t_{t,\sigma} \leq \|B_{s,\gamma^*}(\widetilde{\mu})\|^{t/m'}_{1/(1-\lambda),\gamma^*} \cdot \|S^m_{\widetilde{\mu},\alpha_1} f\|^{t/m}_{q_1,\alpha_1},$$

where m' denotes the conjugate exponent of m, and

$$q_1 = \frac{1}{m/p+1-\lambda}.$$

Since $m > r$, we have $q_1 < (r/p+1-\lambda)^{-1} = q$, and hence

$$\|S^m_{\widetilde{\mu},\alpha_1} f\|_{q_1,\alpha_1} \leq \|S^m_{\widetilde{\mu},\alpha_1} f\|_{q,\alpha_1}.$$

Therefore,

$$\|T^{\beta}_{\widetilde{\mu}} f\|^t_{t,\sigma} \leq \|B_{s,\gamma^*}(\widetilde{\mu})\|^{t/m'}_{1/(1-\lambda),\gamma^*} \cdot \|S^m_{\widetilde{\mu},\alpha_1} f\|^{t/m}_{q,\alpha_1}. \tag{3.14}$$

Now, applying the pointwise estimate for $f \in A^p_{\alpha_2}$, we obtain

$$\begin{aligned}
\|S^m_{\widetilde{\mu},\alpha_1} f\|^q_{q,\alpha_1} &= \int_{\mathbb{B}_n}\left((1-|z|^2)^s \int_{\mathbb{B}_n} \frac{|f(w)|^r|f(w)|^{m-r}}{|1-\langle z,w\rangle|^{n+1+s+\alpha_1}}\,d\widetilde{\mu}(w)\right)^q dv_{\alpha_1}(z)\\
&\leq \|f\|^{q(m-r)}_{p,\alpha_2}\int_{\mathbb{B}_n}\Bigg((1-|z|^2)^s\\
&\qquad\times \int_{\mathbb{B}_n}\frac{|f(w)|^r(1-|w|^2)^{A-(m-r)(n+1+\alpha_2)/p}}{|1-\langle z,w\rangle|^{n+1+s+\alpha_1}}\,d\mu(w)\Bigg)^q dv_{\alpha_1}(z)\\
&= \|f\|^{q(m-r)}_{p,\alpha_2}\cdot\|S^r_{\mu,\alpha_1} f\|^q_{q,\alpha_1}.
\end{aligned}$$

Putting this into (3.14) and using the inequality in part (b), we obtain

$$\begin{aligned}
\|T^{\beta}_{\widetilde{\mu}} f\|_{t,\sigma} &\leq \|B_{s,\gamma^*}(\widetilde{\mu})\|^{1/m'}_{1/(1-\lambda),\gamma^*}\cdot\|f\|^{(m-r)/m}_{p,\alpha_2}\cdot\|S^r_{\mu,\alpha_1} f\|^{1/m}_{q,\alpha_1}\\
&\leq K^{1/m}\cdot\|B_{s,\gamma^*}(\widetilde{\mu})\|^{1/m'}_{1/(1-\lambda),\gamma^*}\cdot\|f\|_{p,\alpha_2}.
\end{aligned}$$

This, together with (3.13), yields

$$\|B_{s,\gamma^*}(\widetilde{\mu})\|_{1/(1-\lambda),\gamma^*} \lesssim K^{1/m} \cdot \|B_{s,\gamma^*}(\widetilde{\mu})\|_{1/(1-\lambda),\gamma^*}^{1/m'},$$

which proves that

$$\|B_{s,\gamma^*}(\widetilde{\mu})\|_{1/(1-\lambda),\gamma^*} \lesssim K \tag{3.15}$$

for μ with compact support on $\mathbb{B}_n$. Then a standard limit argument gives (3.15) for a general positive measure μ.

Now, let $\{a_k\}$ be any lattice in $\mathbb{B}_n$. Since $(n+1+\gamma^*)\lambda - A = (n+1+\gamma)\lambda$, applying Theorem B, we get that

$$\begin{aligned}
\|\mu\|_{\lambda,\gamma}^{1/(1-\lambda)} &\lesssim \sum_k \left(\frac{\mu(D_k)}{(1-|a_k|)^{(n+1+\gamma^*)\lambda - A}} \right)^{1/(1-\lambda)} \\
&\asymp \sum_k \left(\frac{\widetilde{\mu}(D_k)}{(1-|a_k|)^{(n+1+\gamma^*)\lambda}} \right)^{1/(1-\lambda)} \\
&\lesssim \|\widetilde{\mu}\|_{\lambda,\gamma^*}^{1/(1-\lambda)} \lesssim \|B_{s,\gamma^*}(\widetilde{\mu})\|_{1/(1-\lambda),\gamma^*}^{1/(1-\lambda)}.
\end{aligned}$$

Then, from (3.15) we have that μ is a (λ,γ)-Bergman–Carleson measure with

$$\|\mu\|_{\lambda,\gamma} \lesssim K.$$

The proof is complete. □

3.3. *Proof of Theorem 1.1*

The following result, together with Proposition 3.2, concludes the proof of Theorem 1.1.

PROPOSITION 3.4. *Let $\lambda > 0$. If (1.2) holds, then μ is a (λ,γ)-Bergman–Carleson measure. Furthermore, $\|\mu\|_{\lambda,\gamma} \lesssim C$, where C is the constant appearing in (1.2).*

Proof. Assume first that $\lambda \geq 1$. Let

$$f_{i,a}(z) = \frac{(1-|a|^2)^{(n+1+\alpha_i)/p_i}}{(1-\langle z,a\rangle)^{2(n+1+\alpha_i)/p_i}}.$$

Then it can be easily checked that for every $a \in \mathbb{B}_n$ and for all $i = 1,2,\ldots,k$, $\|f_{i,a}\|_{p_i,\alpha_i} \lesssim 1$. Thus, (1.2) implies

$$\int_{\mathbb{B}_n} \prod_{i=1}^k |f_{i,a}(z)|^{q_i}\, d\mu(z) \leq C \prod_{i=1}^k \|f_{i,a}\|_{p_i,\alpha_i}^{q_i} = C, \tag{3.16}$$

where C is a positive constant independent of a. An easy computation shows that

$$\sum_{i=1}^k (n+1+\alpha_i)\frac{q_i}{p_i} = (n+1+\gamma)\lambda.$$

Thus, (3.16) is equivalent to

$$\int_{\mathbb{B}_n} \frac{(1-|a|^2)^{(n+1+\gamma)\lambda}}{|1-\langle z,a\rangle|^{2(n+1+\gamma)\lambda}}\, d\mu(z) \le C.$$

Since $\lambda \ge 1$, by Theorem A we know that μ is a (λ, γ)-Bergman–Carleson measure with $\|\mu\|_{\lambda,\gamma} \lesssim C$.

