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On Computations of Genus 0 Two-Point Descendant Gromov–Witten Invariants

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1. Introduction

Let X be a smooth proper Deligne–Mumford $\mathbb{C}$-stack with projective coarse moduli space. Genus 0 two-point descendant Gromov–Witten invariants of X are invariants of the following kind:

$$\langle a\psi^k, b\psi^l \rangle^X_{0,2,\beta} := \int_{[\overline{\mathcal{M}}_{0,2}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(a)\psi_1^k\, \mathrm{ev}_2^*(b)\psi_2^l, \tag{1.1}$$

where $a, b \in H^*(IX)$, $k, l \in \mathbb{Z}_{\geq 0}$, and $\mathrm{ev}_1, \mathrm{ev}_2 \colon \overline{\mathcal{M}}_{0,2}(X,\beta) \to IX$ are the evaluation maps. We refer to [1] for the basics of the construction of Gromov–Witten invariants for Deligne–Mumford stacks.

Recently, exact computations of genus 0 two-point descendant Gromov–Witten invariants have received much attention because of mirror symmetry for genus 1 and open Gromov–Witten invariants. In the case $X = \mathbb{P}^n$, a formula for the invariants (1.1) is proved in [14]. Formulas for variants of (1.1) involving twists by Euler class and direct sums of line bundles, in the sense of [4], are also proven in [14] and [12] in the toric setting. More recently, a formula for the invariants (1.1) for compact symplectic toric manifolds is proven in [11]. The proofs in [11; 12; 14] follow a strategy that is similar to the one used by Givental in his computation of genus 0 one-point descendant invariants [5; 6]. More precisely, a generating function of invariants (1.1) is proven by virtual localization to satisfy certain recursion relations and certain regularity conditions. The localization computations needed in [11; 12; 14] are somewhat involved.

The purpose of this paper is to discuss a simpler method for explicitly computing (1.1). This method is based on a known fact in topological field theory that relates two-point descendant invariants (1.1) to one-point descendant invariants; see equation (2.5). We explain this method in detail in Section 2. In Section 3 we apply this method to compute two-point descendant invariants for several classes of examples.

Convention. We work over the field of complex numbers. Cohomology groups are taken with rational coefficients. In this paper we consider cohomology only in even degrees.

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2. Method of Computation

In this section we present our method for computing two-point descendant invariants (1.1). We work in the more general context of *twisted orbifold Gromov–Witten theory* as constructed in [13]. We briefly recall this theory, following [13] (but using somewhat different notation).

2.1. Setup

Let X be a smooth proper Deligne–Mumford $\mathbb{C}$-stack with projective coarse moduli space. Let $V \to X$ be a complex vector bundle and $\mathbf{c}(\cdot)$ a multiplicative invertible characteristic class of vector bundles. Given two integers $g, n \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$, let $\overline{\mathcal{M}}_{g,n}(X, \beta)$ be the moduli stack of n-pointed genus g degree β orbifold stable maps to X. For each $i = 1, \dots, n$ there is an evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to IX$ taking values in the inertia stack IX of X. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ be the universal curve and $f : \mathcal{C} \to X$ the universal orbifold stable map. A key ingredient in the construction of the twisted theory is the following element in the K-theory:

$$V_{g,n,\beta} := R\pi_* f^* V \in K^0(\overline{\mathcal{M}}_{g,n}(X, \beta)). \tag{2.1}$$

The $(\mathbf{c}, V)$-twisted orbifold Gromov–Witten invariants of X are defined by

$$\langle a_1 \psi^{k_1}, \dots, a_n \psi^{k_n} \rangle_{g,n,\beta}^{X,(\mathbf{c},V)} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathbf{c}(V_{g,n,\beta}) \prod_{i=1}^{n} \mathrm{ev}_i^*(a_i) \psi_i^{k_i}. \tag{2.2}$$

Here $k_1, \dots, k_n \geq 0$ are integers, $a_1, \dots, a_n \in H^*(IX)$,

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q})$$

is the virtual fundamental class, and $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q})$ are the descendant classes.

2.2. Reduction to One-Point Descendant

Let $\tau \in H^*(IX)$. Consider the linear map

$$R(\tau; z_1, z_2) : H^*(IX) \to H^*(IX)[[z_1^{-1}, z_2^{-1}]]$$

defined by requiring that for $a, b \in H^*(IX)$ we have

$$(a, R(\tau; z_1, z_2)(b))_{(\mathbf{c},V)} := (a, b)_{(\mathbf{c},V)} + \sum_{\beta} \sum_{n} \frac{Q^\beta}{n!} \left\langle \frac{a}{z_1 - \psi}, \tau, \dots, \tau, \frac{b}{z_2 - \psi} \right\rangle_{0,n+2,\beta}^{X,(\mathbf{c},V)}, \tag{2.3}$$

where $(\cdot,\cdot)_{(\mathbf{c},V)}$ is the $(\mathbf{c}, V)$-orbifold Poincaré pairing of X, as defined in [13, Sec. 3.2], and Q^β is an element in the Novikov ring. We can consider $R(\tau; z_1, z_2)$ as a generating function of genus 0 two-point twisted descendant invariants.

Consider the linear map

$$S(\tau; z)\colon H^*(IX) \to H^*(IX)[[z^{-1}]]$$

defined by requiring that for $a, b \in H^*(IX)$ we have

$$(a, S(\tau; z)(b))_{(\mathbf{c},V)} := (a,b)_{(\mathbf{c},V)} + \sum_\beta \sum_n \frac{Q^\beta}{n!} \left\langle a, \tau, \ldots, \tau, \frac{b}{z-\psi} \right\rangle_{0,n+2,\beta}^{X,(\mathbf{c},V)}. \tag{2.4}$$

Likewise we can consider $S(\tau; z)$ as a generating function of genus 0 one-point twisted descendant invariants.

Proposition 2.1.

$$R(\tau; z_1, z_2) = \frac{1}{z_1 + z_2}(S(\tau; z_1)^* S(\tau; z_2) - \mathrm{Id}). \tag{2.5}$$

Here the superscript $*$ indicates the adjoint with respect to the pairing $(\cdot,\cdot)_{(\mathbf{c},V)}$.

This proposition gives a relationship between the linear maps $R(\tau; z_1, z_2)$ and $S(\tau; z)$. This proposition is not new, but for the sake of completeness we present a proof of it as follows.

Proof of Proposition 2.1. The proof of this proposition is a straightforward application of the argument that proves WDVV equations.

