

Quasiconformal Homogeneity and Subgroups of the Mapping Class Group

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ABSTRACT. In the vein of Bonfert-Taylor, Bridgeman, Canary, and Taylor we introduce the notion of quasiconformal homogeneity for closed oriented hyperbolic surfaces restricted to subgroups of the mapping class group. We find uniform lower bounds for the associated quasiconformal homogeneity constants across all closed hyperbolic surfaces in several cases, including the Torelli group, congruence subgroups, and pure cyclic subgroups. Further, we introduce a counting argument providing a possible path to exploring a uniform lower bound for the nonrestricted quasiconformal homogeneity constant across all closed hyperbolic surfaces.

1. Introduction

Let M be a hyperbolic manifold, and $\text{QC}(M)$ be the associated group of quasiconformal homeomorphisms from M to itself. Given any subgroup $\Gamma \leq \text{QC}(M)$, we say that M is Γ -homogeneous if the action of Γ on M is transitive. Furthermore, we say that M is Γ_K -homogeneous for $K \in [1, \infty)$ if the restriction of the action of Γ on M to the subset

$$\Gamma_K = \{f \in \Gamma : K_f \leq K\}$$

on M is transitive, where $K_f = \inf\{K : f \text{ is } K\text{-quasiconformal}\}$ is the dilatation of f .

If $\Gamma = \text{QC}(M)$ and there exists a K such that M is Γ_K -homogeneous, then this manifold is said to be K -quasiconformally homogeneous, or K -qch. In [BTCMT05] it is shown that for each $n \geq 3$, there exists a constant $K_n > 1$ such that if $M \neq \mathbb{H}^n$ is an n -dimensional K -quasiconformally homogeneous hyperbolic manifold, then $K \geq K_n$. This result relies on rigidity in higher dimensions, which does not occur in dimension two. The natural question motivating this paper is as follows.

QUESTION 1.1. *Does there exist a constant $K_2 > 1$ such that every K -qch surface $X \neq \mathbb{H}^2$ satisfies $K \geq K_2$?*

Let $\text{Homeo}^+(S)$ be the group of orientation-preserving homeomorphisms of a surface S . Then the mapping class group of S is defined to be $\pi_0(\text{Homeo}^+(S))$ and is denoted $\text{Mod}(S)$. Given a closed hyperbolic surface X and $f \in \text{QC}(X)$, let $[f] \in \text{Mod}(X)$ denote its homotopy class. Then, the map $\pi : \text{QC}(X) \rightarrow \text{Mod}(X)$

defined by $f \mapsto [f]$ is surjective. If $H \leq \text{Mod}(X)$, then we say that X is H -homogeneous if X is $\pi^{-1}(H)$ -homogeneous. Similarly, we say that X is H_K -homogeneous if it is $\pi^{-1}(H)_K$ -homogeneous.

The focus of this paper will be to restrict ourselves to homogeneity with respect to subgroups of the mapping class group of closed hyperbolic surfaces and find lower bounds for the associated homogeneity constants. We will go about this by leveraging lower bounds on the quasiconformal dilatations for maps in a given homotopy class.

Torelli and Congruence Subgroups

Let S be a closed orientable surface; then $\text{Mod}(S)$ acts on the first homology $H_1(S, \mathbb{Z})$ by isomorphisms, and the kernel of this action is called the *Torelli group*, denoted $\mathcal{I}(S)$. Similarly, the kernel of the action of $\text{Mod}(S)$ on $H_1(S, \mathbb{Z}/r\mathbb{Z})$ is called the *level r congruence subgroup* and is denoted by $\text{Mod}(S)[r]$. The first theorem gives a universal bound on the quasiconformal homogeneity constant with respect to these subgroups for closed hyperbolic surfaces.

THEOREM 1.2. *There exists a constant $K_T > 1$ such that if X is a closed hyperbolic surface that is Γ_K -homogeneous for $\Gamma = \mathcal{I}(X)$ or $\Gamma = \text{Mod}(X)[r]$ with $r \geq 3$, then $K \geq K_T$.*

The case of $\Gamma = \mathcal{I}(X)$ was independently discovered by Greenfield [Gre13].

Since $H_1(S, \mathbb{Z}/r\mathbb{Z})$ is a finite group, so is its automorphism group; hence, $\text{Mod}(S)[r]$ is finite index in $\text{Mod}(S)$. Theorem 1.2 provides an optimistic outlook for answering Question 1.1 in the positive for the case of closed surfaces.

Homogeneity and Teichmüller Space

The rest of the paper is flavored by a technique, introduced in Section 4, which translates questions about homogeneity constants to questions about orbit points under the action of the mapping class group on Teichmüller space. Given a closed hyperbolic surface S , we define its associated *Teichmüller space* $\text{Teich}(S)$ to be the space of equivalence classes of pairs (X, ϕ) , where X is a hyperbolic surface, and $\phi : S \rightarrow X$ is a homeomorphism called the *marking*. Two such pairs (X, ϕ) and (Y, ψ) are equivalent if $\psi \circ \phi^{-1} : X \rightarrow Y$ is homotopic to an isometry (see [Hub06]). The mapping class group $\text{Mod}(S)$ acts on $\text{Teich}(S)$ by changing the marking:

$$[f] \cdot [(X, \phi)] = [(X, \phi \circ f^{-1})].$$

Furthermore, this action is by isometries with respect to the Teichmüller metric on $\text{Teich}(S)$, which is defined by

$$d_T([(X, \phi)], [(Y, \psi)]) = \frac{1}{2} \log(\min K(h)),$$

where the minimum of the quasiconformal dilatation is over all quasiconformal maps $h : X \rightarrow Y$ homotopic to $\psi \circ \phi^{-1}$. The fact that this minimum exists is a well-known theorem of Teichmüller (a proof can be found in [Hub06]).

Our next theorem is a direct result of the technique mentioned before and gives a possible path to finding a lower bound for the quasiconformal homogeneity constant for closed hyperbolic surfaces. It is shown in [BTCMT05] (see Theorem 2.3 below) that surfaces with short curves have large homogeneity constants. We let

$$\text{Teich}_{(\varepsilon, \infty)}(S) = \{(X, \phi) \in \text{Teich}(S) : \ell(X) > \varepsilon\},$$

where $\ell(X)$ is the length of the systole. Also, given a point $\mathcal{X} \in \text{Teich}(S)$, let $B_R(\mathcal{X})$ be the ball of radius R about \mathcal{X} in $(\text{Teich}(S), d_T)$. We let S_g be an oriented closed genus g surface.

THEOREM 1.3. *Suppose there exist constants $\varepsilon, R, C > 0$ such that for any $\mathcal{X} \in \text{Teich}_{(\varepsilon, \infty)}(S_g)$ with $g > 1$,*

$$|\{f \in \text{Mod}(S_g) : f \cdot \mathcal{X} \in B_R(\mathcal{X})\}| \leq Cg.$$

Then, there exists a constant $K_2 > 1$ such that any closed K -qch surface must have $K \geq K_2$.

QUESTION 1.4. *Does there exist such ε, R, C ?*

Note that ε and C can be chosen to be arbitrarily large and R can be chosen to be arbitrarily small.

Finite, Cyclic, and Torsion-Free Subgroups

Returning to more restrictive forms of homogeneity, we use this counting method to consider finite and cyclic subgroups of the mapping class group.

THEOREM 1.5. *There exists a constant $K_F > 1$ such that if a closed hyperbolic surface X is Γ_K -homogeneous, where $\Gamma < \text{Mod}(X)$ has finite order, then $K \geq K_F$. Furthermore, we have*

$$K_F \geq \sqrt{\psi \left(2 \operatorname{arccosh} \left(\frac{1}{42} + 1 \right) \right)} = 1.11469 \dots,$$

where ψ is defined in equation (4.1).

THEOREM 1.6. *There exists a constant $K_C > 1$ such that if a closed hyperbolic surface X is Γ_K -homogeneous, where $\Gamma = \langle [f] \rangle$ with $[f] \in \text{Mod}(X)$ a pure mapping class, then $K \geq K_C$. Furthermore, we have $K_C \geq 1.09297$.*

It is particularly difficult to understand the orbit of points in $\text{Teich}(S)$ under periodic mapping classes; hence, our last theorem deals with torsion-free subgroups of $\text{Mod}(S)$.

THEOREM 1.7. *Let X be a closed hyperbolic surface and suppose $\Gamma < \text{Mod}(X)$ is torsion-free. If X is Γ_K -homogeneous, then*

$$\log K \geq \frac{1}{7,000g^2},$$

where g is the genus of X .

QUESTION 1.8. *Can one find a constant C such that every closed K -qch surface satisfies $K \geq Cg^{-2}$?*

The rest of the paper discusses how to define continuous functions on Teichmüller space and moduli space using subgroups of the mapping class group and the associated homogeneity constants for surfaces.