Next, we consider the case $0 < \lambda < 1$. We use induction on k. If $k = 1$, then (1.2) is just the definition of a Bergman–Carleson measure. Now, let $k \ge 2$ and assume that the result holds for $k-1$ functions. Set $\lambda_k = \lambda$, $\gamma_k = \gamma$ and

$$\lambda_{k-1} = \sum_{i=1}^{k-1} \frac{q_i}{p_i}, \qquad \gamma_{k-1} = \frac{1}{\lambda_{k-1}} \sum_{i=1}^{k-1} \frac{\alpha_i q_i}{p_i}.$$

Considering the measure

$$d\mu_k(z) = |f_k(z)|^{q_k}\, d\mu(z),$$

we see that our condition

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{q_i}\, d\mu(z) \le C \prod_{i=1}^{k} \|f_i\|_{p_i,\alpha_i}^{q_i}$$

is equivalent to the condition

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k-1} |f_i(z)|^{q_i}\, d\mu_k(z) \le C(f_k) \prod_{i=1}^{k-1} \|f_i\|_{p_i,\alpha_i}^{q_i}$$

with $C(f_k) = C \cdot \|f_k\|_{p_k,\alpha_k}^{q_k}$. By induction this implies that μ_k is a $(\lambda_{k-1}, \gamma_{k-1})$-Bergman–Carleson measure with $\|\mu_k\|_{\lambda_{k-1},\gamma_{k-1}} \lesssim C(f_k)$. Since $0 < \lambda_{k-1} < \lambda < 1$, Theorem B implies that $B_{s,\gamma_{k-1}}(\mu_k)$ belongs to $L^{1/(1-\lambda_{k-1}),\gamma_{k-1}}$ for any $s > 0$ with

$$\|B_{s,\gamma_{k-1}}(\mu_k)\|_{1/(1-\lambda_{k-1}),\gamma_{k-1}} \lesssim C(f_k).$$

That is, we have

$$\int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^s |f_k(w)|^{q_k}\, d\mu(w)}{|1-\langle z,w\rangle|^{n+1+s+\gamma_{k-1}}} \right)^{1/(1-\lambda_{k-1})} dv_{\gamma_{k-1}}(z)$$
$$\lesssim (C \cdot \|f_k\|_{p_k,\alpha_k}^{q_k})^{1/(1-\lambda_{k-1})}$$

or, equivalently,

$$\|S_{\mu,\gamma_{k-1}}^{q_k} f_k\|_{1/(1-\lambda_{k-1}),\gamma_{k-1}} \lesssim C \cdot \|f_k\|_{p_k,\alpha_k}^{q_k}$$

whenever f_k is in $A_{\alpha_k}^{p_k}$. Thus, by Lemma 3.3 the measure μ is a (λ^*, γ^*)-Bergman–Carleson measure with $\|\mu\|_{\lambda^*,\gamma^*} \lesssim C$, where

$$\lambda^* = 1 + \frac{q_k}{p_k} - (1-\lambda_{k-1}) \quad \text{and} \quad \gamma^* = \frac{1}{\lambda^*}\left(\gamma_{k-1} + \frac{\alpha_k q_k}{p_k} - \gamma_{k-1}(1-\lambda_{k-1}) \right).$$

Simple algebraic manipulations show that $\lambda^* = \lambda$ and $\gamma^* = \gamma$, concluding the proof. □

4. Vanishing (λ, γ)-Bergman–Carleson Measures

We say that μ is a vanishing (λ, α)-Bergman–Carleson measure if for any two positive numbers p and q satisfying $q/p = \lambda$ and any sequence $\{f_k\}$ in A^p_α with $\|f_k\|_{p,\alpha} \le 1$ and $f_k(z) \to 0$ uniformly on any compact subset of $\mathbb{B}_n$,

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k(z)|^q \, d\mu(z) = 0.$$

It is well known that, for $\lambda \ge 1$, μ is a vanishing (λ, α)-Bergman–Carleson measure if and only if

$$\lim_{|a|\to 1} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{(n+1+\alpha)\lambda+t}} \, d\mu(z) = 0 \tag{4.1}$$

for some (any) $t > 0$. It is also well known that, for $0 < \lambda < 1$, μ is a vanishing (λ, α)-Bergman–Carleson measure if and only if it is a (λ, α)-Bergman–Carleson measure. We refer to [27] for these facts.

THEOREM 4.1. *Let μ be a positive Borel measure on $\mathbb{B}_n$. For any integer $k \ge 1$ and $i = 1, 2, \ldots, k$, let $0 < p_i, q_i < \infty$ and $-1 < \alpha_i < \infty$. Let*

$$\lambda = \sum_{i=1}^k \frac{q_i}{p_i}; \qquad \gamma = \frac{1}{\lambda} \sum_{i=1}^k \frac{\alpha_i q_i}{p_i}.$$

Then the following statements are equivalent.

(i) *μ is a vanishing (λ, γ)-Bergman–Carleson measure.*

(ii) *For any sequence $\{f_{1,l}\}$ in the unit ball of $A^{p_1}_{\alpha_1}$ that is convergent to 0 uniformly in compact subsets of $\mathbb{B}_n$,*

$$\lim_{l\to\infty} F(l) = 0,$$

where

$$F(l) = \sup\left\{ \int_{\mathbb{B}_n} |f_{1,l}(z)|^{q_1} \prod_{i=2}^k |f_i(z)|^{q_i} \, d\mu(z) : \ \|f_i\|_{p_i,\alpha_i} \le 1, i = 2, \ldots, k \right\}.$$

(iii) *For any k sequences $\{f_{1,l}\}, \{f_{2,l}\}, \ldots, \{f_{k,l}\}$ in the unit balls of $A^{p_1}_{\alpha_1}$, $A^{p_2}_{\alpha_2}$, $\ldots$, $A^{p_k}_{\alpha_k}$, respectively, that are all convergent to 0 uniformly in compact subsets of $\mathbb{B}_n$,*

$$\lim_{l\to\infty} \int_{\mathbb{B}_n} |f_{1,l}(z)|^{q_1} |f_{2,l}(z)|^{q_2} \cdots |f_{k,l}(z)|^{q_k} \, d\mu(z) = 0.$$

Proof. By the remark preceding the statement of the theorem, the case $0 < \lambda < 1$ is just a consequence of Theorem 1.1. So, we assume that $\lambda \ge 1$. Let (i) be true, so μ is a vanishing (λ, γ)-Bergman–Carleson measure. Let $\{f_{1,l}\}$ be a sequence in the unit ball of $A^{p_1}_{\alpha_1}$ that is convergent to 0 uniformly in compact subsets of $\mathbb{B}_n$, and let $\{f_i\}$ be arbitrary functions in the unit balls of $A^{p_i}_{\alpha_i}$, $i = 2, 3, \ldots, k$.