By the string equation, we have

$$\begin{aligned}
&\left\langle \frac{a}{z_1-\psi}, \tau, \ldots, \tau, 1, \frac{b}{z_2-\psi} \right\rangle_{0,n+3,\beta}^{X,(\mathbf{c},V)} \\
&= \frac{1}{z_1}\left\langle \frac{a}{z_1-\psi}, \tau, \ldots, \tau, \frac{b}{z_2-\psi} \right\rangle_{0,n+2,\beta}^{X,(\mathbf{c},V)} + \left\langle \frac{a}{z_1-\psi}, \tau, \ldots, \tau, \frac{b}{z_2-\psi} \right\rangle_{0,n+2,\beta}^{X,(\mathbf{c},V)} \frac{1}{z_2} \\
&= \left(\frac{1}{z_1} + \frac{1}{z_2}\right)\left\langle \frac{a}{z_1-\psi}, \tau, \ldots, \tau, \frac{b}{z_2-\psi} \right\rangle_{0,n+2,\beta}^{X,(\mathbf{c},V)}.
\end{aligned} \tag{2.6}$$

In the exceptional case $(n, \beta) = (0, 0)$ we have

$$\left\langle \frac{a}{z_1-\psi}, 1, \frac{b}{z_2-\psi} \right\rangle_{0,3,0}^{X,(\mathbf{c},V)} = \frac{1}{z_1}\frac{1}{z_2}(a,b)_{(\mathbf{c},V)}. \tag{2.7}$$

Let $\{\phi_\alpha\} \subset H^*(IX)$ be an additive basis, and let $\{\phi^\alpha\} \subset H^*(IX)$ be the dual basis with respect to the pairing $(\cdot,\cdot)_{(\mathbf{c},V)}$. The rational equivalence of boundary divisors in $\overline{\mathcal{M}}_{0,4}$ used in the proof of the WDVV equation gives the following:

$$\sum_{n_1+n_2=n,\beta_1+\beta_2=\beta}\binom{n}{n_1}$$
$$\sum_\alpha\left\langle\frac{a}{z_1-\psi},\tau,\ldots,\tau,1,\phi_\alpha\right\rangle^{X,(\mathbf{c},V)}_{0,n_1+3,\beta_1}\left\langle\phi^\alpha,1,\tau,\ldots,\tau,\frac{b}{z_2-\psi}\right\rangle^{X,(\mathbf{c},V)}_{0,n_2+3,\beta_2}$$
$$=\sum_{n_1+n_2=n,\beta_1+\beta_2=\beta}\binom{n}{n_1}$$
$$\sum_\alpha\left\langle\frac{a}{z_1-\psi},\tau,\ldots,\tau,\frac{b}{z_2-\psi},\phi_\alpha\right\rangle^{X,(\mathbf{c},V)}_{0,n_1+3,\beta_1}\langle\phi^\alpha,1,\tau,\ldots,\tau,1\rangle^{X,(\mathbf{c},V)}_{0,n_2+3,\beta_2}$$
$$=\left\langle\frac{a}{z_1-\psi},\tau,\ldots,\tau,\frac{b}{z_2-\psi},\sum_\alpha(\phi^\alpha,1)_{(\mathbf{c},V)}\phi_\alpha\right\rangle^{X,(\mathbf{c},V)}_{0,n+3,\beta}\quad\text{(by string equation)}$$
$$=\left\langle\frac{a}{z_1-\psi},\tau,\ldots,\tau,\frac{b}{z_2-\psi},1\right\rangle^{X,(\mathbf{c},V)}_{0,n+3,\beta}.\tag{2.8}$$

Again by the string equation, we have

$$\begin{aligned}\left\langle\frac{a}{z_1-\psi},\tau,\ldots,\tau,1,\phi_\alpha\right\rangle^{X,(\mathbf{c},V)}_{0,n_1+3,\beta_1}&=\frac{1}{z_1}\left\langle\frac{a}{z_1-\psi},\tau,\ldots,\tau,\phi_\alpha\right\rangle^{X,(\mathbf{c},V)}_{0,n_1+2,\beta_1},\\ \left\langle\phi^\alpha,1,\tau,\ldots,\tau,\frac{b}{z_2-\psi}\right\rangle^{X,(\mathbf{c},V)}_{0,n_2+3,\beta_2}&=\frac{1}{z_2}\left\langle\phi^\alpha,\tau,\ldots,\tau,\frac{b}{z_2-\psi}\right\rangle^{X,(\mathbf{c},V)}_{0,n_2+2,\beta_2},\end{aligned}\tag{2.9}$$

with the exception that

$$\begin{aligned}\left\langle\frac{a}{z_1-\psi},1,\phi_\alpha\right\rangle^{X,(\mathbf{c},V)}_{0,3,0}&=\frac{1}{z_1}(a,\phi_\alpha)_{(\mathbf{c},V)},\\ \left\langle\phi^\alpha,1,\frac{b}{z_2-\psi}\right\rangle^{X,(\mathbf{c},V)}_{0,3,0}&=\frac{1}{z_2}(\phi^\alpha,b)_{(\mathbf{c},V)}.\end{aligned}\tag{2.10}$$

Combining (2.6)–(2.10) and summing over all possible values of n and β, we get

$$\left(\frac{1}{z_1}+\frac{1}{z_2}\right)R(\tau;z_1,z_2)+\frac{1}{z_1z_2}\,\mathrm{Id}=\frac{1}{z_1}\frac{1}{z_2}S(\tau;z_1)^*S(\tau;z_2),\tag{2.11}$$

which is (2.5). □

REMARK 2.2. 1. To avoid notational complications, (2.5) is not stated for equivariant Gromov–Witten invariants. However it is clear from the proof that (2.5) is also valid in equivariant Gromov–Witten theory.

2. It is easy to see that, when X is a compact symplectic toric manifold, the equivariant version of (2.5) recovers [11, Thm. 4.5].

3. A formula for genus 0 *multi-point* Gromov–Witten invariants of $\mathbb{P}^n$ is proved in [15]. It is clear that the method used to prove (2.5) can be applied recursively to prove formulas for genus 0 multi-point Gromov–Witten invariants of more general target space X. We do not pursue this here.

Equation (2.5) expresses $R(\tau; z_1, z_2)$ in terms of $S(\tau; z)$. This reduces the computation of $R(\tau; z_1, z_2)$ to the computation of $S(\tau; z)$.

2.3. One-Point Invariants

As mentioned previously, $S(\tau; z)$ can be considered as a generating function of one-point twisted descendant invariants. When considering one-point twisted descendant invariants, the so-called twisted J-function $J_{X,(\mathbf{c},V)}(\tau; z)$ plays an important role:

$$J_{X,(\mathbf{c},V)}(\tau; z) := z + \tau + \sum_{\beta} \sum_{n} \frac{Q^{\beta}}{n!} \left\langle \tau, \ldots, \tau, \frac{\phi_\alpha}{z - \psi} \right\rangle_{0,n+1,\beta}^{X,(\mathbf{c},V)} \phi^{\alpha}. \tag{2.12}$$

It is easy to see that $J_{X,(\mathbf{c},V)}(\tau; z) = zS(\tau; z)^*(1)$. In other words, the J-function is the "first column" of $S(\tau; z)^*$.

