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Buser–Sarnak Invariants of Prym Varieties

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1. Introduction

Let $A = V/\Lambda$ be an abelian variety with a principal polarization H, viewed as a positive definite Hermitian form on the universal cover V. Recall that the Buser–Sarnak invariant (cf. [L2, Def. 5.3.1]) of (A, H) is defined as

$$m(A, H) := \min_{v \in \Lambda \setminus \{0\}} H(v, v).$$

In their seminal paper [BuSa] (see also [L2, Thm. 5.3.5]), Buser and Sarnak showed that if $A = J$ is the Jacobian of a curve with the natural polarization H_J then its Buser–Sarnak invariant is remarkably small:

$$m(J, H_J) \leq \frac{3}{\pi} \log(4 \dim J + 3). \tag{1.1}$$

Their proof uses hyperbolic geometry of compact Riemann surfaces. A different approach to bounding $m(A, H)$ was discovered by Lazarsfeld ([L1]; see also [L2, Thm. 5.3.6]). He gave an upper bound on $m(A, H)$ in terms of the Seshadri number of (A, H), which is an algebro-geometric invariant. His result can be used to derive an upper bound on $m(A, H)$ depending on the algebro-geometric properties of $m(A, H)$. For Jacobians, this approach yields

$$m(J, H_J) \leq \frac{4}{\pi} \sqrt{\dim J}, \tag{1.2}$$

which is much weaker than (1.1) but is still a nontrivial restriction for Jacobians. Bauer ([Ba]; see also [L2, Rem. 5.3.14]) used this approach to show that, for a Prym variety P with the natural principal polarization H_P,

$$m(P, H_P) \leq \frac{4}{\pi} \sqrt{2 \dim P}. \tag{1.3}$$

Since (1.3) is an analogue of (1.2), it is natural to ask whether there is a better bound for $m(P, H_P)$ of logarithmic order in $\dim P$—that is, an analogue of (1.1). This question was raised explicitly by Bauer [Ba, p. 610].

In this paper, we use the work of [BPS] to answer this question as follows.

Theorem 1.1. *For a Prym variety* (P, H_P),

$$m(P, H_P) \leq 2^{20} \log(2 \dim P).$$

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We remark that the improvement from (1.3) to Theorem 1.1 is not merely of theoretical interest: the bound in Theorem 1.1 can be used to show that some explicit examples of period matrices do not correspond to Prym varieties, whereas such examples are not known for the weaker bound (1.3). Recall from [BuSa, (1.12)] that $m(A, H)$ can be as big as $O(\dim A)$ for some principally polarized abelian variety (A, H). However, it is difficult to write down a period matrix with such large $m(A, H)$. Buser and Sarnak [BuSa, (A.1.7)] exhibited explicit period matrices with $\dim A = 2^n$ and

$$m(A, H) \geq \sqrt{\frac{1}{2} \dim A}.$$

The inequality (1.1) shows that these abelian varieties are not Jacobians if $n \geq 7$; Theorem 1.1 implies that neither are they Prym varieties if $n \geq 53$. To our knowledge, no explicit examples of period matrices are known that can be excluded from Prym varieties by applying the weaker bound (1.3).

The proof of Theorem 1.1 uses the result of [BPS] on short homologically nontrivial loops on compact Riemann surfaces. Their result will be recalled, with a suitable modification, in Section 2. Our new ingredient is considering the $\mathbb{C}$-linear independence of suitable lattice vectors of Jacobians, as presented in Section 3. That independence follows from the interplay between the principal polarization of the Jacobian and the homological intersection of loops on the Riemann surface. The proof of Theorem 1.1 can be obtained as an immediate consequence, which we explain in Section 4.

2. A Variation on Theorem 3.1 of [BPS]

We recall, with a slight change of notation, this result (cf. [BPS, Thm. 2.1]).

THEOREM 2.1. *Let S be a compact Riemann surface of genus h equipped with a hyperbolic metric of curvature -1. Assume that the length of any homologically nontrivial loop on S is at least 1. Then there exist $2h$ loops $\gamma_1, \ldots, \gamma_{2h}$ on S inducing a basis of $H_1(S, \mathbb{Z})$ such that, for each $k \leq 2h$ and each $j \leq k$, the hyperbolic length of γ_j satisfies*

$$\operatorname{length}(\gamma_j) \leq 2^{16} \frac{h}{2h - k + 1} \log(2h - k + 2).$$

Our next theorem is a variation on [BPS, Thm. 3.1]. The main difference is that we replace the number $\eta(g)$ there with $n + 2m$ and the inequality (3.1) of [BPS] with the more precise inequality (2.1). In addition, we have extracted some useful information from the proof in [BPS] that was not included in the theorem's original statement. In particular, highlighting the short loops $\alpha_1, \ldots, \alpha_n$ is crucial for our application.