Let $\mu_r = \mu|_{\mathbb{B}_n \setminus \overline{D}_r}$, where $D_r = \{z \in \mathbb{B}_n : \ |z| < r\}$. Then μ_r is also a (λ, γ)-Bergman–Carleson measure, and

$$\lim_{r \to 1} \|\mu_r\|_{\lambda,\gamma} = 0.$$

(See, p. 130 of [8].) Hence,

$$\begin{aligned}
&\int_{\mathbb{B}_n \setminus \overline{D}_r} |f_{1,l}(z)|^{q_1} |f_2(z)|^{q_2} \cdots |f_k(z)|^{q_k} \, d\mu(z) \\
&\qquad \le \int_{\mathbb{B}_n} |f_{1,l}(z)|^{q_1} |f_2(z)|^{q_2} \cdots |f_k(z)|^{q_k} \, d\mu_r(z) \\
&\qquad \le C \|\mu_r\|_{\lambda,\gamma} \le C\varepsilon
\end{aligned} \tag{4.2}$$

as r sufficiently close to 1. Fix such an r. Since $\{f_{1,l}\}$ converges to 0 uniformly in compact subsets of $\mathbb{B}_n$, there is a constant $K > 0$ such that for any $l > K$, $|f_{1,l}(z)| < \varepsilon$ for any $z \in \overline{D}_r$. Therefore, using Theorem 1.1, we have

$$\begin{aligned}
&\int_{\overline{D}_r} |f_{1,l}(z)|^{q_1} |f_2(z)|^{q_2} \cdots |f_k(z)|^{q_k} \, d\mu(z) \\
&\qquad \le \varepsilon \int_{\mathbb{B}_n} |f_2(z)|^{q_2} \cdots |f_k(z)|^{q_k} \, d\mu(z) \\
&\qquad = \varepsilon \int_{\mathbb{B}_n} |1|^{q_1} |f_2(z)|^{q_2} \cdots |f_k(z)|^{q_k} \, d\mu(z) \\
&\qquad \lesssim \varepsilon \|1\|_{p_1,\alpha_1}^{p_1} \|f_2\|_{p_2,\alpha_2}^{p_2} \cdots \|f_k\|_{p_k,\alpha_k}^{p_k} \lesssim \varepsilon
\end{aligned} \tag{4.3}$$

for any $z \in \overline{D}_r$. Combining (4.2) and (4.3), we get (ii).

It is obvious that (ii) implies (iii). Now let (iii) be true. Let

$$f_{i,a}(z) = \frac{(1 - |a|^2)^{(n+1+\alpha_i)/p_i}}{(1 - \langle z, a\rangle)^{2(n+1+\alpha_i)/p_i}}.$$

Then, as before, we know that for every $a \in \mathbb{B}_n$ and for all $i = 1, 2, \ldots, k$, $\|f_{i,a}\|_{p_i,\alpha_i} \lesssim 1$, and it can be easily checked that

$$\lim_{|a| \to 1} |f_{i,a}(z)| = 0$$

uniformly on any compact subset of $\mathbb{B}_n$. Thus, (iii) implies

$$\lim_{|a| \to 1} \int_{\mathbb{B}_n} \prod_{i=1}^{k} \frac{(1 - |a|^2)^{(n+1+\alpha_i)q_i/p_i}}{|1 - \langle z, a\rangle|^{2(n+1+\alpha_i)q_i/p_i}} \, d\mu(z) = 0.$$

Since $\sum_{i=1}^{k} (n + 1 + \alpha_i) q_i / p_i = (n + 1 + \gamma)\lambda$, this equality is the same as

$$\lim_{|a| \to 1} \int_{\mathbb{B}_n} \frac{(1 - |a|^2)^{(n+1+\gamma)\lambda}}{|1 - \langle z, a\rangle|^{2(n+1+\gamma)\lambda}} \, d\mu(z) = 0.$$

Thus, by (4.1) μ is a vanishing (λ, γ)-Bergman–Carleson measure. The proof is complete. □

Vanishing Bergman–Carleson measures are also useful in order to describe the compactness of Toeplitz operators between weighted Bergman spaces.

THEOREM 4.2. *Let μ be a positive Borel measure on $\mathbb{B}_n$, $0 < p_1, p_2 < \infty$, and $-1 < \alpha_1, \alpha_2 < \infty$. Let β, λ, and γ be as in Theorem 1.2. Then T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$ if and only if μ is a vanishing (λ, γ)-Bergman–Carleson measure.*

Proof. If $0 < \lambda < 1$, then by the remark preceding Theorem 4.1 a vanishing (λ, γ)-Bergman–Carleson measure is the same as a (λ, γ)-Bergman–Carleson measure. Also, since $0 < \lambda < 1$, we have $0 < p_2 < p_1 < \infty$, and therefore the result follows from Theorem 1.2 since in that case T_μ^β is compact from $A_{\alpha_1}^{p_1}$ to $A_{\alpha_2}^{p_2}$ if and only if it is bounded due to a general result of Banach space theory: it is known that, for $0 < p_2 < p_1 < \infty$, every bounded operator from ℓ^{p_1} to ℓ^{p_2} is compact (see, e.g., [12, Thm. I.2.7, p. 31]), and the weighted Bergman space A_α^p is isomorphic to ℓ^p (see [22, Thm. 11, p. 89]; note that the same proof there works for weighted Bergman spaces on the unit ball $\mathbb{B}_n$).

Next, we consider the case $\lambda \geq 1$. If T_μ^β is compact, then $\|T_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0$ for any bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$ converging to zero uniformly on compact subsets of $\mathbb{B}_n$. Let $\{a_k\} \subset \mathbb{B}_n$ with $|a_k| \to 1^-$ and consider the functions

$$f_k(z) = \frac{(1 - |a_k|^2)^{(n+1+\beta)-(n+1+\alpha_1)/p_1}}{(1 - \langle z, a_k \rangle)^{n+1+\beta}}.$$

Due to the conditions on β and Lemma B, we have $\sup_k \|f_k\|_{p_1,\alpha_1} < \infty$, and it is obvious that f_k converges to zero uniformly on compact subsets of $\mathbb{B}_n$. Hence, $\|T_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0$. Therefore, proceeding as in the proof of the case $\lambda \geq 1$ of that (i) implies (ii) in Theorem 1.2, for any $r > 0$, we get

$$\begin{aligned}
\frac{\mu(D(a_k, r))}{(1 - |a_k|^2)^{(n+1+\gamma)\lambda}} &\lesssim (1 - |a_k|^2)^{(n+1+\beta)+(n+1+\alpha_1)/p_1-(n+1+\gamma)\lambda} T_\mu^\beta f_k(a_k) \\
&= (1 - |a_k|^2)^{(n+1+\alpha_2)/p_2} T_\mu^\beta f_k(a_k) \\
&\lesssim \|T_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0.
\end{aligned}$$

Thus, by [27, p. 71], the measure μ is a vanishing (λ, γ)-Bergman–Carleson measure.