Related Results in the Literature

In recent years there have been several papers published that make progress toward understanding quasiconformal homogeneity of surfaces. In [BTBCT07] the authors bound the quasiconformal constant of hyperbolic surfaces having automorphisms with many fixed points away from 1, in particular, all hyperelliptic surfaces. In the same paper, they also consider homogeneity with respect to $\Gamma = \{e\} < \text{Mod}(X)$ and $\text{Aut}(X)$. They prove that a surface is $\{e\}_K$ -homogeneous for some K if and only if it is closed; furthermore, there exists a constant $K_e > 1$ such that $K \geq K_e$. In a similar fashion, the authors find that a hyperbolic surface X is $\text{Aut}(X)_K$ -homogeneous for some K if and only if it is a regular cover of a hyperbolic orbifold; furthermore, there exists a constant $K_{\text{aut}} > 1$ such that $K \geq K_{\text{aut}}$. A sharp bound is found for the constant K_{aut} in [BTMRT11]. The authors in [KM11] show the existence of a lower bound $K_0 > 1$ for the quasiconformal homogeneity constant of genus zero surfaces, which answers a question about quasiconformal homogeneity of planar domains posed by Gehring and Palka in [GP76].

2. Background

2.1. Quasiconformal Geometry

We may think of a quasiconformal map $f : \mathbb{C} \rightarrow \mathbb{C}$ as a function whose derivative df_p sends the unit circle in $T_p\mathbb{C}$ to an ellipse in $T_{f(p)}\mathbb{C}$ whose ratio of the major to minor axis we call $K_f(p)$, wherever the derivative is defined, and $K_f(p)$ is required to be bounded uniformly for all $p \in \mathbb{C}$. We let K_f or $K(f)$ denote the *dilatation* of f , which is defined to be the supremum of $K_f(p)$ over all of \mathbb{C} . Since this is a local condition, this notion holds for Riemann surfaces. Since $K_{f \circ g} \leq K_f \cdot K_g$, we see that

$$\text{QC}(X) = \{f : X \rightarrow X \mid f \text{ is a quasiconformal homeomorphism}\}$$

is a group. We refer the reader to [Hub06] and [FM11] for details.

There are two properties of quasiconformal maps that will play a key role in what follows. The first property shows us that quasiconformal maps retain some of the nicety of conformal maps. Let \mathbb{D} denote the unit disk in \mathbb{C} . The following theorem and proof can be found in [Hub06].

THEOREM 2.1. *Denote by $\mathcal{F}_K(\mathbb{D})$ the set of K -quasiconformal homeomorphisms $f : \mathbb{D} \rightarrow \mathbb{D}$ with $f(0) = 0$. Then $\mathcal{F}_K(\mathbb{D})$ is a normal family.*

We will rely heavily on this theorem for the convergence of sequences of quasiconformal maps, especially in understanding the continuity of the quasiconformal homogeneity constants on the moduli space.

The next property relates the quasiconformal condition of a homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ to the geometry of the hyperbolic plane. We say that $f : \mathbb{D} \rightarrow \mathbb{D}$ is an (A, B) -quasi-isometry if there are constants $A, B > 0$ such that

$$\frac{d(z, w)}{A} - B \leq d(f(z), f(w)) \leq Ad(z, w) + B$$

for all $z, w \in \mathbb{D}$ and where d is the hyperbolic metric on \mathbb{D} . The following theorem can be found in [Vuo88].

THEOREM 2.2. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be K -quasiconformal. Then f is a $(K, K \log 4)$ -quasi-isometry with respect to the hyperbolic metric.*

In particular, the image of a geodesic $\gamma \in \mathbb{D}$ under a K -quasiconformal $f : \mathbb{D} \rightarrow \mathbb{D}$ is a $(K, K \log 4)$ -quasi-geodesic. It is well known (see [Kap01]) that a quasi-geodesic stays within a bounded distance of a geodesic. In our case, we know that there exists some $C(K)$ and some geodesic $\tilde{\gamma}$ such that $f(\gamma) \subset N_{C(K)}(\tilde{\gamma})$, where $N_{C(K)}$ is the $C(K)$ -neighborhood.

2.2. Quasiconformal Homogeneity

The main goal of this section is to state one of the main results of [BTCMT05], which describes how quasiconformal homogeneity interacts with the geometry of a manifold. Though this paper is focused on surfaces, their work deals with arbitrary dimension, so for the moment, we will work in the general setting of hyperbolic manifolds. If M is an orientable hyperbolic n -manifold, then there exists a discrete subgroup $\Gamma < \text{Isom}^+(\mathbb{H}^n)$, called a *Kleinian group*, so that M is isometric to \mathbb{H}^n / Γ . The action of Γ extends to $\partial\mathbb{H}^n = \mathbb{S}^{n-1}$ and acts by conformal automorphisms. The *limit set* $\Lambda(\Gamma)$ is defined to be the intersection of the closure of an orbit of a point $x \in \mathbb{H}^n$ with $\partial\mathbb{H}^n$, that is, $\Lambda(\Gamma) = \overline{\Gamma \cdot x} \cap \partial\mathbb{H}^n$ (note that this definition is independent of the choice of x). See [Thu79] for more on Kleinian groups. Also define $\ell(M)$ to be the infimum of the lengths of homotopically nontrivial curves in M and define $d(M)$ to be the supremum of the diameters of embedded hyperbolic balls in M .

THEOREM 2.3 (Theorem 1.1 in [BTCMT05]). *For each dimension $n \geq 2$ and each $K \geq 1$, there is a positive constant $m(n, K)$ with the following property. Let $M = \mathbb{H}^n / \Gamma$ be a K -quasiconformally homogeneous hyperbolic n -manifold that is not \mathbb{H}^n . Then:*

- (1) $d(M) \leq K\ell(M) + 2K \log 4$.
- (2) $\ell(M) \geq m(n, K)$, that is, there is a lower bound on the injectivity radius of M that only depends on n and K .
- (3) Every nontrivial element of Γ is hyperbolic, and the limit set $\Lambda(\Gamma)$ of Γ is $\partial\mathbb{H}^n$.

In addition, every closed manifold is K -quasiconformally homogeneous for some K (also in [BTCMT05]). These facts tell us that a geometrically finite hyperbolic surface is K -quasiconformally homogeneous for some K if and only if it is closed. Observe that if $G < G' < \text{Mod}(X)$ for some hyperbolic surface X , then if X is G_K -homogeneous, then we have that X is also G'_K -homogeneous. In particular, a geometrically finite hyperbolic surface X is G_K -homogeneous for $G < \text{Mod}(X)$ if and only if X is closed. This fact will be our motivation for stating our theorems in terms of closed surfaces as opposed to the geometrically finite terminology.

The other key tool we will need comes from understanding the quasiconformal homogeneity constant under geometric convergence and the fact that the only hyperbolic n -manifold that is 1-quasiconformally homogeneous is \mathbb{H}^n .

PROPOSITION 2.4 (Proposition 3.2 in [BTBCT07]). *Let $\{M_i\}$ be a sequence of hyperbolic manifolds with M_i being K_i -quasiconformally homogeneous. If $\lim_{i \rightarrow \infty} K_i = 1$, then $\lim_{i \rightarrow \infty} \ell(M_i) = \infty$.*

3. Torelli Groups and Congruence Subgroups

For a closed orientable surface S_g with genus $g \geq 2$, the *Torelli group* $\mathcal{I}(S_g)$ is the kernel of the action of $\text{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$, the first homology with \mathbb{Z} coefficients. We similarly define the *level m congruence subgroup* $\text{Mod}(S_g)[m]$ as the kernel of the action of $\text{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z}/m\mathbb{Z})$. For the rest of this section, all the results stated will hold for both classes of subgroups just mentioned with $m \geq 3$ in the latter case; we will set $\Gamma(S) = \mathcal{I}(S), \text{Mod}(S)[m]$.

An element $f \in \text{Mod}(S)$ is called *pseudo-Anosov* if it has infinite order and no power of f fixes the isotopy class of any essential 1-submanifold. Let $\text{Teich}(S)$ denote the Teichmüller space, the parameterization space of hyperbolic structures associated to S . Given any $f \in \text{Mod}(S)$, define

$$\tau(f) = \inf_{\mathcal{X} \in \text{Teich}(S)} \{d_{\mathcal{T}}(\mathcal{X}, f \cdot \mathcal{X})\}. \quad (3.1)$$

Then f is pseudo-Anosov if and only if $\tau(f) > 0$ and is realized by some $\mathcal{X} \in \text{Teich}(S)$ (see [Ber78]). If f is pseudo-Anosov, then we define its *dilatation* to be $\lambda(f) = \exp(\tau(f))$.

In [FLM08] the authors prove that for a pseudo-Anosov element $f \in \Gamma(S)$, $\log \lambda(f) \geq 0.197$. We would like to have a similar result for reducible elements of these subgroups. We can get such a result directly from the authors' original proof with understanding how their pseudo-Anosov assumption is being used.

In their proof, they use a cone metric on S coming from a quadratic differential with stable and unstable foliations corresponding to the stable and unstable foliations for f . They use this metric to compare lengths of curves. The same proof can be given using a hyperbolic metric on S yielding $2\tau(f) = \log(\lambda(f)^2) \geq 0.197$. The authors' proof over a hyperbolic metric views f as a quasiconformal map and uses Wolpert's lemma.