In various cases of $X, (\mathbf{c}, V)$, the *small* J-function

$$J_{X,(\mathbf{c},V)}(\tau; z)|_{\tau \in H^0(X) \oplus H^2(X)}$$

is explicitly known. For example, when $X = \mathbb{P}^n$ and $(\mathbf{c}, V)$ is vacuous, the small J-function is given by

$$J_{\mathbb{P}^n}(\tau = t_0 1 + tP; z) = ze^{(t_0 1 + Pt)/z} \sum_{d \geq 0} \frac{Q^d e^{dt}}{\prod_{k=1}^{d} (P + kz)^{n+1}}, \tag{2.13}$$

where $1 \in H^0(\mathbb{P}^n)$ and $P \in H^2(\mathbb{P}^n)$ is the hyperplane class. It is evident that

$$J_{\mathbb{P}^n}(\tau = t_0 1 + tP; z)$$

is not the whole $S(\tau; z)$. But in this case one can check that

$$(z\partial/\partial t)^j J_{\mathbb{P}^n}(t_0 1 + tP; z) = z\nabla_{P^j} J_{\mathbb{P}^n}(\tau; z)|_{\tau = t_0 1 + tP}. \tag{2.14}$$

One can also check that the derivative of the full J-function $J_{\mathbb{P}^n}$ along the direction of $P^j \in H^{2j}(\mathbb{P}^n)$, $\nabla_{P^j} J_{\mathbb{P}^n}(\tau; z)$, gives other columns of $S(\tau; z)^*$. Thus by (2.13) and (2.14) we can explicitly compute $S(\tau; z)^*$ for $\tau = t_0 1 + tP$.

The preceding example suggests that we compute $R(\tau; z_1, z_2)|_{\tau \in H^0(X) \oplus H^2(X)}$ as follows. Suppose that the small J-function $J_{X,(\mathbf{c},V)}(\tau; z)|_{\tau \in H^0(X) \oplus H^2(X)}$ is explicitly known, and suppose that we can obtain all columns of $S(\tau; z)^*|_{\tau \in H^0(X) \oplus H^2(X)}$ by successive differentiations along $H^2(X)$; then we can obtain an explicit formula for $S(\tau; z)^*|_{\tau \in H^0(X) \oplus H^2(X)}$. Using (2.5) then yields an explicit formula for $R(\tau; z_1, z_2)|_{\tau \in H^0(X) \oplus H^2(X)}$. Finally, the desired two-point twisted descendant invariants are extracted from $R(\tau; z_1, z_2)|_{\tau \in H^0(X) \oplus H^2(X)}$ after applying string and divisor equations. We next set up this computation scheme in more detail.

2.4. Computation Scheme

Let X and $(\mathbf{c}, V)$ be as in Section 2.1. Suppose we can find the elements

$$\{v_i\}_{i=1,\ldots,N}$$

in $H^*(IX)$ with the following properties.

(1) There exists a permutation $\hat{1}, \dots, \hat{N}$ of $1, \dots, N$ such that the pairing satisfies

$$(v_{\hat{i}}, v_j)_{(\mathbf{c},V)} = m_i \delta_{ij}$$

for a nonzero m_i.

(2) The restriction of $\nabla_{v_j} J_{X,(\mathbf{c},V)}(\tau; z)$ to $H^2(X)$ is known for each v_j.

We know that $\nabla_{v_j} J_{X,(\mathbf{c},V)}(\tau; z)$ is the jth column of $S(\tau; z)^*$. So for $t \in H^2(X)$, the (i, j)-entry of $S(t; z)^*$ is

$$\left(\frac{v_{\hat{i}}}{m_{\hat{i}}}, S(t; z)^*(v_j)\right)_{(\mathbf{c},V)} = S_{ij}(t).$$

Hence the (i, j)-entry of $S(t; z)$ is

$$\left(\frac{v_{\hat{i}}}{m_{\hat{i}}}, S(t; z)(v_j)\right)_{(\mathbf{c},V)} = \frac{1}{m_{\hat{i}}}(v_j, S(t; z)^*(v_{\hat{i}}))_{(\mathbf{c},V)} = \frac{m_j}{m_i} S_{\hat{j}\hat{i}}(t).$$

From this it follows that the (i, j)-entry of $R(t; z_1, z_2)$ is

$$\frac{1}{z_1^{-1} + z_2^{-1}}\left(\sum_k \frac{m_k}{m_j} S_{ik} S_{\hat{j}\hat{k}} - \delta_j^i\right). \tag{2.15}$$

After setting $t = 0$ in (2.15), the coefficient of Q^β gives the desired two-point $(\mathbf{c}, V)$-twisted Gromov–Witten invariant

$$\left\langle \frac{v_{\hat{i}}}{m_i(z_1 - \psi)}, \frac{v_j}{z_2 - \psi} \right\rangle_{0,2,\beta}^{X,(\mathbf{c},V)}.$$

In the next section we investigate some cases in which one can find a cohomology basis with the foregoing properties by using mirror theorems. As in the previous example of $\mathbb{P}^{n-1}$, in all of our applications we can find a collection $\{D_1, D_2, \dots\}$ of first-order linear differential operators with differentiations only along directions in $H^2(X)$ such that, for each $1 \le j \le N$, there exist $i_1, \dots, i_n$ such that

$$z\nabla_{v_j} J_{X,(\mathbf{c},V)}(\tau; z)|_{\tau \in H^2(X)} = zD_{i_1} \circ \cdots \circ zD_{i_n} I_{X,(\mathbf{c},V)} \tag{$*$}$$

after the change of variables in the mirror theorem. Here $I_{X,(\mathbf{c},V)}$ denotes the Givental I-function. Since the J-function takes the form $J_{X,(\mathbf{c},V)}(\tau; z) = z + \tau + O(z^{-1})$, in order for $(*)$ to be true we need to verify the following condition in all our applications.

CONDITION 2.3. For each j and $i_1, \dots, i_n$ as before, the only positive power of z appearing in $zD_{i_1} \circ \cdots \circ zD_{i_{n-1}} I_{X,(\mathbf{c},V)}$ is Az for some $A \in H^*(IX)$.

3. Some Applications

In this section we implement the aforementioned computation scheme for weighted projective spaces and some toric manifolds.