Conversely, let μ be a vanishing (λ, γ)-Bergman–Carleson measure with $\lambda \geq 1$. To prove that T_μ^β is compact, we must show that $\|T_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0$ for any bounded sequence $\{f_k\}$ in $A_{\alpha_1}^{p_1}$ converging to zero uniformly on compact subsets of $\mathbb{B}_n$. If $p_2 > 1$, then, as in the proof of Theorem 1.2, by duality and Theorem 4.1 we have (the numbers p_2' and α_2' are the ones defined by (3.6))

$$\begin{aligned}
\|T_\mu^\beta f_k\|_{p_2,\alpha_2} &\asymp \sup_{\|h\|_{p_2',\alpha_2'} \leq 1} |\langle h, T_\mu^\beta f_k \rangle_\beta| \\
&\leq \sup_{\|h\|_{p_2',\alpha_2'} \leq 1} \int_{\mathbb{B}_n} |f_k(z)||h(z)| \, d\mu(z) \to 0.
\end{aligned}$$

If $0 < p_2 \le 1$, from the estimates obtained in the proof of that (ii) implies (i) in Theorem 1.2 (see (3.8)) it follows that, for any lattice $\{a_j\}$, we have

$$\|T_\mu^\beta f_k\|_{p_2,\alpha_2}^{p_2} \lesssim \sum_{j=1}^\infty \left(\frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}}\right)^{p_2} \left(\int_{\tilde{D}_j} |f_k(z)|^{p_1}\, dv_{\alpha_1}(z)\right)^{p_2/p_1}. \tag{4.4}$$

Let $\varepsilon > 0$. Since μ is a vanishing (λ,γ)-Bergman–Carleson measure, due to [27, p. 71], there is $0 < r_0 < 1$ such that

$$\sup_{|a_j|>r_0} \frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}} < \varepsilon. \tag{4.5}$$

Split the sum appearing in (4.4) in two parts: one over the points a_j with $|a_j| \le r_0$ and the other over the points with $|a_j| > r_0$. Since $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{B}_n$, it is clear that the sum over the points a_j with $|a_j| \le r_0$ (a finite sum) goes to zero as k goes to infinity. On the other hand, by (4.5) and since $p_2 \ge p_1$ (because $\lambda \ge 1$), we have

$$\begin{aligned} &\sum_{j:\ |a_j|>r_0} \left(\frac{\mu(D_j)}{(1-|a_j|^2)^{(n+1+\gamma)\lambda}}\right)^{p_2} \left(\int_{\tilde{D}_j} |f_k(z)|^{p_1}\, dv_{\alpha_1}(z)\right)^{p_2/p_1} \\ &\quad < \varepsilon^{p_2} \sum_{j:\ |a_j|>r_0} \left(\int_{\tilde{D}_j} |f_k(z)|^{p_1}\, dv_{\alpha_1}(z)\right)^{p_2/p_1} \le \varepsilon^{p_2} \|f_k\|_{p_1,\alpha_1}^{p_2} \le C\varepsilon^{p_2}. \end{aligned}$$

Thus, $\|T_\mu^\beta f_k\|_{p_2,\alpha_2} \to 0$, finishing the proof. □

5. Applications

As a direct consequence of Theorem 1.1 and Theorem 4.1, we have the following result.

COROLLARY 5.1. *Let μ be a positive Borel measure on $\mathbb{B}_n$. Let $p, q > 0$, $s \ge 0$, and $\alpha, \delta > -1$ be given constants such that $q/p + s/(n+1+\delta) \ge 1$. Let*

$$\lambda = \frac{q}{p} + \frac{s}{(n+1+\delta)} \quad \textit{and} \quad \gamma = \frac{1}{\lambda}\left(\frac{\alpha q}{p} + \frac{\delta s}{n+1+\delta}\right).$$

Then μ is a (λ,γ)-Bergman–Carleson measure if and only if for any $f \in A_\alpha^p$ and for some (any) $t > 0$,

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f(z)|^q \frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{s+t}}\, d\mu(z) \lesssim \|f\|_{p,\alpha}^q; \tag{5.1}$$

and μ is a vanishing (λ,γ)-Bergman–Carleson measure if and only if for some (any) $t > 0$ and for any sequence $\{f_k\}$ in A_α^p with $\|f_k\|_{p,\alpha} \le 1$ and $f_k(z) \to 0$ uniformly on any compact subset of $\mathbb{B}_n$,

$$\lim_{k\to\infty} \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f_k(z)|^q \frac{(1-|a|^2)^t}{|1-\langle z,a\rangle|^{s+t}}\, d\mu(z) = 0. \tag{5.2}$$

REMARK. Note that (5.1) does not depend on δ, which means that, in this corollary, we can choose any real number $\delta > -1$ satisfying $q/p + s/(n+1+\delta) \geq 1$ for λ and γ. Furthermore, if $s = 0$, then we can also take $t = 0$ since then the result reduces to the definition of (vanishing) Bergman–Carleson measures.

Proof of Corollary 5.1. We begin with the first part. The case $s = 0$ follows directly from the definition of Bergman–Carleson measures and the trivial inequality $|1 - \langle z, a\rangle| \geq (1 - |a|)$ for $z \in \mathbb{B}_n$. So we assume that $s > 0$. Since $s/(n+1+\delta) > 0$, we can choose two positive numbers p_2 and q_2 such that $s/(n+1+\delta) = q_2/p_2$. Then

$$\lambda = \frac{q}{p} + \frac{s}{n+1+\delta} = \frac{q}{p} + \frac{q_2}{p_2} \geq 1$$

and

$$\gamma = \frac{1}{\lambda}\left(\frac{\alpha q}{p} + \frac{\delta s}{n+1+\delta}\right) = \frac{1}{\lambda}\left(\frac{\alpha q}{p} + \frac{\delta q_2}{p_2}\right).$$

Let μ be a (λ, γ)-Bergman–Carleson measure. Then, from the previous observation and Theorem 1.1 we know that for any $f \in A^p_\alpha$ and $g \in A^{p_2}_\delta$, we have

$$\int_{\mathbb{B}_n} |f(z)|^q |g(z)|^{q_2}\, d\mu(z) \lesssim \|f\|^q_{p,\alpha} \cdot \|g\|^{q_2}_{p_2,\delta}. \tag{5.3}$$

For any $t > 0$, let

$$g(z) = g_a(z) = \frac{(1-|a|^2)^{t/q_2}}{(1 - \langle z, a\rangle)^{(n+1+\delta)/p_2 + t/q_2}}.$$

Using Lemma B, we easily check that $g_a \in A^{p_2}_\delta$ and $\sup_{a\in\mathbb{B}_n} \|g_a\|_{p_2,\delta} \lesssim 1$. Put $g = g_a$ in equation (5.3), take the supremum over all $a \in \mathbb{B}_n$, and we get (5.1).