LEMMA 3.1 (Wolpert’s lemma, Lemma 12.5 in [FM11]). *Let X, Y be hyperbolic surfaces, and let $f : X \rightarrow Y$ be a K -quasiconformal homeomorphism. For any isotopy class c of simple closed curves in X , the following holds:*

$$\frac{\ell_X(c)}{K} \leq \ell_Y(f(c)) \leq K \ell_X(c),$$

where $\ell_X(c)$ denotes the length of the unique geodesic representative of c in X .

This is also explained in a remark in [FLM08]. By replacing the cone metric coming from the pseudo-Anosov with a hyperbolic metric we remove the first instance of the pseudo-Anosov assumption.

The second way that the pseudo-Anosov assumption is used is to state that f does not fix the homotopy class of a shortest curve. We can remove this assumption by looking at mapping classes that do not fix a shortest curve.

THEOREM 3.2 (Farb, Leininger, and Margalit [FLM08]). *Let X be a hyperbolic surface, and γ the homotopy class of a shortest curve in X . If $f : X \rightarrow X$ is a quasiconformal homeomorphism with $[f] \in \mathcal{I}(X)$ or $[f] \in \text{Mod}(X)[m]$ for some $m \geq 3$ such that $f(\gamma) \neq \gamma$, then $\log K(f) \geq 0.197$.*

For studying quasiconformal homogeneity with respect to $\Gamma(S)$, this theorem will allow us to discard any elements not fixing a shortest curve. This will be enough to prove our theorem. We start with a lemma describing the situation for large genus surfaces.

LEMMA 3.3. *There exists g_0 such that if X is a closed hyperbolic surface of genus $g > g_0$ and X is Γ_K -homogeneous for either $\Gamma = \mathcal{I}(X)$ or $\Gamma = \text{Mod}(X)[m]$ for $m \geq 3$, then $\log K > 0.197$.*

Proof. From Theorem 2.2 we know that if $f : X \rightarrow X$ is K -quasiconformal, then f is a $(K, K \log 4)$ -quasi-isometry. In particular, there is some $C(K) \geq 0$ such that if γ is a geodesic in X , then $f(\gamma)$ is contained in a $C(K)$ -neighborhood of $\tilde{\gamma}$, call it $N_{C(K)}(\tilde{\gamma})$, for some geodesic $\tilde{\gamma}$ in X . Define $C_0 = C \exp(0.197)$. Also, if X is a genus g hyperbolic surface, then $\ell(X) \leq A \log g$, where A is a constant independent of genus (this is Gromov’s inequality for surfaces, see [Gro83]). Now choose g_0 such that

$$\frac{4\pi(g_0 - 1)}{A \log g_0} > 2 \sinh C_0.$$

Assume that the genus of X is $g > g_0$ and that X is Γ_K -homogeneous. Let γ be a closed geodesic in X of shortest length; then it satisfies $\ell_X(\gamma) \leq A \log g$. For every $y \in X$ and $x \in \gamma$, there exists $f : X \rightarrow X$ such that $[f] \in \Gamma_K$ and $f(x) = y$. If $\log K < 0.197$, then $[f(\gamma)] = [\gamma]$, implying that every point of X must be in the C_0 -neighborhood of γ . Let us identify the universal cover of X with \mathbb{H}^2 , so that $X = \mathbb{H}^2/G$ for $G < \text{Isom}^+(\mathbb{H}^2)$. In the upper half-plane model, we can translate a lift of γ to be the imaginary axis, so that the geodesic segment $[i, ie^{\ell_X(\gamma)}]$ maps onto γ . If U is a C_0 -neighborhood of this segment in \mathbb{H}^2 , then by

the preceding we know that there exists a fundamental domain for the action of G on \mathbb{H}^2 contained in U . In particular, this implies $\text{Area}(U) \geq \text{Area}(X)$. However,

$$\text{Area}(U) = 2\ell_X(\gamma) \sinh C_0 < 2A \log(g) \sinh(C_0) < 4\pi(g - 1).$$

But $4\pi(g - 1) = \text{Area}(X)$; hence, we found $\text{Area}(U) < \text{Area}(X)$. This is a contradiction; thus, we must have $\log K > 0.197$. \square

THEOREM 1.2. *There exists a constant $K_T > 1$ such that if X is a closed hyperbolic surface that is Γ_K -homogeneous for $\Gamma = \mathcal{I}(X)$ or $\Gamma = \text{Mod}(X)[r]$ with $r \geq 3$, then $K \geq K_T$.*

Proof. Given a sequence of hyperbolic surfaces $\{X_n\}$, let g_n be the genus of X_n and $\Gamma_n = \mathcal{I}(X_n)$, $\text{Mod}(X_n)[m]$ for $m \geq 3$. We proceed by contradiction: Suppose the statement is false; then there exists a sequence of hyperbolic surfaces $\{X_n\}$ that are $(\Gamma_n)_{K_n}$ -homogeneous such that $\lim_{n \rightarrow \infty} K_n = 1$. Since $K_n \rightarrow 1$, Proposition 2.4 tells us that $\ell(X_n) \rightarrow \infty$, and Gromov's inequality implies that $g_n \rightarrow \infty$. Pick N such that $g_N > g_0$, where g_0 is from Lemma 3.3. For all $n > N$, we have $\log K_n > 0.197$, contradicting $K_n \rightarrow 1$. This completes the proof. \square

4. A Counting Problem in Teichmüller Space

For the rest of the paper, our main method of studying quasiconformal homogeneity will be to translate the problem of understanding the homogeneity constants to one of counting orbit points in Teichmüller space under the action of the mapping class group. Before stating the lemma that will allow us to accomplish this, we recall a proposition in [BTBCT07].

PROPOSITION 4.1 (Proposition 6.2 in [BTBCT07]). *Let $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a quasiconformal map that extends to the identity on $\partial_\infty \mathbb{H}^2$, and let $x \in \mathbb{H}^2$. Then $K(f) \geq \psi(d(x, f(x)))$, where $\psi : [0, \infty) \rightarrow [1, \infty)$ is the increasing homeomorphism given by the function*

$$\psi(d) = \coth^2\left(\frac{\pi^2}{4\mu(e^{-d})}\right) = \coth^2 \mu(\sqrt{1 - e^{-2d}}), \quad (4.1)$$

where $\overline{\mu}(r)$ is the modulus of the Grötsch ring whose complementary components are \mathbb{B}^2 and $[1/r, \infty)$ for $0 < r < 1$.

The explicit formula for ψ was originally due to Teichmüller [Tei44]. In what follows, we will define $K(\phi)$ for $\phi \in \text{Mod}(X)$ by

$$K(\phi) = \min\{K_f : f \in \text{QC}(X) \text{ and } [f] = \phi\},$$

where $[f]$ denotes the homotopy class of f .

LEMMA 4.2. *Let X be a genus g closed hyperbolic surface and $\Gamma < \text{Mod}(X)$ such that X is Γ_K -homogeneous. If the set*

$$\{\phi \in \Gamma : K(\phi) < K\}$$

is finite with cardinality n , then

$$K \geq \sqrt{\psi \left(2 \operatorname{arccosh} \left(\frac{2}{n} (g-1) + 1 \right) \right)},$$

where ψ is defined in (4.1).

Proof. Since the action of $\operatorname{Mod}(X)$ on $\operatorname{Teich}(X)$ is properly discontinuous, there can only be finitely many mapping classes with dilatation less than K . Let ϕ_1, \dots, ϕ_n be the n elements in Γ such that $K(\phi_i) \leq K$. Fix $a \in X$ and let

$$U_i = \{x \in X : \exists f \in \operatorname{QC}_K(X) \text{ such that } [f] = \phi_i \text{ and } f(a) = x\}.$$

In particular, $X = \bigcup_{i=1}^n U_i$. Now $\operatorname{Area}(X) = 4\pi(g-1) \leq \sum \operatorname{Area}(U_i)$; hence, there exists $k \in \{1, \dots, n\}$ such that $U = U_k$ satisfies $\operatorname{Area}(U) \geq \frac{4\pi}{n}(g-1)$. Let d be the diameter of U , so that

$$2\pi \left(\cosh \frac{d}{2} - 1 \right) \geq \operatorname{Area}(U) \geq \frac{4\pi}{n}(g-1),$$

where the leftmost term is the area of the hyperbolic ball of diameter d . This implies

$$d \geq 2 \operatorname{arccosh} \left(\frac{2}{n} (g-1) + 1 \right).$$

For $\varepsilon > 0$, let $x, y \in U$ be such that $d_X(x, y) = d - \varepsilon$ and pick $f, g \in \operatorname{QC}_K(X)$ with $[f] = [g] = \phi_i$ such that $f(a) = x$ and $g(a) = y$. Then $h = g \circ f^{-1}$ is isotopic to the identity, and $h(x) = y$. Let $\tilde{h} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a lift of h that extends to the identity on $\partial_\infty \mathbb{H}^2$. Proposition 4.1 implies

$$K(\tilde{h}) = K(h) \geq \psi(d(x, y)) = \psi(d - \varepsilon).$$

We now have

$$K^2 \geq K(f) \cdot K(g^{-1}) \geq K(f \circ g^{-1}) = K(h) \geq \psi(d_X(x, y)) = \psi(d - \varepsilon).$$

The result follows by letting ε tend to zero and the fact that ψ is increasing. \square

Let us wrap Lemma 4.2 in the language of Teichmüller theory. Given $\mathcal{X} = (X, \phi) \in \operatorname{Teich}(S_g)$, we can identify $f \in \operatorname{Mod}(S_g)$ with $\phi \circ f \circ \phi^{-1} \in \operatorname{Mod}(X)$. Then

$$|\{g \in \operatorname{Mod}(X) : K(g) < K\}| = |\{f \in \operatorname{Mod}(S_g) : f \cdot \mathcal{X} \in B_{\log \sqrt{K}}(\mathcal{X})\}|,$$

where $B_R(\mathcal{X})$ is the ball of radius R in the Teichmüller metric centered at $\mathcal{X} \in \operatorname{Teich}(S_g)$. This allows us to think about orbits in $\operatorname{Teich}(S_g)$. Lemma 4.2 provides a possible route to proving that there exists an universal constant $K_2 > 1$ such that if X is a K -quasiconformally homogeneous closed hyperbolic surface, then $K \geq K_2$.