3.1. Weighted Projective Space $X = \mathbb{P}(w_0, w_1, \dots, w_n)$

Here we mostly follow the notation in [3]. Let $P \in H^2(X)$ be the hyperplane class, and let $N = w_0 + \cdots + w_n$. Denote by $\langle a \rangle$ the fractional part of the rational number a. The small J-function of the weighted projective spaces was computed in [3, Thm. 1.7] as

$$J_X(t;z) = ze^{Pt/z} \sum_{d\geq 0;\,\langle d\rangle\in F} \frac{e^{dt}Q^d}{\prod_{i=0}^{n}\prod_{0<b\leq dw_i;\,\langle b\rangle=\langle dw_i\rangle}(w_iP+bz)} 1_{\langle d\rangle},$$

where

$$F = \{k/w_i : 0 \leq k < w_i \text{ and } 0 \leq i \leq n\}$$

and $c_1,\ldots,c_N$ are defined to be the sequence obtained by arranging the terms

$$\frac{0}{w_0}, \frac{1}{w_0}, \ldots, \frac{w_0-1}{w_0}, \frac{0}{w_1}, \frac{1}{w_1}, \ldots, \frac{w_1-1}{w_1}, \ldots, \frac{0}{w_n}, \frac{1}{w_n}, \ldots, \frac{w_n-1}{w_n}$$

in increasing order. The connected components of IX are indexed by the elements of F. For any $f \in F$, let 1_f be the fundamental class of the corresponding component of IX. By [3, Lemma 5.1] there exists a basis $\mathcal{B} = \{v_1,\ldots,v_N\}$ for $H^*(IX)$ given by $v_j = \sigma_j P^{r_j} 1_{c_j}$, where

$$\sigma_j = \frac{\prod_{m:c_m<c_j}(c_j-c_m)}{\prod_{i=0}^{n}\prod_{b:\langle b\rangle=\langle c_jw_i\rangle,\,0<b\leq c_jw_i} b},$$

and $r_j = \#\{i \mid i<j, c_i=c_j\}$. Define

$$d_j = \#\{i \mid c_i = c_j\},$$
$$m_j = \prod_{\{i|c_jw_i\in\mathbb{Z}\}} w_i;$$

then the dual basis of $\mathcal{B}$ is given by $\{v^1,\ldots,v^N\}$, where

$$v^j = \frac{m_j}{\sigma_j}P^{d_j-r_j}1_{\langle 1-c_j\rangle} = \frac{m_j}{\sigma_j\sigma_{\hat{j}}}v_{\hat{j}}.$$

Note that we define $\hat{j}$ by the second of these equalities.

We know that $\nabla_{v_j}J_X(\tau;z)$ is the jth column of $S(\tau;z)^*$, and by [3, Lemma 5.1] there exist explicitly given linear differential operators $D_1,\ldots,D_N$ such that

$$\nabla_{v_j}J(\tau;z)|_{\tau=tP} = z^{-1}D_jJ(t;z).$$

So if we denote by $J_k(\tau;z)$ the component of the J-function along v_k, then the (k,j)-entry of $S(tP;z)^*$ is

$$\langle v^k, S(tP;z)^*(v_j)\rangle = z^{-1}D_jJ_k(t;z).$$

Hence the (k,j)-entry of $S(tP;z)$ is

$$\begin{aligned}\langle v^k, S(tP;z)(v_j)\rangle = \langle v_j, S(tP;z)^*(v^k)\rangle &= \frac{m_k\sigma_j\sigma_{\hat{j}}}{m_{\hat{j}}\sigma_k\sigma_{\hat{k}}}\langle v^{\hat{j}}, S(tP;z)^*(v_{\hat{k}})\rangle \\ &= \frac{m_k\sigma_j\sigma_{\hat{j}}}{m_{\hat{j}}\sigma_k\sigma_{\hat{k}}}z^{-1}D_{\hat{k}}J_{\hat{j}}(t).\end{aligned}$$

From this it follows that the (k,j)-entry of the $R(t;z_1,z_2)$ is

$$\frac{1}{z_1^{-1}+z_2^{-1}}\left(\sum_i \frac{m_i\sigma_j\sigma_{\hat{j}}}{m_{\hat{j}}\sigma_i\sigma_{\hat{i}}} D_i J_k(t;z_1) D_{\hat{i}} J_{\hat{j}}(t;z_2) - \delta_k^j\right). \tag{3.1}$$

After we set $t = 0$ in (3.1), the coefficient of Q^d gives the desired two-point descendant Gromov–Witten invariant

$$\left\langle \frac{v^j}{z_1-\psi_1}, \frac{v_k}{z_2-\psi_2}\right\rangle_{0,d}^X.$$

We know from [3, Proof of Lemma 5.1] that

$$D_j J(0;z) = z \sum_{d\geq 0;\ \langle d\rangle \in F} \frac{\prod_{m=1}^{j-1}(P+(d-c_m)z)}{\prod_{i=0}^n \prod_{0<b\leq dw_i;\ \langle b\rangle = \langle dw_i\rangle}(w_i P + bz)} 1_{\langle d\rangle} Q^{d-c_j}. \tag{3.2}$$

So in order to compute the right-hand side of (3.1), we need to read off the coefficients of $D_i J$ and $D_{\hat{i}} J$ along the specific basis elements. For this we introduce the new variables H_1, H_2 and x_1, x_2 while keeping track of powers of P and the indices of $1_{\langle d\rangle}$ in $D_i J$ and $D_{\hat{i}} J$, respectively. Now using (3.1) and (3.2), for $d > 0$ we can write

$$\sum_{j,k=1}^N \left\langle \frac{v_j}{z_1-\psi_1}, \frac{v_k}{z_2-\psi_2}\right\rangle_{0,d} \frac{m_j^2}{\sigma_j^2\sigma_{\hat{j}}^2} H_1^{r_j} H_2^{r_k} x_1^{c_j} x_2^{c_k} = \frac{1}{z_1+z_2}\sum_{s=1}^N \sum_{\substack{d_1,d_2\geq 0\\ \langle d_1\rangle,\langle d_2\rangle\in F\\ d_1+d_2=d+c_s+c_{\hat{s}}}} \Phi_{d_1,d_2},$$

where

$$\Phi_{d_1,d_2} = \frac{m_s \prod_{m=1}^{s-1}(H_1+(d_1-c_m)z_1)\prod_{\hat{m}=1}^{\hat{s}-1}(H_2+(d_2-c_{\hat{m}})z_2)}{\sigma_s\sigma_{\hat{s}}\prod_{i=0}^n \prod_{\substack{0<b_1\leq d_1 w_s;\ \langle b_1\rangle = \langle d_1 w_s\rangle\\ 0<b_2\leq d_2 w_{\hat{s}};\ \langle b_2\rangle=\langle d_2 w_{\hat{s}}\rangle}} (w_s H_1 + b_1 z_1)(w_{\hat{s}} H_2 + b_2 z_2)} x_1^{\langle d_1\rangle} x_2^{\langle 1-\langle d_2\rangle\rangle}.$$

This specializes to [14, Thm. 1].