THEOREM 2.2. *Let M be a compact Riemann surface of genus g equipped with a hyperbolic metric of curvature -1. Denote by n the maximal number of homologically independent closed geodesics of length less than $2 \operatorname{arcsinh}(1)$ on M. For any* *positive integer $m < g-n$, there are $n+2m$ closed geodesics $\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{2m}$ on M with the following properties.*

(i) *The homology classes of $\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{2m}$ are part of a basis of $H_1(M, \mathbb{Z})$.*

(ii) *The pair α_i and α_j are disjoint for any $1 \le i \ne j \le n$, and $\gamma_1 \cup \cdots \cup \gamma_{2m}$ is disjoint from $\alpha_1 \cup \cdots \cup \alpha_n$.*

(iii) *For each i, $1 \le i \le n$,*

$$\text{length}(\alpha_i) \le 2 \operatorname{arcsinh}(1);$$

and for each j, $1 \le j \le 2m$,

$$\text{length}(\gamma_j) \le 2^{15} \frac{g + (n+m)}{g - (n+m)} \log(2g - 2(n+m) + 2).$$

(iv) *Each member of $\{\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{2m}\}$ has a collar of width at least $w_0 = \frac{1}{2} \operatorname{arcsinh}(1)$.*

Proof. The proof follows that of [BPS, Thm. 3.1] with a few minor modifications. Instead of repeating the full proof, we shall sketch the argument with precise references to the proof in [BPS] and will indicate where modifications are needed.

Let $\alpha_1, \dots, \alpha_n$ be homologically independent closed geodesics on M of length less than $2 \operatorname{arcsinh}(1)$. By the collar lemma, these geodesics are disjoint and there is a collar of width $1 > w_0$ around each of them, which verifies all of the points in (i)–(iv) that are concerned only with the α_i.

The key idea of [BPS] is to introduce an auxiliary compact Riemann surface S of genus $g+n$ with the property that all homologically nontrivial loops on S have length at least $2 \operatorname{arcsinh}(1) > 1$. This surface S is constructed by attaching fat tori to a certain deformation of $M \setminus \{\alpha_1, \dots, \alpha_n\}$. We refer the reader to the second and the third paragraphs in the proof of Theorem 3.1 in [BPS] for the construction of S.

Applying Theorem 2.1 to S with $h = g + n$, we can choose $2m + 4n \le 2(g+n)$ homologically independent loops $\gamma_1, \dots, \gamma_{2m+4n}$ on S satisfying

$$\begin{aligned}\text{length}(\gamma_k) &\le 2^{16} \frac{g+n}{2(g+n) - (2m+4n) + 1} \log(2(g+n) - (2m+4n) + 2) \\ &= 2^{16} \frac{g+n}{2g - 2n - 2m + 1} \log(2g - 2n - 2m + 2),\end{aligned}$$

which implies

$$\text{length}(\gamma_k) \le 2^{15} \frac{g + (n+m)}{g - (n+m)} \log(2g - 2(n+m) + 2). \tag{2.1}$$

Among the loops $\gamma_1, \dots, \gamma_{2m+4n}$, we can choose $2m$ members (say, $\gamma_1, \dots, \gamma_{2m}$) that naturally correspond to simple closed geodesics on M; these geodesics on M are likewise denoted by $\gamma_1, \dots, \gamma_{2m}$. This is explained, modulo replacing $\eta(g)$ there by $2m + n$, in the second paragraph of [BPS, p. 52]. Here the number $2m$ is obtained from $(2m + 4n) - 2 \times (2n) = 2m$, where $2n$ is the number of fat tori attached to $M \setminus \{\alpha_1, \dots, \alpha_n\}$ in the construction of S.

The property (i) is proved in the third paragraph of [BPS, p. 52], and the first paragraph there confirms property (ii). Property (iii) follows from the same reasoning as that in the fourth paragraph of [BPS, p. 52] if one replaces (3.2) of [BPS] with our (2.1). Finally, Step 2 of the proof of Theorem 3.1 in [BPS] proves property (iv). □

3. Result on Jacobians

Theorem 2.2 enables us to obtain the following result.