THEOREM 1.3. *Suppose there exist constants $\varepsilon, R, C > 0$ such that for any $\mathcal{X} \in \operatorname{Teich}_{(\varepsilon, \infty)}(S_g)$ with $g > 1$,*

$$|\{f \in \operatorname{Mod}(S_g) : f \cdot \mathcal{X} \in B_R(\mathcal{X})\}| \leq Cg.$$

Then, there exists a constant $K_2 > 1$ such that any closed K -qch surface must have $K \geq K_2$.

Proof. We proceed by contradiction: Assume that there exists a sequence of closed hyperbolic surfaces $\{X_n\}$ such that X_n is K_n -quasiconformally homogeneous and $K_n \rightarrow 1$. This implies $\ell(X_n) \rightarrow \infty$ by Proposition 2.4 and $g_n \rightarrow \infty$ by Gromov's inequality, where g_n is the genus of X_n . By Lemma 4.2 and the cardinality assumption we have that

$$K_n \geq \sqrt{\psi\left(2 \operatorname{arccosh}\left(\frac{2}{Cg_n}(g_n - 1) + 1\right)\right)}.$$

(Note that we use that both ψ and $\operatorname{arccosh}$ are increasing functions.) In particular, we have

$$\lim_{n \rightarrow \infty} K_n \geq \sqrt{\psi\left(2 \operatorname{arccosh}\left(\frac{2}{C} + 1\right)\right)} > 1.$$

This contradicts the assumption $K_n \rightarrow 1$, which completes the proof. \square

5. Finite Subgroups

For a closed orientable surface S with negative Euler characteristic, there are well-known bounds for the order of finite groups and elements in $\operatorname{Mod}(S)$: it is a theorem of Hurwitz that the group $\operatorname{Isom}^+(X)$ for a closed hyperbolic surface X of genus $g \geq 2$ has order bounded above by $84(g - 1)$. Also, it was proved by Wiman [Wim95] that any element in $\operatorname{Isom}^+(X)$ has order bounded above by $4g + 2$ (both of these are proved in [FM11]). In addition, the Nielsen realization theorem proved by Kerckhoff [Ker83] tells us that a finite subgroup of $\operatorname{Mod}(S)$ can be realized as a subgroup of $\operatorname{Isom}^+(X)$ for some hyperbolic surface X homeomorphic to S . Combining these results with Lemma 4.2, we get the following results.

THEOREM 1.5. *There exists a constant $K_F > 1$ such that if a closed hyperbolic surface X is Γ_K -homogeneous, where $\Gamma < \operatorname{Mod}(X)$ has finite order, then $K \geq K_F$. Furthermore, we have*

$$K_F \geq \sqrt{\psi\left(2 \operatorname{arccosh}\left(\frac{1}{42} + 1\right)\right)} = 1.11469\dots,$$

where ψ is defined in equation (4.1).

Proof. From the preceding discussion we know that $|\Gamma| \leq 84(g - 1)$. The result follows by setting $n = 84(g - 1)$ in Lemma 4.2. \square

THEOREM 5.1. *There exists a constant $K_P > 1$ such that if a closed hyperbolic surface X is Γ_K -homogeneous, where $\Gamma = \langle f \rangle$ and $f \in \operatorname{Mod}(X)$ is periodic, then*

$K \geq K_P$. In particular, we have

$$K_P \geq \sqrt{\psi\left(2 \operatorname{arccosh}\left(\frac{6}{5}\right)\right)} = 1.35547\dots$$

Proof. From the preceding discussion we know that $|\phi| \leq 4g + 2$, so we can use Lemma 4.2 with $n = 4g + 2$. We see that the worst case is $n = 4g + 2$ when $g = 2$. □

6. Pure Cyclic Subgroups

We follow [Iva92] in calling a homeomorphism $f : S \rightarrow S$ *pure* if for some closed one-dimensional submanifold C of S , the following are true:

- (1) the components of C are nontrivial,
- (2) $f|_C$ is the identity,
- (3) f does not rearrange the components of $S \setminus C$, and
- (4) f induces on each component of S cut along C a homeomorphism either homotopic to a pseudo-Anosov or the identity homeomorphism.

An element of $\operatorname{Mod}(S)$ is called *pure* if the homotopy class contains a pure homeomorphism. Note that we allow $C = \emptyset$, so that pseudo-Anosov homeomorphisms are pure. Recall that for a mapping class $f \in \operatorname{Mod}(S)$, we let $\tau(f)$ denote its translation length in $\operatorname{Teich}(S)$. We can then break pure mapping class elements into three categories along the lines of Bers’s classification of surface diffeomorphisms: if $f \in \operatorname{Mod}(S)$ is pure, then:

- (i) $\tau(f) > 0$ and realized, so that f is a (full) pseudo-Anosov,
- (ii) $\tau(f) > 0$ and not realized, so that f induces a pseudo-Anosov homeomorphism on some component of S cut along the canonical reduction system for f (we will call these *partial pseudo-Anosov*), or
- (iii) $\tau(f) = 0$ and not realized, so that f is a Dehn twist about a multicurve, which we will call a *multitwist*.

We will consider homogeneity with respect to cyclic subgroups generated by each type of pure mapping class in turn.

6.1. Full and Partial Pseudo-Anosov Mapping Classes

Let S be a closed surface, and $f \in \operatorname{Mod}(S)$ be a pure partial pseudo-Anosov mapping class. Then there exists a multicurve C and a representative of f , which we will also call f , such that f fixes C pointwise. Let R be a component of the (possibly disconnected) surface resulting from cutting S along C such that $f|_R$ is pseudo-Anosov. We can build a punctured surface F by gluing punctured disks to each of the boundary components of R , so that R is embedded in F . Furthermore, since f restricted to ∂R is the identity, we can extend $f|_R$ to a map $\hat{f} : F \rightarrow F$ by defining $\hat{f}|_R = f|_R$ and $\hat{f}|_{F \setminus R} = id$. We have constructed \hat{f} so that $[\hat{f}] \in \operatorname{Mod}(F)$ is a full pseudo-Anosov map on a punctured surface, and our

first goal will be to relate the translation length $\tau(f)$ of f in $\text{Teich}(S)$ to the translation length $\tau(\hat{f})$ of \hat{f} in $\text{Teich}(F)$.

LEMMA 6.1. *Let S, f, F, \hat{f} be defined as before. Then $\tau(f) \geq \tau(\hat{f})$.*

Proof. Recall that $\tau(f)$ is not realized, so let $\{(X_n, \phi_n)\}$ be a sequence in $\text{Teich}(S)$, and $f_n : X_n \rightarrow X_n$ be the Teichmüller map in the homotopy class of $\phi_n \circ f \circ \phi_n^{-1}$ so that $\lim_{n \rightarrow \infty} K(f_n) = e^{2\tau(f)}$. Define R_n to be the geometric straightening of $\phi_n(R)$ in X_n so that ∂R_n is a disjoint union of simple closed geodesics. The collaring lemma provides disjoint neighborhoods around each boundary component of R_n ; let N_n be the union of these neighborhoods. We can then pick points $x_n \in R_n \setminus N_n$ such that $f_n(x_n) \in R_n \setminus N_n$. The sequence—possibly a subsequence—of pointed surfaces (X_n, x_n) converges geometrically to (X_∞, x_∞) , where X_∞ is homeomorphic to F since the collection of curves permuted by f must be pinched. This convergence is clear since this limit agrees with the visual limit from the viewpoint of x_n . With this setup we will construct a quasiconformal map on X_∞ that has the same translation length in $\text{Teich}(F)$ as \hat{f} and smaller dilatation than $\lim_{n \rightarrow \infty} K(f_n)$.