REMARK 3.1. Using the mirror theorems stated in [3], our method can be applied to compute the twisted two-point Gromov–Witten invariants of a complete intersection inside a weighted projective space if it satisfies Condition 2.3. However, since the mirror theorem usually involves a nontrivial change of variables, the formulas we get are less explicit.

3.2. *Toric Manifolds*

In this section we discuss applications of the aforementioned method to compute genus 0 two-point descendant Gromov–Witten invariants of a smooth projective toric variety.

Let X be a smooth projective toric variety. The (small) J-function J_X of X is determined by the toric mirror theorem [5; 6; 8]. How *explicitly* the J-function of X is determined depends on X. If X is Fano (i.e., $-K_X$ is ample), then J_X is equal to the combinatorially defined *I-function* I_X. If X is semi-Fano but not Fano (i.e., $-K_X$ is nef and big but not ample), then J_X is equal to I_X after a change of variables (the inverse mirror map) that is often given by a power series with recursively determined coefficients. If X is not semi-Fano, the situation is quite complicated.

In Section 3.2.1 we discuss how to use toric mirror theorems to compute the necessary generating function $S(\tau; z)$ for toric manifolds X. The outcome is not very explicit, as it involves some recursively determined quantities. In Section 3.3.2 we discuss an approach to yield more explicit formulas in the toric Fano case.

It is worth mentioning that the discussions in this section in principle work for toric Deligne–Mumford (DM) stacks as well, given the appropriate mirror theorem for them. A mirror theorem for toric DM stacks will be proved in [2] (see [9, Sec. 4.1] for some details of the result).

3.2.1. *Using the Mirror Theorem in General*

Let X be a smooth projective toric manifold. According to [7], the totality of genus 0 Gromov–Witten invariants of X can be encoded in a Lagrangian submanifold $\mathcal{L}_X$ in a suitable symplectic vector space. Following [6], one can write down a cohomology-valued formal function $I_X(t; z)$ called the *I-function* of X. The toric mirror theorem in this generality (see [8]) states that the family

$$t \mapsto I_X(t; z), \quad t \in H^2(X),$$

lies on $\mathcal{L}_X$. By general properties of the Lagrangian submanifold $\mathcal{L}_X$ (see [7]), this implies that $\mathcal{L}_X$ (and consequently the genus 0 Gromov–Witten theory of X) is determined by $I_X(t; z)$. On the other hand, the family

$$\tau \mapsto J_X(\tau; z)$$

defined by the J-function also lies on $\mathcal{L}_X$.

Thus it is possible to determine J_X from I_X. However, in this generality the process of determining J_X from I_X involves *Birkhoff factorization,* as explained in [4, pp. 29–30]. Moreover, for the computations of two-point Gromov–Witten invariants, we need to determine not only the J-function J_X but also other columns of $S(\tau; z)^*$. To do this, we need to use the fact that differentiation along any direction in the cohomology $H^*(X)$ can be expressed as a higher-order differential operator involving only differentiations along directions in $H^2(X)$ (this is true because $H^*(X)$ is multiplicatively generated by $H^2(X)$). To summarize, there exist differential operators P_i, $i = 1, \ldots, \dim H^*(X)$, involving only differentiations in $H^2(X)$ directions and satisfying the following property. Let $(P_i I_X(t; z))$ be the matrix whose columns are $P_i I_X$. Then there exists a matrix-valued formal series $B(\tau; z)$ in z such that

$$(P_i I_X(t; z)) = S(\tau; z)^* B(\tau; z). \tag{3.3}$$

We refer to [10, Prop. 5.6] for more detailed discussions on this.

Together, (3.3) and (2.5) allow us to express $R(\tau; z_1, z_2)$ as follows:

$$R(\tau; z_1, z_2) = \frac{1}{z_1 + z_2}\big((P_i I_X(t; z_1))B(\tau; z_1)^{-1}(B(\tau; z_2)^*)^{-1}(P_i I_X(t; z_2))^* - \mathrm{Id}\big). \tag{3.4}$$

Unfortunately, equation (3.4) is not very explicit because the differential operators, the Birkhoff factorizations, and the generalized mirror map $\tau = \tau(t)$ can only be determined recursively. It may be possible to produce recursive algorithms for computing two-point Gromov–Witten invariants using (3.4), but we do not pursue it here.

3.2.2. *Fixed Points Set Method*
Let X be an n-dimensional smooth toric variety whose toric fan is generated by the rays $r_1, \dots, r_N$. In this section we mostly follow the notation in [6]. If $\{P_1, \dots, P_k\}$ is a basis for $H^2(X)$ dual to the generators of the semi-group Λ of the curve classes in X then, in the equivariant cohomology ring of X, the class of the divisor corresponding to the ray r_j is given by

$$R_j = \sum_{i=1}^{k} m_{ij} P_i - \lambda_j \quad \text{for } j = 1, \dots, N,$$

where $\lambda_1, \dots, \lambda_N$ are the equivariant parameters. Note that $n = N - k$, and we can choose the basis $\{P_1, \dots, P_k\}$ so that $(m_{ij})_{j=1,\dots,k}$ is the identity matrix. For any $\{i_1, \dots, i_n\} \subset \{1, \dots, N\}$ such that $r_{i_1}, \dots, r_{i_n}$ generate a cone in the fan, let

$$v_{\{i_1,\dots,i_n\}} = R_{i_1} \cdots R_{i_n}$$

be the class of the corresponding fixed point in the equivariant cohomology ring of X. Denote by F the set of $\{i_1, \dots, i_n\} \subset \{1, \dots, N\}$ such that $r_{i_1}, \dots, r_{i_n}$ generate a cone in the fan. For any $\{i_1, \dots, i_n\} \in F$, let $n_{\{i_1,\dots,i_n\}}$ be the equivariant Euler class of the tangent bundle at the corresponding fixed point. This class is given by

$$n_{\{i_1,\dots,i_n\}} = R_{i_1} \cdots R_{i_n}|_{P_1=x_1,\dots,P_k=x_k},$$

where $x_1, \dots, x_k$ uniquely solve the system of equations

$$\sum_{i=1}^{k} m_{ij} x_i = \lambda_j \quad \text{for } j \in \{1, \dots, N\} - \{i_1, \dots, i_n\}.$$

Also, for the same $\{i_1, \dots, i_n\} \in F$ and any $j \in \{i_1, \dots, i_n\}$, define

$${}_j n_{\{i_1,\dots,i_n\}} = \left.\frac{R_{i_1} \cdots R_{i_n}}{R_j}\right|_{P_1=x_1,\dots,P_k=x_k}$$

for $x_1, \dots, x_k$ defined as before.