THEOREM 3.1. *Let M be a compact Riemann surface M of genus $g \geq 2$, and let $(V/\Lambda, H)$ be its Jacobian (with the conventions $V = H^0(M, K_M)^*$ and $\Lambda = H_1(M, \mathbb{Z})$). Then, for any positive integer $k < g$, there exist lattice vectors $v_1, \dots, v_k \in \Lambda \subset V$ that are linearly independent over $\mathbb{C}$ and satisfy*

$$H(v_i, v_i) \leq 2^{15} \frac{g+k}{g-k} \log(2g - 2k + 2)$$

for every $i \in \{1, 2, \dots, k\}$.

We will use the following result of [BuSa].

LEMMA 3.2. *Let M be a compact Riemann surface with hyperbolic metric of curvature -1 and let $(V/\Lambda, H)$ be its Jacobian. Suppose there exists a simple closed geodesic $\gamma \subset M$ with a collar of width at least $w_0 = \frac{1}{2}\operatorname{arcsinh}(1)$ such that the homology class of γ is an element of a basis of $H_1(M, \mathbb{Z})$. Then the lattice vector $v \in \Lambda$ determined by the homology class of γ satisfies*

$$H(v, v) \leq \operatorname{length}(\gamma).$$

Proof. This result is contained in [BuSa, pp. 36–37] and also in the proof of [BPS. Thm. 3.7]. We briefly recall the argument for the reader's convenience.

Let $\mathcal{U}$ be the collar of width w_0 around γ. Let F be any smooth function defined on $\mathcal{U}$ that takes the value 0 on one connected component of $\partial\mathcal{U}$ and the value 1 on the other. Define the 1-form ω with integral periods by setting ω equal to dF on $\mathcal{U}$ and to 0 outside of $\mathcal{U}$. Then the cohomology class of ω becomes (up to sign) the Poincaré dual of the homology class of γ. Thus,

$$H(v, v) \leq \int_M \omega \wedge *\omega \leq \int_{\mathcal{U}} dF \wedge *dF.$$

The minimum of the last integral among possible choices of F is the capacity of the collar. By [BuSa, (3.4)],

$$H(v, v) \leq \frac{\operatorname{length}(\gamma)}{\pi - 2\theta_0} \leq \operatorname{length}(\gamma);$$

here $\theta_0 = \arcsin(1/\cosh(w_0))$. This completes the proof. □

For the $\mathbb{C}$-linear independence, we need the following lemma.

LEMMA 3.3. *In the setting of Theorem 2.2, let $(V/\Lambda, H)$ be the Jacobian of M. Let $u_i \in \Lambda$ (resp. $w_j \in \Lambda$) be the lattice vector corresponding to the homology class of α_i (resp. γ_j). Then there are m vectors among $w_1, \dots, w_{2m}$ (call them $w_1, \dots, w_m$) such that*

$$u_1, \dots, u_n, w_1, \dots, w_m$$

are linearly independent over $\mathbb{C}$ as vectors in V.

Proof. Denote by $J: V \to V$ the complex structure of V corresponding to multiplication by $\sqrt{-1}$. We recall that $E = \operatorname{Im} H$ is a real symplectic form on V and that $S(u, v) := E(J(u), v)$ is a positive definite real symmetric form on V. Moreover, E takes $\mathbb{Z}$-values on Λ corresponding to the intersection product on cohomology.

Define two real subspaces U, W of V as

$$U := \mathbb{R}u_1 + \cdots + \mathbb{R}u_n \quad \text{and} \quad W := \mathbb{R}w_1 + \cdots + \mathbb{R}w_{2m}$$

with $\dim_{\mathbb{R}} U = n$ and $\dim_{\mathbb{R}} W = 2m$. For any pair $1 \le i, i' \le n$, the loops α_i and $\alpha_{i'}$ are disjoint and so $E(u_i, u_{i'}) = 0$. That is, U is isotropic with respect to the symplectic form E.

We claim that $U \cap J(U) = 0$—in other words, that U is totally real. In fact, if $u = J(u')$ for some $u, u' \in U$ then

$$0 = E(u, u') = E(J(u'), u') = S(u', u').$$

This implies $u' = 0$ by the positive definiteness of S, proving the claim. Hence $u_1, \dots, u_n$ are linearly independent over $\mathbb{C}$ and

$$U^{\mathbb{C}} := \mathbb{C}u_1 + \cdots + \mathbb{C}u_n$$

has complex dimension n.

Next we claim that $U^{\mathbb{C}} \cap W = 0$. If $w \in U^{\mathbb{C}} \cap W$ then we can write $w = u + J(u')$ for some $u, u' \in U$. Observe that $E(W, U) = 0$ because $\gamma_1 \cup \cdots \cup \gamma_{2m}$ is disjoint from $\alpha_1 \cup \cdots \cup \alpha_n$. Therefore,

$$0 = E(w, u') = E(u, u') + E(J(u'), u') = S(u', u').$$

By the positive definiteness of S, we obtain $u' = 0$. Then

$$w = u \in U \cap W = 0$$

establishes that $w = 0$, proving the claim.