We will want to work in the hyperbolic plane; in particular, we will use the disk model (\mathbb{D}, d_H) , where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and d_H is the hyperbolic metric. Let us identify the universal cover of (X_n, x_n) with $(\mathbb{D}, 0)$, and let $\Gamma_n < \text{Isom}(\mathbb{D})$ be such that $X_n = \mathbb{D}/\Gamma_n$. We may assume that our marking $\phi_n : S \rightarrow X_n$ induces the representation $\rho_n = (\phi_n)_* : \pi_1 S \rightarrow \Gamma_n$. We note that the Γ_n converge to a group Γ_∞ such that $\mathbb{H}^2/\Gamma_\infty = X_\infty$. Let \tilde{y}_n be a lift of $f(x_n)$ such that $d_H(0, \tilde{y}_n) = d_X(x_n, f(x_n))$, then choose a lift $\tilde{f}_n : \mathbb{D} \rightarrow \mathbb{D}$ of f_n with $\tilde{f}_n(0) = \tilde{y}_n$. By compactness, the sequence of points $\{\tilde{y}_n\}$ must have a convergent subsequence, which we also call $\{\tilde{y}_n\}$, in $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Set $\tilde{y}_\infty = \lim_{n \rightarrow \infty} \{\tilde{y}_n\}$, then since the $x_n \in R_n$ have been chosen to avoid going up the cusp, we see that $\tilde{y}_\infty \in \mathbb{D}$. Let $y_n \in X_n$ be the projection of \tilde{y}_∞ to X_n . Define $h_n : X_n \rightarrow X_n$ such that h_n is isotopic to the identity, $h_n(f(x_n)) = y_n$, and $\lim_{n \rightarrow \infty} K(h_n) = 1$. Now $g_n = h_n \circ f_n : X_n \rightarrow X_n$ with $g_n(x_n) = y_n$; in particular, we can choose lifts $\tilde{g}_n : \mathbb{D} \rightarrow \mathbb{D}$ of the g_n with $\tilde{g}_n(0) = \tilde{y}_\infty$.

The family of K -quasiconformal maps

$$\{g : \mathbb{D} \rightarrow \mathbb{D} : K(g) \leq K \text{ and } g(0) = \tilde{y}_\infty\}$$

is normal [Hub06]; therefore, the sequence $\{\tilde{g}_n\}$ of quasiconformal maps has a convergent subsequence, which we also call $\{\tilde{g}_n\}$. Define $\tilde{g}_\infty = \lim_{n \rightarrow \infty} \{\tilde{g}_n\}$, so that $\tilde{g}_\infty(0) = \tilde{y}_\infty$ and

$$K(\tilde{g}_\infty) = \lim_{n \rightarrow \infty} K(\tilde{g}_n) = \lim_{n \rightarrow \infty} K(g_n) \leq \lim_{n \rightarrow \infty} [K(h_n) \cdot K(f_n)] = e^{2\tau(f)}.$$

It is left to show that \tilde{g}_∞ descends to a map $g_\infty : X_\infty \rightarrow X_\infty$ and $\tau(\hat{f}) \leq \frac{1}{2} \log K(g_\infty)$.

In order to finish the proof, we will look at a particular definition of the geometric limit (details for geometric limits can found in §E.1 in [BP92]). Let $p_n : \mathbb{H}^2 \rightarrow X_n$ be the canonical projections (where we identify $X_n = \mathbb{H}^2/\Gamma_n$).

Since the sequence (X_n, x_n) converges to (X_∞, x_∞) geometrically, we can find bi-Lipschitz maps $\tilde{\psi}_n : \overline{B(0, r_n)} \rightarrow \mathbb{H}^2$, where $B(z, r)$ is the ball of radius r about z , such that $\tilde{\psi}_n(0) = 0$, the $\tilde{\psi}_n$ converge to the identity on \mathbb{H}^2 , and for all z_1, z_2 in the domain of $\tilde{\psi}_n$,

$$p_\infty(z_1) = p_\infty(z_2) \iff p_n(\tilde{\psi}_n(z_1)) = p_n(\tilde{\psi}_n(z_2)). \tag{6.1}$$

In particular, the maps $\tilde{\psi}_n^{-1} \circ \tilde{g}_n \circ \tilde{\psi}_n$ converge to \tilde{g}_∞ . Combining (6.1) with the fact that \tilde{g}_n is Γ_n -equivariant we see that $\tilde{\psi}_n^{-1} \circ \tilde{g}_n \circ \tilde{\psi}_n$ is Γ_∞ -equivariant on its domain. This implies that \tilde{g}_∞ is Γ_∞ -equivariant and descends to $g_\infty : X_\infty \rightarrow X_\infty$.

It is left to show $\tau(\hat{f}) \leq \frac{1}{2} \log K(g_\infty)$. Condition (6.1) implies that the maps $\tilde{\psi}_n$ descend to $\psi_n : K_n \hookrightarrow X_n$, where K_n is a compact set in X_∞ . By the preceding we know that the domain of $\psi_n^{-1} \circ g_n \circ \psi_n$ is converging to X_∞ and $\psi_n^{-1} \circ g_n \circ \psi_n$ is converging to g_∞ . Choose N such that for $n > N$, removing the domain of $\psi_n^{-1} \circ g_n \circ \psi_n$ from X_∞ results in a disjoint union of punctured disks. We can then extend $\psi_n^{-1} \circ g_n \circ \psi_n : X_\infty \rightarrow X_\infty$ without affecting convergence. We therefore see that for large n , $\psi_n^{-1} \circ g_n \circ \psi_n$ is homotopic to g_∞ , which implies

$$g_\infty \simeq \psi_n^{-1} \circ \phi_n \circ f \circ \phi_n^{-1} \circ \psi_n.$$

On the domain of interest, we are really looking at restricting the ϕ_n and f to R and then extending. In fact, we see that

$$g_\infty \simeq \psi_n^{-1} \circ \phi_n \circ \hat{f} \circ \phi_n^{-1} \circ \psi_n.$$

We can think of an extension of $\psi_n^{-1} \circ \phi_n|_R$ as a marking $F \rightarrow X_\infty$, which implies $\tau(\hat{f}) \leq \frac{1}{2} \log K(g_\infty) \leq \tau(f)$ as desired. \square

We will consider both full and partial pseudo-Anosov homeomorphisms at the same time. We will rely on a result of Penner [Pen91], which provides a lower bound for the dilatation of a pseudo-Anosov $f \in \text{Mod}(S)$:

$$\log \lambda(f) \geq \frac{\log 2}{|\chi(S)|},$$

where $\chi(S)$ denotes the Euler characteristic of S . This holds for both closed and punctured surfaces.

THEOREM 6.2. *There exists a constant $K_A > 1$ such that if a closed hyperbolic surface X is Γ_K -homogeneous, where $\Gamma = \langle f \rangle$ with $f \in \text{Mod}(X)$ either pseudo-Anosov or partial pseudo-Anosov, then $K \geq K_A$. In particular, we have $K_A \geq 1.42588$.*

Proof. Let $[f] \in \text{Mod}(X)$, and let $R \subseteq X$ be a connected subsurface such that $f|_R$ is pseudo-Anosov and $f(R)$ is isotopic to R . Note that in the case f is not reducible, $R = X$. We will keep with our notation as before, so that we can extend $f|_R$ to $\hat{f} : F \rightarrow F$, where F is a punctured surface in the reducible case or again $F = X$ and $\hat{f} = f$ in the pseudo-Anosov case. If we let $\tau(\hat{f})$ denote

the translation length of \hat{f} in $\text{Teich}(F)$, then, since $|\chi(F)| \leq |\chi(X)|$, we have $\tau(\hat{f}) \geq \frac{\log 2}{12(g-1)}$, where g is the genus of X (see [Pen91]). Let $m \in \mathbb{Z}$ be such that

$$\frac{m \log 2}{6(g-1)} \geq \log K.$$

Since $(\hat{f}^2) = \hat{f}^2$ and $\tau(\hat{f}^2) = 2\tau(\hat{f})$, we find

$$\tau(f^m) \geq \tau(\hat{f}^m) = m\tau(\hat{f}) \geq \frac{m \log 2}{12(g-1)} \geq \frac{1}{2} \log K.$$

In particular, $K(f^m) \geq K$. We can now appeal to Lemma 4.2 with $n \leq 2m + 1$ (accounting for negative powers and the identity) to find that

$$K \geq \mu_g(K),$$

where we define

$$\mu_g(K) = \sqrt{\psi \left(2 \operatorname{arccosh} \left(\frac{2 \log 2}{12(g-1) \log K + \log 2} (g-1) + 1 \right) \right)}.$$

Since $\mu_g(K)$ increases with g , we have that $K \geq \mu_2(K)$. For $K \geq 1$, we see that $\mu_2(K)$ is decreasing and so there exists a unique solution to $K - \mu_2(K) = 0$, call it K_A . A computation shows that $K_A = 1.42588\dots$, and the result follows. \square