For any $S_1, S_2 \in F$ we have

$$v_{S_1} \cdot v_{S_2} = \begin{cases} n_{S_1} v_{S_1} & \text{if } S_1 = S_2, \\ 0 & \text{otherwise}; \end{cases}$$

and for any $j \in \{1, \dots, N\}$ we have

$$R_j = \sum_{\substack{S \in F \\ S \ni j}} \frac{v_S}{{}_j n_S}$$

in the equivariant cohomology ring. For any $j = 1, \dots, N$, define the operator

$$D_j = \sum_{i=1}^{k} m_{ij} \frac{\partial}{\partial t_i} - \lambda_j \frac{\partial}{\partial t_0}.$$

The mirror theorem is expressed most simply for the smooth Fano toric variety. In order to make our formulas as explicit as possible, for simplicity we assume hereafter that X is Fano. For any $\beta \in \Lambda$, let $\beta_i = \int_\beta P_i$ and $R_j(\beta) = \int_\beta R_j$. By the equivariant mirror theorem [6], the equivariant small J-function of X is given by

$$\begin{aligned} &J_X(t_0, t_1, \dots, t_k; z) \\ &\quad = z e^{(t_0 + t_1 P_1 + \cdots + t_k P_k)/z} \sum_{\beta \in \Lambda} e^{t_1 \beta_1 + \cdots + t_k \beta_k} \prod_{j=1}^{N} \frac{\prod_{m=-\infty}^{0} (R_j + mz)}{\prod_{m=-\infty}^{R_j(\beta)} (R_j + mz)}. \end{aligned}$$

We can compute, for any nonnegative integer r,

$$\begin{aligned} &z D_{i_1} \circ \cdots \circ z D_{i_r} J_X(t; z) \\ &\quad = z e^{(t_0 + t_1 P_1 + \cdots + P_k t_k)/z} \sum_{\beta \in \Lambda} e^{t_1 \beta_1 + \cdots + t_k \beta_k} (R_{i_1} + z R_{i_1}(\beta)) \cdots (R_{i_r} + z R_{i_r}(\beta)) \\ &\qquad \times \prod_{j=1}^{N} \frac{\prod_{m=-\infty}^{0} (R_j + mz)}{\prod_{m=-\infty}^{R_j(\beta)} (R_j + mz)}. \end{aligned}$$

LEMMA 3.2. *If* $\dim X \leq 3$, *then Condition 2.3 holds for the operators* D_i *and for the fixed points set basis defined previously.*

Proof. We prove the case $\dim X = 3$; the other cases are similar. It suffices to show that the only positive power of z appearing in $z D_{i_1} \circ z D_{i_2} J_X(t; z)$ is Az for some cohomology class A. We claim that the power of $1/z$ in the product

$$\prod_{j=1}^{N} \frac{\prod_{m=-\infty}^{0} (R_j + mz)}{\prod_{m=-\infty}^{R_j(\beta)} (R_j + mz)}$$

is at least $2 - \delta_{0, R_{i_1}(\beta)} - \delta_{0, R_{i_2}(\beta)}$. For any $1 \leq j \leq N$, the power of $1/z$ in

$$\frac{\prod_{m=-\infty}^{0} (R_j + mz)}{\prod_{m=-\infty}^{R_j(\beta)} (R_j + mz)}$$

is at least $R_j(\beta)$ if $R_j(\beta)$ is nonnegative and is at least $1 + R_j(\beta)$ if $R_j(\beta)$ is negative. If for all $1 \leq j \leq N$ we have $R_j(\beta) \geq 0$, then clearly the claim holds. If for some $1 \leq j_0 \leq N$ we have $R_{j_0}(\beta) < 0$, then $1 + \sum_{j=1}^{N} R_j(\beta) = 1 - K_X \cdot \beta \geq 2$ because X is Fano by assumption—so again the claim holds and hence the lemma follows. □

REMARK 3.3. From the proof of Lemma 3.2 one can give the following geometric criterion for ensuring that Condition 2.3 holds for a general smooth Fano toric variety. Let

$$j_X = \min_{C \subset X \text{ a rational curve}} (-K_X \cdot \beta + \#\{j \mid R_j(C) < 0\}).$$

Then Condition 2.3 holds if $j_X \geq \dim X - 1$.

From now on we assume that X is such that Condition 2.3 holds for the operators D_i and the fixed points set basis already defined. Then, by the construction, $v_{\{i_1,\dots,i_n\}}$ is the coefficient of z in

$$zD_{i_1}\circ\cdots\circ zD_{i_n}J(t_0,t_1,\dots,t_k;z)$$

for any $\{i_1,\dots,i_n\}\in F$ and, moreover,

$$z\nabla_{v_{\{i_1,\dots,i_n\}}}J(\tau;z)|_{H^0(X)\oplus H^2(X)} = zD_{i_1}\circ\cdots\circ zD_{i_n}J(t_0,t_1,\dots,t_k;z).$$

For given $S_1,S_2\in F$, our computation scheme shows that the (S_1,S_2)-entry of $R(t;z_1,z_2)$ is given by

$$\begin{aligned}&\left\langle\frac{v_{S_1}}{n_{S_1}}, R(t;z_1,z_2)v_{S_2}\right\rangle\\ &= -\delta^{S_1}_{S_2} + \frac{1}{z_1^{-1}+z_2^{-1}}\\ &\quad\times\sum_{\{i_1,\dots,i_n\}\in F}\frac{n_{S_2}}{n_{\{i_1,\dots,i_n\}}}[zD_{i_1}\circ\cdots\circ zD_{i_n}J(t;z)]_{v_{S_1}}[zD_{i_1}\circ\cdots\circ zD_{i_n}J(t;z)]_{v_{S_2}},\end{aligned}$$

where $[\cdot]_{v_S}$ is the coordinate along the basis element v_S.