Let $\phi: V \to V/U^{\mathbb{C}}$ be the $\mathbb{C}$-linear projection of complex vector spaces. Since $U^{\mathbb{C}} \cap W = 0$, it follows that the image $\phi(W)$ is isomorphic to W. Hence $\phi(w_1), \dots, \phi(w_{2m})$ are $\mathbb{R}$-linearly independent vectors in the complex vector space $V/U^{\mathbb{C}}$. After renumbering, then, we can assume that $\phi(w_1), \dots, \phi(w_m)$ are linearly independent over $\mathbb{C}$.

To complete the proof, it suffices to show that

$$u_1, \dots, u_n, w_1, \dots, w_m$$

are linearly independent over $\mathbb{C}$. Suppose that

$$\sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j w_j = 0$$

for some $a_i, b_j \in \mathbb{C}$. Applying ϕ to both sides, we obtain $b_j = 0$ by the $\mathbb{C}$-linear independence of $\phi(w_1), \dots, \phi(w_m)$. Then the $\mathbb{C}$-linear independence of $u_1, \dots, u_n$ implies $a_i = 0$. $\square$

Proof of Theorem 3.1. First we consider the case when $k > n$. Setting $m = k - n$, we choose $v_1, \dots, v_k$ as

$$u_1, \dots, u_n, w_1, \dots, w_m$$

(using the terminology of Lemma 3.3). These vectors are linearly independent over $\mathbb{C}$ by Lemma 3.3. From Theorem 2.2 and Lemma 3.2, we can deduce that $H(v_i, v_i)$ is bounded by

$$2^{15} \frac{g+k}{g-k} \log(2g - 2k + 2).$$

When $k \leq n$, we choose $v_i := u_i$ for $1 \leq i \leq k$. Then $v_1, \dots, v_k$ are linearly independent over $\mathbb{C}$ by Lemma 3.3. Applying Lemma 3.2 now yields the required bound on $H(v_i, v_i)$. $\square$

4. Application to Prym Varieties

We recall the definition (in [Be]) of Prym varieties and fix our notation for the proof of Theorem 1.1. Let $f : \tilde{C} \to C$ be an étale double cover of a compact Riemann surface C of genus $p + 1 \geq 3$. The genus of $\tilde{C}$ is $2p + 1$. Let $J(\tilde{C}) = V/\Lambda$ be the Jacobian of C with the polarization H. The involution of $\tilde{C}$ exchanging the two sheets of f induces an involution $\sigma : V \to V$ that preserves both the lattice Λ and the polarization H of $J(\tilde{C})$. Write $V = V^+ \oplus V^-$ for the decomposition of V into $(+1)$-eigenspace and (-1)-eigenspace of σ. Then $\dim V^+ = p+1$, $\dim V^- = p$, and $P := V^-/(\Lambda \cap V^-)$ is an abelian subvariety of $J(\tilde{C})$ that we call the *Prym variety* of the cover f. There exists a principal polarization H_P on P satisfying $2H_P = H|_P$.

Proof of Theorem 1.1. Apply Theorem 3.1 to $M = \tilde{C}$, $g = 2p + 1$, and $k = p + 2$. We thus obtain $\mathbb{C}$-linearly independent vectors $v_1, \dots, v_{p+2} \in \Lambda$ with

$$H(v_i, v_i) \leq 2^{15} \frac{3p+3}{p-1} \log(2p) \leq 2^{19} \log(2p).$$

Using the involution σ on V, define

$$y_i := v_i - \sigma(v_i) \in \Lambda \cap V^-.$$

Now suppose that $y_i = 0$ for all i, $1 \leq i \leq p + 2$. Then $v_i = \sigma(v_i) \in V^+$. Since $\dim V^+ = p + 1$, we obtain a contradiction to the $\mathbb{C}$-linear independence of $v_1, \dots, v_{p+2}$. Thus there must be some nonzero y_i. Since $y_i \in \Lambda \cap V^-$, it is a nonzero lattice vector for the Prym variety P with

$$\sqrt{H(y_i, y_i)} \le \sqrt{H(v_i, v_i)} + \sqrt{H(\sigma(v_i), \sigma(v_i))} = 2\sqrt{H(v_i, v_i)}.$$

Then

$$H_P(y_i, y_i) = \frac{1}{2} H(y_i, y_i) \le 2H(v_i, v_i) \le 2^{20} \log(2p),$$

which proves Theorem 1.1. □

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