6.2. Multitwists

We start this section with finding a lower bound for the dilatation of a quasiconformal homeomorphism homotopic to a multitwist. We do this by understanding the map induced on the boundary of the hyperbolic plane. Let X be a closed hyperbolic surface, and let $f \in \text{QC}(X)$. Then by identifying the universal cover of X with \mathbb{H}^2 we can choose $\tilde{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ to be a lift of f . Furthermore, we can extend \tilde{f} to the boundary of \mathbb{H}^2 continuously, which we identify with \mathbb{R} . Let $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ be the restriction of \tilde{f} to $\mathbb{R} = \partial\mathbb{H}^2$. We can choose \tilde{f} such that $\bar{f}(\infty) = \infty$. In this setup there exists an M such that \bar{f} is \mathbb{R} -quasi-symmetric with modulus M , that is,

$$\frac{1}{M} \leq \frac{\bar{f}(x+t) - \bar{f}(x)}{\bar{f}(x) - \bar{f}(x-t)} \leq M$$

for all $x \in \mathbb{R}$ and $t > 0$ (see §4.9 of [Hub06]). Sharp bounds are known for the modulus M associated to a K -quasiconformal homeomorphism of \mathbb{H}^2 : define

$$\lambda(K) = \frac{1}{(\mu^{-1}(\pi K/2))^2} - 1,$$

where $\mu(r)$ is the modulus of the Grötsch ring whose complementary components are $\overline{\mathbb{B}^2}$ and $[1/r, \infty]$ for $0 < r < 1$. Then (see [LV73]) we have

$$\frac{1}{\lambda(K(f))} \leq \frac{\bar{f}(x+t) - \bar{f}(x)}{\bar{f}(x) - \bar{f}(x-t)} \leq \lambda(K(f)). \quad (6.2)$$

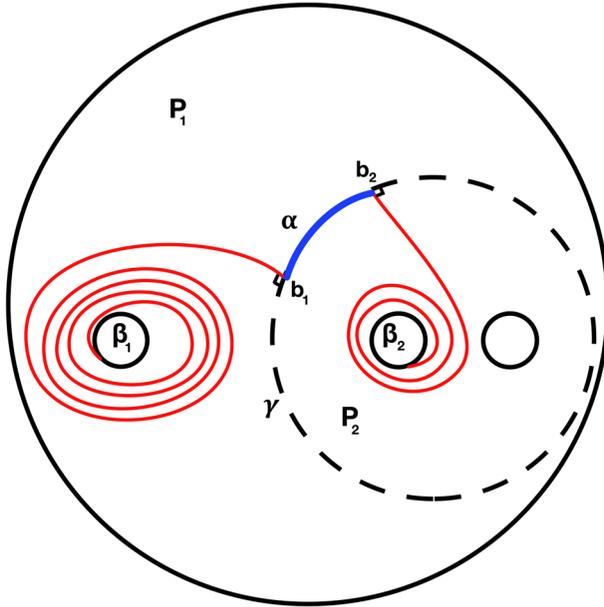


Figure 1 A 4-punctured sphere in X with γ bounding two embedded pairs of pants. The curve α intersects γ once and spirals toward both β_1 and β_2 so that it is disjoint from all boundary components

If f is homotopic to a multitwist, then this is enough information to produce a lower bound for $K(f)$ in terms of the lengths of the curves f twists about.

LEMMA 6.3. *Let X be a closed hyperbolic surface, and $f \in \text{QC}(X)$ be homotopic to a multitwist T_C about a multicurve $C = \{\gamma_1, \dots, \gamma_n\}$, so that $T_C = T_{\gamma_1}^{m_1} \circ \dots \circ T_{\gamma_n}^{m_n}$. If $m = |m_k|$ and $\ell = \ell_X(\gamma_k)$ is such that $m\ell = \max_i \{|m_i| \cdot \ell_X(\gamma_i)\}$, then*

$$K(f) \geq \frac{2}{\pi} \mu \left(\sqrt{\frac{2}{2 + e^{(m-1)\ell} + e^{(m-1/2)\ell}}} \right),$$

where $\mu(r)$ is the modulus of the Grötsch ring whose complementary components are $\overline{\mathbb{B}^2}$ and $[1/r, \infty]$ for $0 < r < 1$.

Proof. Let $\gamma = \gamma_k$ so that $\ell = \ell_X(\gamma)$ and extend the collection $C = \{\gamma_1, \dots, \gamma_n\}$ of disjoint simple closed curves to a maximal collection, call it C' , giving a pants decomposition for X . We want to construct an infinite simple complete geodesic in X that does not intersect any element of C' other than γ . First assume that γ bounds two pairs of pants P_1 and P_2 as in Figure 1. Let β_i be a component of ∂P_i for $i = 1, 2$ such that $\beta_i \neq \gamma$. Then there exists a geodesic ray in P_i spiraling toward β_i and meeting γ perpendicularly at b_i . In X , P_1 and P_2 are glued together with a twist along γ , so we can create a geodesic α by connecting the two rays via an arc on γ connecting the images of b_1 and b_2 in X and pulling this curve

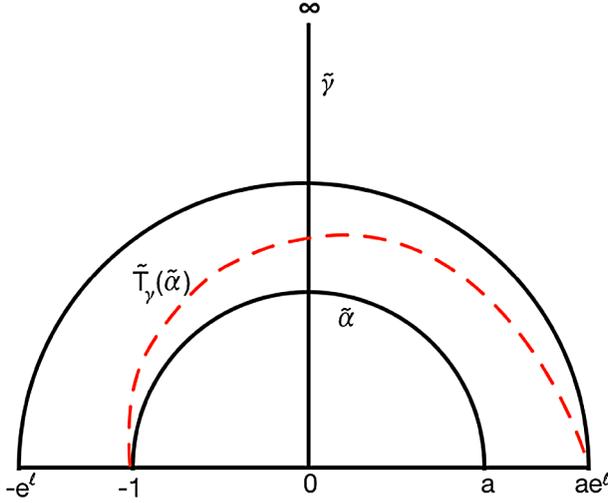


Figure 2 Lifts of α and γ in the upper half-plane. Also drawn is a copy of $\tilde{\alpha}$ under a translation by the element of $\pi_1 X$ representing γ . The dotted geodesic is the image of $\tilde{\alpha}$ under the lift of a Dehn twist about γ

tight. The other possibility is that γ bounds a single pair of pants P . In P we have two copies of γ and one other boundary component. There exists a ray emanating perpendicularly from each copy of γ spiraling toward this other component such that these two rays are disjoint. We then construct α from these rays as before. We see that α is our desired complete geodesic.

We can identify the universal cover \tilde{X} of X with the upper half-plane $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$ so that we have lifts $\tilde{\gamma}$, $\tilde{\alpha}$ of γ , α in the configuration showed in Figure 2. Let $T_\gamma : X \rightarrow X$ be a left Dehn twist about γ , and let $\tilde{T}_\gamma : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a lift of T_γ fixing $\tilde{\gamma}$. Let $[x, y]$ denote the geodesic in \mathbb{H}^2 with endpoints $x, y \in \partial\mathbb{H}^2$. In our setup, $\tilde{\alpha} = [-1, a]$, and we see that $\tilde{T}_\gamma(\tilde{\alpha})$ is homotopic to the dotted curve shown in Figure 2 and has endpoints $[-1, ae^\ell]$. By iterating this map, we can construct a family of geodesics $\{\alpha_n\}$ in X that are the projection of $\tilde{T}_\gamma^n(\tilde{\alpha}) = [-1, ae^{n\ell}]$. Furthermore, every α_n is an infinite simple complete geodesic in X that does not intersect any element of C' other than γ . We can then find an integer k such that $ae^{k\ell} \in [\frac{1}{2}(e^{-\ell} + e^{-\ell/2}), \frac{1}{2}(1 + e^{\ell/2})]$; define $\tilde{\beta} = [-1, ae^{k\ell}] = [-1, b]$, so that the image of $\tilde{\beta}$ is $\beta = \alpha_k$.

We now want to investigate $K = K(f)$ by studying $\bar{f} : \partial\mathbb{H}^2 \rightarrow \partial\mathbb{H}^2$, which is the induced boundary map from the lift $\tilde{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ fixing $0, -1, \infty$. Since two homotopic maps induce the same boundary map on \mathbb{H}^2 , we have $\bar{f} = \bar{T}_C$ (it is convenient to think of \bar{f} as the map on $\partial\mathbb{H}^2$ coming from a left earthquake along the complete lift of the multicurve C ; see [Ker83] for the definition of an earthquake). Let us assume for now that $\frac{1}{2}(e^{-\ell} + e^{-\ell/2}) \leq b \leq 1$ and that f twists

left about γ (if not, we can just study f^{-1}). By construction, β is infinite in X , β intersects γ exactly once, and $\beta \cap \gamma_i = \emptyset$ for $i \neq k$; this implies that $\tilde{\gamma}$ is the only geodesic in the full lift of C that $\tilde{\beta}$ intersects. Therefore, we know that $\bar{f}(b) = be^{m\ell}$ and also that $[-1, b^{m\ell}]$ and $[-1, \bar{f}(1)]$ do not intersect as $[-1, b]$ and $[-1, 1]$ do not. In particular, we must have that $\bar{f}(1) \geq b^{m\ell}$. This yields:

$$\lambda(K) \geq \frac{\bar{f}(1) - \bar{f}(0)}{\bar{f}(0) - \bar{f}(-1)} = \bar{f}(1) \geq be^{m\ell} \geq \frac{1}{2}(e^{(m-1)\ell} + e^{(m-1/2)\ell}).$$