For any $S\in F$ we introduce the variables X_S, Y_S with the relations

$$X_{S_1}X_{S_2} = \begin{cases} n_{S_1}X_{S_1} & \text{if } S_1=S_2,\\ 0 & \text{otherwise;}\end{cases}\qquad Y_{S_1}Y_{S_2} = \begin{cases} n_{S_1}Y_{S_1} & \text{if } S_1=S_2,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, for any $j\in\{1,\dots,N\}$, define the new variables U_j and V_j by

$$U_j = \sum_{\substack{S\in F\\ S\ni j}}\frac{X_S}{jn_S}\quad\text{and}\quad V_j = \sum_{\substack{S\in F\\ S\ni j}}\frac{Y_S}{jn_S}.$$

Then for $\beta\in\Lambda-\{0\}$ we get

$$\begin{aligned}&\sum_{S_1,S_2\in F}\left\langle\frac{v_{S_1}}{n_{S_1}(z_1-\psi_1)},\frac{v_{S_2}}{n_{S_2}(z_2-\psi_2)}\right\rangle_{0,\beta}X_{S_1}Y_{S_2}\\ &\quad=\frac{1}{z_1+z_2}\sum_{\{i_1,\dots,i_n\}\in F}\frac{1}{n_{\{i_1,\dots,i_n\}}}\sum_{\beta_1+\beta_2=\beta}\prod_{j=1}^{n}(U_{i_j}+zR_{i_j}(\beta_1))(V_{i_j}+zR_{i_j}(\beta_2))\\ &\quad\quad\times\prod_{r=1}^{N}\frac{\prod_{m=-\infty}^{0}(U_r+mz_1)(V_r+mz_2)}{\prod_{m=-\infty}^{R_r(\beta_1)}(U_r+mz_1)\prod_{m=-\infty}^{R_r(\beta_2)}(V_r+mz_2)}.\end{aligned}$$

EXAMPLE: $X=\mathbb{P}^n$. In this case, $N=n+1$ and $H^2(X)$ is generated by the hyperplane class denoted by P. It can be easily seen that X satisfies the condition in Remark 3.3. According to [5, Thm. 9.5], the equivariant J-function of X is

$$J(t_0,t_1;z) = ze^{(t_0+Pt_1)/z}\sum_{d=0}^{\infty}e^{dt_1}\frac{1}{\prod_{m=1}^{d}(R_1+mz)\cdots(R_{n+1}+mz)},$$

where $R_j = P-\lambda_j$. In this case, $D_j = \frac{\partial}{\partial t_1}-\lambda_j\frac{\partial}{\partial t_0}$ and one can compute

$$zD_{i_1} \circ \cdots \circ zD_{i_n} J(t_0, t_1; z) = ze^{(t_0+Pt_1)/z} \sum_{d=0}^{\infty} e^{dt_1} \frac{(R_{i_1} - dz) \cdots (R_{i_n} - dz)}{\prod_{m=1}^{d} (R_1 + mz) \cdots (R_{n+1} + mz)}.$$

Here F is the set of the subsets of $\{1, \ldots, n+1\}$ with n elements. For any $S \in F$, let $s \in \{1, \ldots, n+1\} - S$; then $n_S = \prod_{i \in S} (\lambda_s - \lambda_i)$. For $d > 0$ we have

$$\begin{aligned}\sum_{S_1, S_2 \in F} &\left\langle \frac{v_{S_1}}{n_{S_1}(z_1 - \psi_1)}, \frac{v_{S_2}}{n_{S_2}(z_2 - \psi_2)} \right\rangle_{0,d} X_{S_1} Y_{S_2} \\ &= \frac{1}{z_1 + z_2} \sum_{\{i_1, \ldots, i_n\} \in F} \frac{1}{n_{\{i_1, \ldots, i_n\}}} \\ &\quad \times \sum_{d_1 + d_2 = d} \frac{(U_{i_1} - d_1 z_1) \cdots (U_{i_n} - d_1 z_1)(V_{i_1} - d_2 z_2) \cdots (V_{i_n} - d_2 z_2)}{\prod_{m=1}^{d_1} (U_1 + m z_1) \cdots (U_{n+1} + m z_1) \times \prod_{m=1}^{d_2} (V_1 + m z_2) \cdots (V_{n+1} + m z_2)}.\end{aligned}$$

REMARK 3.4. We know that, for any $l = 0, \ldots, n$,

$$P^l = \sum_{S \in F} \frac{\lambda_s^l}{n_S} v_S.$$

One can therefore get the ordinary two-point invariants

$$\left\langle \frac{P^{l_1}}{z_1 - \psi_1}, \frac{P^{l_2}}{z_2 - \psi_2} \right\rangle_{0,d}$$

from the foregoing equivariant two-point invariants by taking the nonequivariant limits.

REMARK 3.5. This example can be easily generalized to the case $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ for $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$. For this, first note that X satisfies the condition in Remark 3.3. Now if $P_1, \ldots, P_k \in H^2(X)$ are the pullbacks of the hyperplane class from each factor, then for any $1 \le r \le k$ one can take

$$R_{j_r} = P_r - \lambda_{j_r} \quad \text{for } 1 \le j_r \le n_r + 1$$

and proceed as before to recover the formula in [11, Thm. 1.1].

EXAMPLES OF SEMI-FANO TORIC MANIFOLDS. We first consider the toric manifold $X_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. To see if our method works here, we check Condition 2.3 directly. Let P_1 and P_2 be, respectively, the fiber class and the universal divisor on X. Then the class of the equivariant divisors consists of $R_1 = P_1 - \lambda_1$, $R_2 = P_1 - \lambda_2$, $R_3 = P_2 - \lambda_3$, $R_4 = P_2 - P_1 - \lambda_4$, and $R_5 = P_2 - P_1 - \lambda_5$. By [6], the equivariant I-function is

$$ze^{t_0 + t_1 P_1 + t_2 P_2 / z} \sum_{d_1, d_2 = 0}^{\infty} e^{t_1 d_1 + t_2 d_2} \times \frac{\prod_{m=-\infty}^{0} (R_4 + mz)(R_5 + mz)}{\prod_{m=1}^{d_1} (R_1 + mz)(R_2 + mz) \prod_{m=1}^{d_2} (R_3 + mz) \prod_{m=-\infty}^{d_2 - d_1} (R_4 + mz)(R_5 + mz)}$$

and coincides with the equivariant small J-function.

We have

$$D_1 = \frac{\partial}{\partial t_1} - \lambda_1 \frac{\partial}{\partial t_0},\ D_3 = \frac{\partial}{\partial t_2} - \lambda_3 \frac{\partial}{\partial t_0},\ D_4 = \left(\frac{\partial}{\partial t_2} - \frac{\partial}{\partial t_1}\right) - \lambda_4 \frac{\partial}{\partial t_0}, \ldots.$$

We denote the fraction in the previous sum by $I(d_1, d_2)$. If $d_2 \geq d_1$ then, up to a constant factor, $I(d_1, d_2)$ is $1/z^{3d_2}(1 + o(1/z))$; if $d_2 < d_1$ then, up to a constant factor, $I(d_1, d_2)$ is $1/z^{3d_2+2}(1 + o(1/z))$.