Conversely, suppose (5.1) holds for some $t > 0$. Given an arbitrary $t_1 > 0$, let

$$f_a(z) = \frac{(1-|a|^2)^{t_1/q}}{(1 - \langle z, a\rangle)^{(n+1+\alpha)/p + t_1/q}}.$$

As before, it is easy to check that $f_a \in A^p_\alpha$ and $\|f_a\|_{p,\alpha} \lesssim 1$. It is clear that

$$(n+1+\gamma)\lambda = s + (n+1+\alpha)q/p,$$

and therefore, due to (5.1), we get

$$\int_{\mathbb{B}_n} \frac{(1-|a|^2)^{t+t_1}}{|1 - \langle z, a\rangle|^{(n+1+\gamma)\lambda + t + t_1}}\, d\mu(z) = \int_{\mathbb{B}_n} |f_a(z)|^q \frac{(1-|a|^2)^t}{|1 - \langle z, a\rangle|^{s+t}}\, d\mu(z) \lesssim 1.$$

Therefore,

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^{t+t_1}}{|1 - \langle z, a\rangle|^{(n+1+\gamma)\lambda + t + t_1}}\, d\mu(z) < \infty.$$

Since $\lambda \geq 1$, by Theorem A we see that μ is a (λ, γ)-Bergman–Carleson measure.

Next, we deal with the part concerning vanishing Bergman–Carleson measures. If $s = 0$, then the result follows easily from the definition of vanishing Bergman–Carleson measures, and so we assume that $s > 0$. If μ is a vanishing (λ, γ)-Bergman–Carleson measure, then, proceeding as in the first part, but using Theorem 4.1 instead of Theorem 1.1, we obtain (5.2). Conversely, suppose that

(5.2) holds for some $t > 0$. Let $\{a_k\} \subset \mathbb{B}_n$ with $|a_k| \to 1$ and, for arbitrary $t_1 > 0$, consider the functions

$$f_k(z) = \frac{(1-|a_k|^2)^{t_1/q}}{(1-\langle z, a_k\rangle)^{(n+1+\alpha)/p+t_1/q}}.$$

Then $\sup_k \|f_k\|_{p,\alpha} \le C$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{B}_n$, and using (5.2), we see that

$$\begin{aligned}
\int_{\mathbb{B}_n} \frac{(1-|a_k|^2)^{t+t_1}}{|1-\langle z, a_k\rangle|^{(n+1+\gamma)\lambda+t+t_1}}\, d\mu(z) &= \int_{\mathbb{B}_n} |f_k(z)|^q \frac{(1-|a_k|^2)^t}{|1-\langle z, a_k\rangle|^{s+t}}\, d\mu(z) \\
&\le \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f_k(z)|^q \frac{(1-|a|^2)^t}{|1-\langle z, a\rangle|^{s+t}}\, d\mu(z) \\
&\longrightarrow 0.
\end{aligned}$$

Since $\lambda \ge 1$, it follows from (4.1) that μ is a vanishing (λ, γ)-Bergman–Carleson measure. The proof is complete. $\square$

5.1. *Applications to Extended Cesàro Operators*

For $g \in H(\mathbb{B}_n)$, the radial derivative is defined by

$$Rg(z) = \sum_{k=1}^n z_k \frac{\partial g}{\partial z_k}(z),$$

and the extended Cesàro operator is defined by

$$J_g f(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}, \qquad f \in H(\mathbb{B}_n).$$

In the case of one variable, the operator is the same as

$$J_g f(z) = \int_0^z f(\xi) g'(\xi)\, d\xi,$$

which is also called the Riemann–Stieltjes operator. Here we are considering boundedness and compactness of these operators from a weighted Bergman space into the general space $F(p,q,s)$ on the unit ball, which is defined as the space of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^q (1-|\phi_a(z)|^2)^s\, dv(z) < \infty,$$

where $0 < p < \infty$, $-n-1 < q < \infty$, $0 \le s < \infty$, and $q+s > -1$. We also say that $f \in F_0(p,q,s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^q (1-|\phi_a(z)|^2)^s\, dv(z) = 0.$$

Here, for our purpose, we point out that if $s > n$ and $\alpha > 0$, then for any $p > 0$, the space $F(p, p\alpha - n - 1, s) = B^\alpha$, the α-Bloch space, which means the space of all functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_{B^\alpha} = \sup_{z\in\mathbb{B}_n} |Rf(z)|(1-|z|^2)^\alpha < \infty.$$

When $\alpha = 1$, $B^1 = B$, the classical Bloch space.

For the case of the unit disk, this result can be found in [25]. For the case of the unit ball $\mathbb{B}_n$, the result may be also known, but we were not able to find a reference, so we provide a brief proof here. First, we show that $F(p,q,s)$ are all subspaces of some α-Bloch space.

PROPOSITION 5.2. *Let* $0 < p < \infty$, $-n-1 < q < \infty$, $0 \le s < \infty$, *and* $q+s > -1$. *Then* $F(p,q,s) \subseteq B^{(n+1+q)/p}$.

Proof. Let $f \in F(p,q,s)$. By subharmonicity we have that, for a fixed r, $0 < r < 1$,

$$|Rf(a)|^p \lesssim \frac{1}{(1-|a|^2)^{n+1}} \int_{D(a,r)} |Rf(z)|^p\, dv(z).$$

Let $\alpha = (n+1+q)/p$. Then $q = p\alpha - n - 1$. Hence,

$$\begin{aligned} |Rf(a)|^p(1-|a|^2)^{p\alpha} &\lesssim \frac{1}{(1-|a|^2)^{n+1-p\alpha}} \int_{D(a,r)} |Rf(z)|^p\, dv(z) \\ &\lesssim \int_{D(a,r)} |Rf(z)|^p (1-|z|^2)^{p\alpha-n-1}\, dv(z). \end{aligned}$$

Since $|1-\langle z,a\rangle| \asymp (1-|z|^2) \asymp (1-|a|^2)$ for $z \in D(a,r)$, we know from (2.1) that $1-|\phi_a(z)|^2 \asymp 1$ for $z \in D(a,r)$, and so, for $s \ge 0$,

$$\begin{aligned} |Rf(a)|^p(1-|a|^2)^{p\alpha} &\lesssim \int_{D(a,r)} |Rf(z)|^p (1-|z|^2)^{p\alpha-n-1}(1-|\phi_a(z)|^2)^s\, dv(z) \\ &\lesssim \int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^{p\alpha-n-1}(1-|\phi_a(z)|^2)^s\, dv(z). \end{aligned}$$

This clearly implies $F(p, p\alpha-n-1, s) \subseteq B^\alpha$, or $F(p,q,s) \subseteq B^{(n+1+q)/p}$. □

PROPOSITION 5.3. *Let* $0 < p < \infty$, $-n-1 < q < \infty$, $0 \le s < \infty$, *and* $q+s > -1$. *If* $s > n$, *then* $F(p,q,s) = B^{(n+1+q)/p}$.