By the preceding we can write

$$K = \frac{2}{\pi} \mu \left(\sqrt{\frac{1}{\lambda(K) + 1}} \right),$$

and since μ is a decreasing function (see [LV73]), we have

$$K \geq \frac{2}{\pi} \mu \left(\sqrt{\frac{2}{2 + e^{(m-1)\ell} + e^{(m-1/2)\ell}}} \right).$$

Now assume that $1 \leq b \leq \frac{1}{2}(1 + e^\ell)$. Furthermore, since $K(f) = K(f^{-1})$ for any quasiconformal map, we may assume that f twists to the right along γ . We have the same exact setup as before, except that this time the inequality is as follows:

$$\frac{1}{\lambda(K)} \leq \frac{\bar{f}(1) - \bar{f}(0)}{\bar{f}(0) - \bar{f}(-1)} = \bar{f}(1) \leq \bar{f}(b) = be^{-m\ell} \leq \frac{1}{2}(e^{-m\ell} + e^{(1/2-m)\ell}),$$

yielding

$$K \geq \frac{2}{\pi} \mu \left(\sqrt{\frac{1 + e^{\ell/2}}{1 + e^{\ell/2} + 2e^{m\ell}}} \right).$$

Since μ is decreasing, for $\ell \geq 0$, the first inequality for K is always smaller. \square

We saw in Theorem 2.3 that a hyperbolic surface X with a short curve has a large homogeneity constant. We leverage this with the above lemma to get a universal bound for the homogeneity constant with respect to a subgroup of $\text{Mod}(X)$ generated by a multitwist.

THEOREM 6.4. *There exists a constant $K_D > 1$ such that if a closed hyperbolic surface X is Γ_K -homogeneous, where $\Gamma = \langle f \rangle < \text{Mod}(X)$ with f being a multitwist, then $K \geq K_D$. In particular, we have $K_D \geq 1.09297$.*

Proof. Let $\ell = \ell(X)$ be the systole of X . From the definition of $m(2, K)$ in Theorem 2.3 given in [BTCMT05] and the inequality $\ell \geq m(2, K)$ we have

$$K \geq \frac{\log((1/2) \tanh(\ell/2)) - \log 2e}{\log((1/2) \tanh(d_2/2)) - \log 2e} \equiv \Phi(\ell), \tag{6.3}$$

where d_2 is defined such that every closed hyperbolic surface contains an embedded hyperbolic disk of diameter d_2 . It is shown in [Yam81] that we can take

$d_2 = 2 \log(1 + \sqrt{2})$. From Lemma 6.3 we have

$$K(f) \geq \frac{2}{\pi} \mu \left(\sqrt{\frac{2}{3 + e^{\ell/2}}} \right) \equiv \Psi(\ell). \quad (6.4)$$

Now, Φ is decreasing on $\mathbb{R}^{>0}$ with $\Phi(0) = +\infty$, and Ψ is an increasing function on $\mathbb{R}^{>0}$ with $\Psi(0) = 1$; hence, there exists a unique value L such that $\Phi(L) = \Psi(L)$. We note that $L \approx 1.33994$ and $\Phi(L) \approx 1.09297$. If $\ell \leq L$, then $K \geq \Phi(L)$. Assume that $\ell \geq L$ and $K < \Psi(L)$. Then $K(f) \geq \Psi(L)$ and every element in Γ_K is isotopic to the identity: this case is handled in [BTBCT07] and tells us that it must be $K \geq 1.626 > \Psi(L)$. This contradiction proves the theorem. \square

Theorem 1.6 is now just a corollary of the previous two sections with setting $K_C = \min\{K_D, K_A\}$.

7. Torsion-Free Subgroups

In this section we investigate a lower bound for the homogeneity constant of a surface in terms of its genus. The idea is to find a lower bound for the dilatation of a quasiconformal map on a thick surface. Periodic elements create serious difficulties that we do not know how to deal with, so we will restrict ourselves to the torsion-free case.

THEOREM 1.7. *Let X be a closed hyperbolic surface and suppose $\Gamma < \text{Mod}(X)$ is torsion-free. If X is Γ_K -homogeneous, then*

$$\log K \geq \frac{1}{7,000g^2},$$

where g is the genus of X .

Proof. Let $\mathcal{F} = \{f \in \Gamma : \log K(f) < 7,000^{-1}g^{-2}\}$. Then our goal is to show that $\mathcal{F} = \{id\}$. The first observation is that \mathcal{F} cannot contain any pseudo-Anosov or pure partial pseudo-Anosov elements. This is seen by combining the bounds in [Pen91] already mentioned and Lemma 6.1.

We can find ℓ_0 such that $\log \Phi(\ell_0) > 1$, where Φ is defined in (6.3); in particular, we can take $\ell_0 = 1.8$. Furthermore, we know that if $\ell(X) < \ell_0$, then $K > \Phi(\ell_0) > \exp(g^{-2})$. Therefore, we may assume that $\ell(X) > \ell_0$, and so \mathcal{F} cannot contain any multitwists since any multitwist will have dilatation bigger than $\Psi(\ell_0) = 1.12$, where Ψ is defined in (6.4). We are left with mapping classes of the form f where some power of f is either a partial pseudo-Anosov or multitwist.

Let us first consider the partial pseudo-Anosov case: we can find a subsurface $R \subset X$ and a $k > 0$ such that f^k fixes the isotopy class of R and $f^k|_R$ is pseudo-Anosov. There are at most $\chi(X)/\chi(R)$ copies of R permuted by f in X ; therefore, we may choose $k \leq \chi(X)/\chi(R)$. We then have

$$\log K(f^k) \geq \frac{\log 4}{|\chi(R)|}.$$

It follows that

$$k \cdot |\chi(R)| \cdot \log K(f) \geq |\chi(R)| \cdot \log K(f^k) \geq \log 4$$

and

$$\log K(f) \geq \frac{\log 4}{k \cdot |\chi(R)|} \geq \frac{\log 4}{|\chi(X)|}.$$

This shows that $f \notin \mathcal{F}$.

We may now suppose that some power of f is a multitwist. Recall that $\ell(X) > \ell_0$. Choose a simple closed curve γ and $k > 0$ such that $f^k([\gamma]) = [\gamma]$. Define R_1 and R_2 to be the subsurfaces on either side of γ (possibly $R_1 = R_2$) such that there exists $n > 0$ with $f^n|_{R_1} = f^n|_{R_2} = id$. Let $R = R_1 \cup R_2$. Then we can choose $k < \chi(X)/\chi(R)$; furthermore, f^{2k} fixes the isotopy classes of both R_1, R_2 . Now choose m_i such that $f^{2km_i}|_{R_i} = id$ for $i = 1, 2$. By doubling R_i we see that

$$m_i \leq 4|\chi(R_i)| + 6 \leq 10|\chi(R_i)|$$

(recall that for a periodic element $h \in \text{Mod}(S_g)$, $|\langle h \rangle| \leq 4g + 2 = 2|\chi(S_g)| + 6$). This implies $2km_1m_2 < 800g^2$. The same line of argument as before tells us that

$$2 \cdot k \cdot m_1 \cdot m_2 \cdot \log K(f) \geq \log \Phi(\ell_0)$$

and

$$\log K(f) \geq \frac{\log \Phi(\ell_0)}{800g^2} > \frac{1}{7,000g^2}.$$

Again we see that $f \notin \mathcal{F}$.

We have exhausted all the torsion-free elements in $\text{Mod}(S_g)$; hence, $\mathcal{F} = \{id\}$, as claimed. If $\log K < 7,000^{-1}g^{-2}$, then we can proceed by contradiction as we did in the cyclic multitwist case: we must have that the elements in Γ_K are isotopic to the identity: this case is handled in [BTBCT07] and implies $K \geq 1.626$, which is larger than our assumption. This is a contradiction, so we see that $\log K > 7,000^{-1}g^{-2}$. \square

8. Functions on Teichmüller Space and Moduli Space

This section looks at building functions on Teichmüller space out of measuring the homogeneity constant at a given point. The statements and techniques follow the related results in [BTBCT07]. For the entirety of this section, let S be a closed orientable surface with $\chi(S) < 0$. Let $\mathcal{X} = [(X, \phi)] \in \text{Teich}(S)$. Then, given $\Gamma < \text{Mod}(S)$, define

$$\Gamma_\phi = \{[\phi \circ f \circ \phi^{-1}] : f \in \text{Homeo}^+(S) \text{ and } [f] \in \Gamma\} < \text{Mod}(X).$$

We then define $K_\Gamma : \text{Teich}(S) \rightarrow (1, \infty)$ by

$$K_\Gamma([(X, \phi)]) = \min\{K : X \text{ is } (\Gamma_\phi)_K\text{-homogeneous}\}.$$

LEMMA 8.1. *Given $\Gamma < \text{Mod}(S)$, the function $K_\Gamma : \text{Teich}(S) \rightarrow (1, \infty)$ exists and is well defined.*

Proof. We first need to prove that K_Γ exists, that is, that the minimum exists. Let X be a hyperbolic surface, and let $\phi : S \rightarrow X$ be a diffeomorphism. Set

$$K = \inf\{Q : X \text{ is } (\Gamma_\phi)_Q\text{-homogeneous}\}.$$

We can then find a sequence $\{K_j\}$ converging to K such that X is $(\Gamma_\phi)_{K_j}$ -homogeneous. We want to show that X is $(\Gamma_\phi)_K$ -homogeneous.