By symmetry we need only check Condition 2.3 for $zD_3 \circ zD_1 I$ and $zD_4 \circ zD_1 I$. We can then conclude that

$$z\nabla_{R_4 R_3 R_1} J = zD_4 \circ zD_3 \circ zD_1 I, \qquad z\nabla_{R_5 R_3 R_1} J = zD_5 \circ zD_3 \circ zD_1 I,$$

and so forth; hence we can follow the rest of the computations in Section 3.2.2 for X_1 without change. We have

$$zD_3 \circ zD_1 I = ze^{t_0 + p_1 t_1 + p_2 t_2/z} \sum_{d_1, d_2 = 0}^{\infty} e^{t_1 d_1 + t_2 d_2} (R_1 + d_1 z)(R_3 + d_2 z) I(d_1, d_2).$$

Comparing the power of z in $(R_1 + d_1 z)(R_3 + d_2 z)$ with the power of $1/z$ in $I(d_1, d_2)$, we see that Condition 2.3 holds. A similar analysis shows that the same condition holds for $zD_4 \circ zD_1 I$.

The second example we consider is $X_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. As in the previous example, we demonstrate how we can check Condition 2.3. Again let P_1 and P_2 be, respectively, the fiber class and the universal divisor on X. Then the class of the equivariant divisors consists of $R_1 = P_1 - \lambda_1$, $R_2 = P_1 - \lambda_2$, $R_3 = P_2 - \lambda_3$, $R_4 = P_2 - \lambda_4$, and $R_5 = P_2 - 2P_1 - \lambda_5$. By [6], the equivariant I-function is

$$ze^{t_0 + t_1 P_1 + t_2 P_2/z} \sum_{d_1, d_2 = 0}^{\infty} e^{t_1 d_1 + t_2 d_2}$$
$$\times \frac{\prod_{m=\infty}^{0} (R_5 + mz)}{\prod_{m=1}^{d_1} (R_1 + mz)(R_2 + mz) \prod_{m=1}^{d_2} (R_3 + mz)(R_4 + mz) \prod_{m=-\infty}^{d_2 - 2d_1} (R_5 + mz)}.$$

This time, the mirror transformation involves a nontrivial change of variables. Following the notation in [6], let

$$f(Q) = \sum_{d=1}^{\infty} \frac{(2d-1)!}{(d!^2)} Q^d.$$

We now modify our derivation operators in Section 3.2.2 according to the mirror transformation:

$$D_1 = \frac{\partial}{\partial T_1} - \lambda_1 \frac{\partial}{\partial t_0}, \quad D_2 = \frac{\partial}{\partial T_1} - \lambda_2 \frac{\partial}{\partial t_0}, \quad D_3 = \frac{\partial}{\partial T_2} - \lambda_3 \frac{\partial}{\partial t_0},$$
$$D_4 = \frac{\partial}{\partial T_2} - \lambda_4 \frac{\partial}{\partial t_0}, \quad D_5 = \frac{\partial}{\partial T_2} - 2\frac{\partial}{\partial T_1} - \lambda_5 \frac{\partial}{\partial t_0};$$

here

$$\frac{\partial}{\partial T_1} = \frac{1}{1+2e^{t_1}f'(e^{t_1})}\frac{\partial}{\partial t_1} \quad \text{and} \quad \frac{\partial}{\partial T_2} = \frac{1}{1-e^{t_2}f'(e^{t_1})}\frac{\partial}{\partial t_2}.$$

We denote the fraction in the preceding sum by $I(d_1, d_2)$. If $d_2 \geq 2d_1$ then, up to a constant factor, $I(d_1, d_2)$ is $1/z^{3d_2}(1 + o(1/z))$; if $d_2 < 2d_1$ then, up to a constant factor, $I(d_1, d_2)$ is $1/z^{3d_2+1}(1 + o(1/z))$. By symmetry we need only check Condition 2.3 for $zD_3 \circ zD_1 I$ and $zD_4 \circ zD_1 I$. We then conclude that

$$z\nabla_{R_4R_3R_1} J = zD_4 \circ zD_3 \circ zD_1 I, \qquad z\nabla_{R_5R_3R_1} J = zD_5 \circ zD_3 \circ zD_1 I,$$

and so forth after the change of variables (see [6] for the details). So we can follow the rest of computations of Section 3.2.2 for X_2 as well. We have

$$\begin{aligned} &zD_3 \circ zD_1 I \\ &= \frac{ze^{t_0+P_1t_1+P_2t_2/z}}{(1+2e^{t_1}f'(e^{t_1}))(1-e^{t_2}f'(e^{t_1}))} \sum_{d_1,d_2=0}^{\infty} e^{t_1d_1+t_2d_2} \\ &\quad \times (P_1 - (1+2e^{t_1}f'(e^{t_1}))\lambda_1 + d_1 z)(P_2 - (1-e^{t_2}f'(e^{t_1}))\lambda_3 + d_2 z) I(d_1, d_2). \end{aligned}$$

Comparing the power of $1/z$ in $I(d_1, d_2)$ with the power of z in the factor of $I(d_1, d_2)$, we can again verify Condition 2.3. Similar analysis shows that the same is true for $zD_4 \circ zD_1 I$.

Remark 3.6. Note that Condition 2.3 does not hold for $zD_5 \circ zD_1 I$. In fact,

$$\begin{aligned} &zD_5 \circ zD_1 I \\ &\quad = \frac{ze^{t_0+P_1t_1+P_2t_2/z}}{1+2e^{t_1}f'(e^{t_1})} \sum_{d_1,d_2=0}^{\infty} e^{t_1d_1+t_2d_2}(P_1 - (1+2e^{t_1}f'(e^{t_1}))\lambda_1 + d_1 z) \\ &\quad\quad \times \left(\frac{P_2}{1-e^{t_2}f'(e^{t_1})} - \frac{2P_1}{1+2e^{t_1}f'(e^{t_1})} - \lambda_5 \right. \\ &\quad\quad\quad \left. + \left(\frac{d_2}{1-e^{t_2}f'(e^{t_1})} - \frac{2d_1}{1+2e^{t_1}f'(e^{t_1})} \right) z \right) I(d_1, d_2) \\ &\quad\quad + \text{other terms}. \end{aligned}$$

One can therefore see that, in $I(d_1, 0)$ for $d_1 > 0$, there are terms of z-degree equal to -1 and the factor of $I(d_1, 0)$ has terms of z-degree 2. This means that $z\nabla_{R_4R_5R_1} J \neq zD_4 \circ zD_5 \circ zD_1 I$ whereas, by the previous paragraph, $z\nabla_{R_5R_4R_1} J = zD_5 \circ zD_4 \circ zD_1 I$.

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