Proof. Let $\alpha = (n+1+q)/p$. The inclusion $F(p,q,s) \subseteq B^\alpha$ has been proved in the previous proposition. Now we are proving the opposite inclusion. Let $f \in B^\alpha$ and assume that $s > n$. Then, by Lemma B,

$$\begin{aligned} \|f\|^p_{F(p,q,s)} &= \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |Rf(z)|^p (1-|z|^2)^{p\alpha-n-1}(1-|\phi_a(z)|^2)^s\, dv(z) \\ &\le \|f\|^p_{B^\alpha} \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} (1-|z|^2)^{-n-1}(1-|\phi_a(z)|^2)^s\, dv(z) \\ &\lesssim \|f\|^p_{B^\alpha} \sup_{a\in\mathbb{B}_n} (1-|a|^2)^s \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{s-n-1}}{|1-\langle z,a\rangle|^{2s}}\, dv(z) \\ &\lesssim \|f\|^p_{B^\alpha}, \end{aligned}$$

and so $f \in F(p, q, s)$. The proof is complete. $\square$

In a similar way we can prove that, under the same restrictions on the parameters, $F_0(p, q, s) \subseteq B_0^{(n+1+q)/p}$, and $F_0(p, q, s) = B_0^{(n+1+q)/p}$ if $s > n$, where, for $\alpha > 0$, B_0^α is the closed subspace of B^α that consists of functions $f \in H(\mathbb{B}_n)$ such that

$$\lim_{|z| \to 1} |Rf(z)|(1 - |z|^2)^\alpha = 0$$

and is called the little α-Bloch space. We will frequently use the following well-known result [28, Exer. 7.7] for the α-Bloch space: for $\alpha > 1$, an analytic function $f \in B^\alpha$ if and only if

$$\sup_{z \in \mathbb{B}_n} |f(z)|(1 - |z|^2)^{\alpha - 1} < \infty,$$

and the norm of f in B^α is

$$|f(0)| + \|f\|_{B^\alpha} \asymp \sup_{z \in \mathbb{B}_n} |f(z)|(1 - |z|^2)^{\alpha - 1}, \quad \alpha > 1. \tag{5.4}$$

THEOREM 5.4. *Let $0 < p, t, \alpha < \infty$, $-1 < \beta < \infty$, $0 \le s < \infty$, with $p\beta + s > n$. Let $g \in H(\mathbb{B}_n)$ and suppose that $\beta - (n + 1 + \alpha)/t > 0$ and $p/t + s/(n + 1 + \delta) \ge 1$ for some $\delta > -1$. Then*

(a) *J_g is a bounded operator from A_α^t into $F(p, p\beta - n - 1, s)$ if and only if $g \in B^{\beta - (n+1+\alpha)/t}$;*
(b) *J_g is a compact operator from A_α^t into $F(p, p\beta - n - 1, s)$ if and only if $g \in B_0^{\beta - (n+1+\alpha)/t}$.*

Proof. An easy computation shows that $R(J_g f) = f Rg$. By definition, J_g is bounded from A_α^t into $F(p, p\beta - n - 1, s)$ if and only if, for any $f \in A_\alpha^t$,

$$\begin{aligned}
&\|J_g f\|_{F(p, p\beta - n - 1, s)}^p \\
&\quad = \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |f(z)|^p |Rg(z)|^p (1 - |z|^2)^{p\beta - n - 1} (1 - |\phi_a(z)|^2)^s \, dv(z) \\
&\quad = \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |f(z)|^p |Rg(z)|^p (1 - |z|^2)^{s + p\beta - n - 1} \frac{(1 - |a|^2)^s}{|1 - \langle z, a\rangle|^{2s}} \, dv(z) \\
&\quad = \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |f(z)|^p \frac{(1 - |a|^2)^s}{|1 - \langle z, a\rangle|^{2s}} \, d\mu_g(z) \\
&\quad \le C \|f\|_{t,\alpha}^p,
\end{aligned}$$

where $d\mu_g(z) = |Rg(z)|^p (1 - |z|^2)^{s + p\beta - n - 1} \, dv(z)$. By Corollary 5.1, this is equivalent to that μ_g is an (λ, γ)-Bergman–Carleson measure, where

$$\lambda = \frac{p}{t} + \frac{s}{n + 1 + \delta} \quad \text{and} \quad \gamma = \frac{1}{\lambda}\left(\frac{\alpha p}{t} + \frac{\delta s}{n + 1 + \delta}\right).$$

Then, by the condition in the theorem, $\lambda \geq 1$, and it is easy to check that $\gamma > -1$. Thus, by Theorem A the boundedness of J_g is equivalent to

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^{(n+1+\gamma)\lambda}}{|1-\langle z,a\rangle|^{2(n+1+\gamma)\lambda}} |Rg(z)|^p (1-|z|^2)^{s+p\beta-n-1}\, dv(z) < \infty. \tag{5.5}$$

An easy computation shows that

$$(n+1+\gamma)\lambda = s + (n+1+\alpha)\frac{p}{t},$$

and (5.5) becomes

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^{s+(n+1+\alpha)p/t}}{|1-\langle z,a\rangle|^{2(s+(n+1+\alpha)p/t)}} |Rg(z)|^p (1-|z|^2)^{s+p\beta-n-1}\, dv(z) < \infty,$$

which is the same as

$$\begin{aligned}\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} & |Rg(z)|^p (1-|z|^2)^{p(\beta-(n+1+\alpha)/t)-n-1} \\ & \times (1-|\phi_a(z)|^2)^{s+(n+1+\alpha)p/t}\, dv(z) < \infty.\end{aligned}$$

Thus, the operator J_g is bounded from A^t_α into $F(p, p\beta-n-1, s)$ if and only if

$$g \in F\left(p, q, s + (n+1+\alpha)\frac{p}{t}\right) \tag{5.6}$$

with

$$q = p\left(\beta - \frac{n+1+\alpha}{t}\right) - n - 1.$$

Since $\lambda \geq 1$ and $\gamma > -1$, we know that

$$s + (n+1+\alpha)\frac{p}{t} = (n+1+\gamma)\lambda \geq n+1+\gamma > n,$$

and so, by Proposition 5.3, condition (5.6) is equivalent to $g \in B^{\beta-(n+1+\alpha)/t}$, which proves part (a).

Using the second part of Corollary 5.1, the criterion for the compactness in part (b) is proved in the same way. We omit the details. □

Our next result is for the integral operator

$$I_g f(z) = \int_0^1 Rf(tz)g(tz)\frac{dt}{t}.$$

This operator can be considered as a companion of the operator J_g.