Let $x, y \in X$. Then we can find a K_j -quasiconformal homeomorphisms f_j such that $f_j(x) = y$. Pick lifts $\tilde{x}, \tilde{y} \in \mathbb{H}^2$ and $\tilde{f}_j : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ of x, y , and f_j , respectively, such that $\tilde{f}_j(\tilde{x}) = \tilde{y}$. We recall that the family of all Q -quasiconformal homeomorphisms of \mathbb{H}^2 sending \tilde{x} to \tilde{y} is normal (see Cor. 4.4.3 in [Hub06]). Therefore, there exists a subsequence of $\{\tilde{f}_j\}$ that converges to a K -quasiconformal homeomorphism $\tilde{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ with $\tilde{f}(\tilde{x}) = \tilde{y}$. Furthermore, \tilde{f} descends to a K -quasiconformal mapping $f : X \rightarrow X$. It is left to show that $[f] \in \Gamma_\phi$. Since the connected components of $\text{QC}(X)$ are given by isotopy classes, we must have that for j large, $[f_j] = [f]$, and since each $[f_j] \in \Gamma_\phi$, so is $[f]$. This shows that the minimum exists.

Since a point in Teichmüller space is an equivalence class, we must check that K_Γ is well defined. Let $(X, \phi) = (X, \psi) \in \text{Teich}(S)$, so that ϕ and ψ are isotopic. Since $\text{Mod}(X)$ is defined up to isotopy, it is clear that $\Gamma_\phi = \Gamma_\psi$ and $K_\Gamma((X, \phi)) = K_\Gamma((X, \psi))$. Now let $(X, \phi) = (Y, \xi) \in \text{Teich}(S)$, so that $\phi \circ \xi^{-1} \simeq I$ for some conformal map $I : Y \rightarrow X$. Since conformal maps preserve quasiconformal dilatations, it is clear that $K_\Gamma((Y, \xi)) = K_\Gamma((X, I \circ \xi))$. By definition $I \circ \xi \simeq \psi$, so that by the previous argument $K_\Gamma((X, \phi)) = K_\Gamma((Y, \xi))$. This shows that $K_\Gamma : \text{Teich}(S) \rightarrow (1, \infty)$ is well defined. \square

We now associated to each subgroup of the mapping class group a continuous function of Teichmüller space. In the following we closely adhere to the proof of Lemma 7.1 in [BTBCT07].

PROPOSITION 8.2. *For $\Gamma < \text{Mod}(S)$, the function $K_\Gamma : \text{Teich}(S) \rightarrow (1, \infty)$ is continuous.*

Proof. We will prove the continuity in two steps: we will first prove that K_Γ is lower semicontinuous and then that it is upper semicontinuous. We make the following definitions for the entirety of the proof: Let $\{\mathcal{X}_n\} = \{(X_n, \phi_n)\}$ be a sequence in $\text{Teich}(S)$ converging to $\mathcal{X} = (X, \phi) \in \text{Teich}(S)$. Let $f_n = \phi \circ \phi_n^{-1} : X_n \rightarrow X$ and observe that $\lim_{n \rightarrow \infty} K(f_n) = 1$.

Pick $x, y \in X$ and set $x_n = f_n^{-1}(x)$ and $y_n = f_n^{-1}(y)$. Then there is a $K_\Gamma(X_n)$ -qc mapping $g_n : X_n \rightarrow X_n$ such that $g_n(x_n) = y_n$ with $[g_n] \in \Gamma_{\phi_n}$. Let $\{\mathcal{X}_{n_j}\}$ be a subsequence of $\{\mathcal{X}_n\}$ such that $\lim K_\Gamma(\mathcal{X}_{n_j}) = \liminf K_\Gamma(\mathcal{X}_n)$. Since $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1} : X \rightarrow X$ with $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}(x) = y$ and $\lim K(f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}) \leq \liminf K(f_{n_j})^2 K(g_{n_j}) = \liminf K(g_{n_j})$, we can pass to another subsequence, still labelled $\{\mathcal{X}_{n_j}\}$, such that $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}$ converges to a quasiconformal mapping $g : X \rightarrow X$ such that $g(x) = y$ (this is again due to normality as in Lemma 8.1). For j large, we must have that $f_{n_j} \circ g_{n_j} \circ f_{n_j}^{-1}$ is homotopic to g ,

again since the connected components of $\text{QC}(X)$ are given by isotopy classes. Since $g_{n_j} \in \Gamma_{\phi_{n_j}}$, we have $[g] \in \Gamma_{f_{n_j} \circ \phi_{n_j}}$, but $f_{n_j} \circ \phi_{n_j} = \phi$, so that $[g] \in \Gamma_\phi$. By our setup we now have

$$K(g) \leq \liminf K(g_{n_j}) \leq \lim K_\Gamma(\mathcal{X}_n) = \liminf(\mathcal{X}_n).$$

Since x, y were arbitrary,

$$K_\Gamma(\mathcal{X}) \leq \liminf K_\Gamma(\mathcal{X}_n).$$

Therefore, K_Γ is lower semicontinuous.

It is left to show that K_Γ is upper semicontinuous. Fix n , choose $x_n, y_n \in X_n$, and set $x = f_n(x_n)$ and $y = f_n(y_n)$. Then there exists a $K_\Gamma(\mathcal{X})$ -qc mapping $g_n : X \rightarrow X$ such that $g_n(x) = y$. We then have that $h_n = f_n^{-1} \circ g_n \circ f_n$ is a qc mapping of X_n such that $h_n(x_n) = y_n$ and $[h_n] \in \Gamma_{\phi_n}$. Furthermore,

$$K(h_n) \leq K(f_n)^2 K(g_n) \leq K(f_n)^2 K_\Gamma(\mathcal{X}).$$

Since x_n, y_n were arbitrary, we have that

$$K_\Gamma(\mathcal{X}_n) \leq K(f_n)^2 K_\Gamma(\mathcal{X})$$

and thus

$$\limsup K_\Gamma(\mathcal{X}_n) \leq \lim K(f_n)^2 K_\Gamma(\mathcal{X}) = K_\Gamma(\mathcal{X}).$$

Therefore, K_Γ is upper semicontinuous. □

It is natural to ask when these functions descend to functions on the moduli space. Recall that if $X \in \mathcal{M}(S)$, then two points $\mathcal{X}, \mathcal{Y} \in \text{Teich}(S)$ are in the preimage of X under the projection $\text{Teich}(S) \rightarrow \mathcal{M}(S)$ if there exists $[f] \in \text{Mod}(S)$ with $\mathcal{Y} = [f] \cdot \mathcal{X}$. If $\mathcal{X} = [(X, \phi)]$, then $[f] \cdot \mathcal{X} = [(X, \psi)]$ with $\psi = \phi \circ f^{-1}$. Given a normal subgroup $\Gamma \triangleleft \text{Mod}(S)$, then by definition we have

$$\begin{aligned} \Gamma_\psi &= \{[\psi \circ g \circ \psi^{-1}] : g \in \text{Homeo}^+(S) \text{ and } [g] \in \Gamma\} \\ &= \{[\phi \circ f^{-1} \circ g \circ f \circ \psi^{-1}] : g \in \text{Homeo}^+(S) \text{ and } [g] \in \Gamma\} \\ &= \{[\phi \circ g' \circ \phi^{-1}] : g' \in \text{Homeo}^+(S) \text{ and } [g'] \in \Gamma\} \\ &= \Gamma_\phi. \end{aligned}$$

Since $\Gamma_\psi = \Gamma_\phi$, it is clear that $K_\Gamma(\mathcal{X}) = K_\Gamma(f \cdot \mathcal{X})$. This proves the following:

PROPOSITION 8.3. *For a normal subgroup $\Gamma \triangleleft \text{Mod}(S)$, the function $K_\Gamma : \text{Teich}(S) \rightarrow (1, \infty)$ descends to a continuous function $K_\Gamma : \mathcal{M}(S) \rightarrow (1, \infty)$.*

REMARK 8.4. The normality of the subgroup in Lemma 8.1 is required: Dehn twists about curves with different lengths have different dilatations, and all Dehn twists about nonseparating simple closed curves are conjugates. If we take $\Gamma = \langle f \rangle$ where $f \in \text{Mod}(S)$ is a Dehn twist about a curve γ , then for $X \in \mathcal{M}(S)$ with $\ell(X)$ very small, we can choose $\phi : S \rightarrow X$ and $\psi : S \rightarrow X$ and some K such that $|(\Gamma_\phi)_K| = 1$ (where $\ell(\phi(\gamma))$ is very large) and $|(\Gamma_\psi)_K| = 1,000$ (where $\ell(\psi(\gamma))$ is very small). In the latter case, you have more quasiconformal maps at your disposal.

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