THEOREM 5.5. *Let $0 < p, t, \beta < \infty$, $-1 < \alpha < \infty$, $0 \leq s < \infty$ with $p\beta + s > n$. Let $g \in H(\mathbb{B}_n)$ and suppose that $p/t + s/(n+1+\delta) \geq 1$ for some $\delta > -1$. Then I_g is a bounded operator from A^t_α into $F(p, p\beta-n-1, s)$ if and only if*

(i) *$g \in B^{\beta-(n+1+\alpha)/t}$ for $\beta > 1 + (n+1+\alpha)/t$;*
(ii) *$g \in H^\infty$ for $\beta = 1 + (n+1+\alpha)/t$;*
(iii) *$g \equiv 0$ for $0 < \beta < 1 + (n+1+\alpha)/t$.*

Proof. Assume that either (i), or (ii), or (iii) holds. We proceed to show that the operator $I_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$ is bounded. First, we consider case (i), that is, when $\beta > 1 + (n + 1 + \alpha)/t$. Let $g \in B^{\beta-(n+1+\alpha)/t}$. An easy computation shows that $R(I_g f) = gRf$. Hence, due to (5.4), for any $f \in A^t_\alpha$, we have

$$\begin{aligned}
&\|I_g f\|^p_{F(p,p\beta-n-1,s)} \\
&\quad = \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |g(z)|^p |Rf(z)|^p (1-|z|^2)^{p\beta-n-1}(1-|\phi_a(z)|^2)^s \, dv(z) \\
&\quad = \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |g(z)|^p |Rf(z)|^p (1-|z|^2)^{s+p\beta-n-1} \frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{2s}} \, dv(z) \\
&\quad \le \|g\|^p_{B^{\beta-(n+1+\alpha)/t}} \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |Rf(z)|^p \frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{2s}} \, d\mu(z),
\end{aligned} \tag{5.7}$$

where $d\mu(z) = (1-|z|^2)^{(n+1+\alpha)p/t+p+s-n-1} \, dv(z)$. By Lemma B, for any $\eta > 0$, we have

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|a|^2)^\eta}{|1-\langle z,a\rangle|^{\eta+(n+1+\alpha)p/t+p+s}} \, d\mu(z) \\
&\qquad = \sup_{a\in\mathbb{B}_n} (1-|a|^2)^\eta \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{(n+1+\alpha)p/t+p+s-n-1}}{|1-\langle z,a\rangle|^{\eta+(n+1+\alpha)p/t+p+s}} \, dv(z) \\
&\qquad < \infty.
\end{aligned} \tag{5.8}$$

Notice that the application of Lemma B here is correct since if we let

$$\lambda = \frac{p}{t} + \frac{s}{n+1+\delta} \quad \text{and} \quad \gamma = \frac{1}{\lambda}\left(\frac{(t+\alpha)p}{t} + \frac{\delta s}{n+1+\delta}\right),$$

then, by the condition in the theorem, $\lambda \ge 1$, and it is easy to check that $\gamma > -1$. Also, an easy computation shows that

$$(n+1+\gamma)\lambda = s + (n+1+\alpha+t)\frac{p}{t} = (n+1+\alpha)\frac{p}{t} + p + s. \tag{5.9}$$

Thus, we have

$$\begin{aligned}
(n+1+\alpha)\frac{p}{t} + p + s - n - 1 &= (n+1+\gamma)\lambda - n - 1 \\
&\ge n+1+\gamma-n-1 = \gamma > -1.
\end{aligned}$$

Hence, due to (5.9), condition (5.8) means that μ is a (λ,γ)-Bergman–Carleson measure, and so by (5.7), Corollary 5.1, and [28, Thm. 2.16] we have that

$$\|I_g f\|^p_{F(p,p\beta-n-1,s)} \lesssim \|g\|^p_{B^{\beta-(n+1+\alpha)/t}} \|Rf\|^p_{t,t+\alpha} \asymp \|g\|^p_{B^{\beta-(n+1+\alpha)/t}} \|f\|^p_{t,\alpha},$$

and so $I_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$ is bounded.

Case (ii) is proved in the exactly same way as the proof for case (i) with $\|g\|_{B^{\beta-(n+1+\alpha)/t}}$ replaced by $\|g\|_{H^\infty}$. Case (iii) is trivial.

Conversely, suppose that $I_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$ is bounded. Then, by Proposition 5.2 the operator $I_g : A^t_\alpha \to B^\beta$ is also bounded. For $\eta > 0$ and

$a \in \mathbb{B}_n$, let

$$f_a(z) = \frac{(1-|a|^2)^\eta}{(1-\langle z,a\rangle)^{\eta+(n+1+\alpha)/t}}.$$

It is easy to check that $\sup_{a\in\mathbb{B}_n} \|f_a\|_{t,\alpha} \le C$. An easy computation shows that

$$Rf_a(z) = \frac{1}{\eta+(n+1+\alpha)/t}\frac{(1-|a|^2)^\eta}{(1-\langle z,a\rangle)^{\eta+(n+1+\alpha)/t+1}}.$$

Note that $R(I_g f_a)(z) = Rf_a(z)g(z)$, and therefore

$$\begin{aligned}(1-|a|^2)^{\beta-(n+1+\alpha)/t-1}|g(a)| &\asymp |Rf_a(a)||g(a)|(1-|a|^2)^\beta \\ &\le \sup_{z\in\mathbb{B}_n} |Rf_a(z)||g(z)|(1-|z|^2)^\beta \\ &= \|I_g f_a\|_{B^\beta} \le C\|I_g\|. \qquad (5.10)\end{aligned}$$

This directly gives (ii), and, by the maximum principle, we also obtain (iii). Part (i) follows from (5.4). The proof is complete. □

Similarly, by a standard method, we can prove the following compactness result.

THEOREM 5.6. *Let* $0 < p, t, \beta < \infty$, $-1 < \alpha < \infty$, $0 \le s < \infty$ *with* $p\beta + s > n$. *Let* $g \in H(\mathbb{B}_n)$ *and suppose that* $p/t + s/(n+1+\delta) \ge 1$ *for some* $\delta > -1$. *Then* I_g *is a compact operator from* A^t_α *into* $F(p, p\beta-n-1, s)$ *if and only if*

(i) $g \in B_0^{\beta-(n+1+\alpha)/t}$ *for* $\beta > 1 + (n+1+\alpha)/t$;
(ii) $g \equiv 0$ *for* $0 < \beta \le 1 + (n+1+\alpha)/t$.

Proof. If I_g is a compact operator from A^t_α into $F(p, p\beta-n-1, s)$, then $I_g : A^t_\alpha \to B^\beta$ is also compact due to Proposition 5.2. Let $\{a_k\} \subset \mathbb{B}_n$ with $|a_k| \to 1$ and, for $\eta > 0$, consider the sequence of holomorphic functions $\{f_k\}$ given by

$$f_k(z) = \frac{(1-|a_k|^2)^\eta}{(1-\langle z,a_k\rangle)^{\eta+(n+1+\alpha)/t}}.$$

As before, $\sup_k \|f_k\|_{t,\alpha} \le C$, and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{B}_n$. Since I_g is compact, from (5.10) we get

$$(1-|a_k|^2)^{\beta-(n+1+\alpha)/t-1}|g(a_k)| \lesssim \|I_g f_k\|_{B^\beta} \to 0.$$

This gives (ii) by the maximum principle and also (i) since, as in (5.4), for $\sigma > 1$, a function $f \in H(\mathbb{B}_n)$ is in B_0^σ if and only if $\lim_{|z|\to 1^-}(1-|z|^2)^{\sigma-1}|f(z)| = 0$.

Conversely, assume that (i) holds, that is, $\beta > 1 + (n+1+\alpha)/t$ and $g \in B_0^{\beta-(n+1+\alpha)/t}$. Then, given $\varepsilon > 0$, there is $0 < r_0 < 1$ such that

$$\sup_{r_0<|z|<1} (1-|z|^2)^{\beta-(n+1+\alpha)/t-1}|g(z)| < \varepsilon. \qquad (5.11)$$

Let $\{f_k\}$ be a bounded sequence in A^t_α converging to zero uniformly on compact subsets of $\mathbb{B}_n$. From (5.7) we get

$$\|I_g f_k\|^p_{F(p,p\beta-n-1,s)} = I_1(k) + I_2(k) \qquad (5.12)$$

with

$$I_1(k) := \sup_{a\in\mathbb{B}_n} \int_{|z|\le r_0} |g(z)|^p |Rf_k(z)|^p (1-|z|^2)^{s+p\beta-n-1} \frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{2s}}\, dv(z)$$

and

$$I_2(k) := \sup_{a\in\mathbb{B}_n} \int_{r_0<|z|<1} |g(z)|^p |Rf_k(z)|^p (1-|z|^2)^{s+p\beta-n-1} \frac{(1-|a|^2)^s}{|1-\langle z,a\rangle|^{2s}}\, dv(z).$$

Since $\{Rf_k\}$ also converges to zero uniformly on compact subsets of $\mathbb{B}_n$, there is a positive integer k_0 such that $\sup_{|z|\le r_0} |Rf_k(z)| < \varepsilon$ for $k \ge k_0$. Then, using (5.4), we easily see that

$$I_1(k) \le C\varepsilon^p \|g\|^p_{B^{\beta-(n+1+\alpha)/t}}.$$

On the other hand, by (5.11) and arguing as in the proof of Theorem 5.5, we obtain

$$I_2(k) < C\varepsilon^p \|f_k\|^p_{t,\alpha} \le C\varepsilon^p. \tag{5.13}$$

This shows that $\|I_g f_k\|_{F(p,p\beta-n-1,s)} \to 0$, proving that I_g is compact. Since case (ii) is trivial, the proof is complete. □

5.2. *Pointwise Multipliers*

For an holomorphic function g in $\mathbb{B}_n$, the pointwise multiplication operator M_g is defined as follows: $M_g f = gf$ for $f \in H(\mathbb{B}_n)$.

LEMMA 5.7. *Let $-1 < \alpha < \infty$, $0 < t, \beta < \infty$, and suppose that $M_g : A^t_\alpha \to B^\beta$ is bounded. Then*

(i) $g \in B^{\beta-(n+1+\alpha)/t}$ *if* $\beta > 1 + (n+1+\alpha)/t$;
(ii) $g \in H^\infty$ *if* $\beta = 1 + (n+1+\alpha)/t$;
(iii) $g \equiv 0$ *if* $0 < \beta < 1 + (n+1+\alpha)/t$.

Proof. By definition it is easy to see that $B^{\beta_1} \subseteq B^{\beta_2}$ for $\beta_1 < \beta_2$. Hence, in case (iii) we may assume that $1 < \beta < 1 + (n+1+\alpha)/t$. For $\eta > 0$ and $a \in \mathbb{B}_n$, let

$$f_a(z) = \frac{(1-|a|^2)^\eta}{(1-\langle z,a\rangle)^{\eta+(n+1+\alpha)/t}}.$$

We have seen before that $\{f_a\}$ is uniformly bounded in A^t_α. Since $M_g : A^t_\alpha \to B^\beta$ is bounded, we know that

$$\sup_{a\in\mathbb{B}_n} (|g(0)f_a(0)| + \|gf_a\|_{B^\beta}) \lesssim \sup_{a\in\mathbb{B}_n} \|f_a\|_{t,\alpha} < \infty.$$

However, since $\beta > 1$, by (5.4) we get

$$\begin{aligned} |g(0)f_a(0)| + \|gf_a\|_{B^\beta} &= |g(0)|(1-|a|^2)^\eta + \sup_{z\in\mathbb{B}_n} |g(z)||f_a(z)|(1-|z|^2)^{\beta-1} \\ &\ge |g(a)||f_a(a)|(1-|a|^2)^{\beta-1} \\ &= |g(a)|(1-|a|^2)^{\beta-1-(n+1+\alpha)/t}. \end{aligned}$$

Hence, we get

$$\sup_{a\in\mathbb{B}_n} |g(a)|(1-|a|^2)^{\beta-1-(n+1+\alpha)/t} < \infty,$$

which gives (i) and (ii) and also gives (iii) by the maximum principle. □

Now we are ready to prove the following characterizations for bounded pointwise multiplication operators from A^t_α to $F(p,q,s)$ spaces.

THEOREM 5.8. *Let* $0 < p, t, \beta < \infty$, $-1 < \alpha < \infty$, $0 \le s < \infty$ *with* $p\beta + s > n$. *Let* $g \in H(\mathbb{B}_n)$ *and suppose that* $p/t + s/(n+1+\delta) \ge 1$ *for some* $\delta > -1$. *Then* $M_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$ *is bounded if and only if*

(i) $g \in B^{\beta-(n+1+\alpha)/t}$ *for* $\beta > 1 + (n+1+\alpha)/t$;
(ii) $g \in H^\infty$ *for* $\beta = 1 + (n+1+\alpha)/t$;
(iii) $g \equiv 0$ *for* $0 < \beta < 1 + (n+1+\alpha)/t$.

Proof. Suppose that either (i), or (ii), or (iii) is satisfied. Then, by Theorem 5.4 and Theorem 5.5 we know that both J_g and I_g are bounded from A^t_α to $F(p, p\beta - n - 1, s)$. Since

$$R(M_g f) = fRg + gRf = R(I_g f) + R(J_g f),$$

we easily see that M_g is also bounded from A^t_α to $F(p, p\beta - n - 1, s)$.

Conversely, suppose that $M_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$ is bounded. By Proposition 5.2 we know that $M_g : A^t_\alpha \to B^\beta$ is also bounded, and so by Lemma 5.7 we directly get (i), (ii), and (iii). □

The result on the compactness of the multiplication operator is stated next.

THEOREM 5.9. *Let* $0 < p, t, \beta < \infty$, $-1 < \alpha < \infty$, $0 \le s < \infty$ *with* $p\beta + s > n$. *Let* $g \in H(\mathbb{B}_n)$ *and suppose that* $p/t + s/(n+1+\delta) \ge 1$ *for some* $\delta > -1$. *Then* $M_g : A^t_\alpha \to F(p, p\beta - n - 1, s)$ *is compact if and only if*

(i) $g \in B_0^{\beta-(n+1+\alpha)/t}$ *for* $\beta > 1 + (n+1+\alpha)/t$;
(ii) $g \equiv 0$ *for* $0 < \beta \le 1 + (n+1+\alpha)/t$.

This follows, using standard arguments, arguing in a similar way as in Theorem 5.8. We omit the proof here.

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