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Remarks on the Metric Induced by the Robin Function II

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1. Introduction

Let D be a C^∞-smoothly bounded domain in $\mathbf{C}^n$ $(n \geq 2)$. For $p \in D$, let $G(z,p)$ be the Green function for D with pole at p associated to the standard Laplacian

$$\Delta = 4\sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}$$

on $\mathbf{C}^n \approx \mathbf{R}^{2n}$. Then $G(z,p)$ is the unique function of $z \in D$ satisfying the conditions that $G(z,p)$ is harmonic on $D \setminus \{p\}$, $G(z,p) \to 0$ as $z \to \partial D$, and $G(z,p) - |z-p|^{-2n+2}$ is harmonic near p. Thus

$$\Lambda(p) = \lim_{z\to p}(G(z,p) - |z-p|^{-2n+2})$$

exists and is called the *Robin constant* for D at p. The function

$$\Lambda : p \to \Lambda(p)$$

is called the *Robin function* for D.

The Robin function for D is negative and real-analytic, and it tends to $-\infty$ near ∂D (see [10]). Furthermore, if D is pseudoconvex then, by a result of Levenberg and Yamaguchi [7], $\log(-\Lambda)$ is a strongly plurisubharmonic function on D. Therefore,

$$ds^2 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta}\, dz_\alpha \otimes d\bar{z}_\beta$$

is a Kähler metric on D, which is called the Λ-*metric*. Recall that the holomorphic sectional curvature of ds^2 at $z \in D$ along the direction $v \in \mathbf{C}^n$ is given by

$$R(z,v) = \frac{R_{\alpha\bar{\beta}\gamma\bar{\delta}} v^\alpha \bar{v}^\beta v^\gamma \bar{v}^\delta}{g_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta};$$

here

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z_\gamma \partial \bar{z}_\delta} + g^{\nu\bar{\mu}} \frac{\partial g_{\alpha\bar{\mu}}}{\partial z_\gamma} \frac{\partial g_{\nu\bar{\beta}}}{\partial \bar{z}_\delta}$$

are the components of the curvature tensor,

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$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta}$$

are the components of ds^2, and $g^{\alpha\bar{\beta}}$ are the entries of the matrix $(g_{\alpha\bar{\beta}})^{-1}$. In the preceding formulas, we follow the standard convention of summing over all indices that appear once in the upper and lower position.

Now let v be a vector in $\mathbf{C}^n$. At each point $z \in \partial D$, there is a canonical splitting $\mathbf{C}^n = H_z(\partial D) \oplus N_z(\partial D)$ along the complex tangential and normal directions at z and so v can uniquely be written as $v = v_H(z) + v_N(z)$, where $v_H(z) \in H_z(\partial D)$ and $v_N(z) \in N_z(\partial D)$. Also, the smoothness of ∂D implies that if $z \in D$ is sufficiently close to ∂D then there is a unique point $\pi(z) \in \partial D$ that is closest to it; that is, $d(z, \partial D) = |z - \pi(z)|$. Therefore, v can uniquely be written as $v = v_H(\pi(z)) + v_N(\pi(z))$. We will abbreviate $v_H(\pi(z))$ as $v_H(z)$ and $v_N(\pi(z))$ as $v_N(z)$ and call them, respectively, the *horizontal* and *normal* components of v at z. For a strongly pseudoconvex domain D, the boundary behavior of $R(z, v_N(z))$ was calculated in [1] in a special case—namely, when $z \to z_0 \in \partial D$ along the inner normal to ∂D at z_0. One goal of this paper is to remove the restriction that $z \to z_0$ along the inner normal when obtaining the boundary behavior of $R(z, v_N(z))$. More generally, we have the following theorem.

THEOREM 1.1. *Let $\{D_\nu\}$ be a sequence of C^∞-smoothly bounded pseudoconvex domains in $\mathbf{C}^n$ that converges in the C^∞-topology to a C^∞-smoothly bounded strongly pseudoconvex domain D in $\mathbf{C}^n$. If $p_\nu \in D_\nu$ and if $\{p_\nu\}$ converges to a point $p_0 \in \partial D$ then, for any $v \in \mathbf{C}^n$ with $v_N(p_0) \neq 0$,*

$$\lim_{\nu\to\infty} R_\nu(p_\nu, v_{N\nu}(p_\nu)) = -\frac{1}{n-1},$$

where R_ν is the holomorphic sectional curvature of the Λ-metric on D_ν and $v_{N\nu}(p_\nu)$ is the normal component of v at p_ν relative to the domain D_ν.

In this theorem and henceforth, the C^∞-convergence of the sequence $\{D_\nu\}$ to D has the following standard meaning: there exist C^∞-smooth defining functions ψ_ν for D_ν and ψ for D such that $\{\psi_\nu\}$ converges in the C^∞-toplogy on compact subsets of $\mathbf{C}^n$ to ψ. With the same meaning, sometimes it is also said that $\{D_\nu\}$ is a C^∞-perturbation of D. An immediate consequence of this theorem is the following result.

COROLLARY 1.2. *Let D be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$. Fix $z_0 \in \partial D$ and let $v \in \mathbf{C}^n$ be such that $v_N(z_0) \neq 0$. Then, for $z \in D$,*

$$\lim_{z\to z_0} R(z, v_N(z)) = -\frac{1}{n-1}.$$

To understand the difficulty in the computation, let us first normalize the data in Theorem 1.1 as follows.

(a) Since the Λ-metric is invariant under translation and unitary rotation [1, Lemma 5.1], we will assume without loss of generality that $p_0 = 0$ and that the normal to ∂D at p_0 is along the $\Re z_n$-axis.

(b) If ν is sufficiently large, then the distance between p_ν and ∂D_ν, denoted δ_ν, is realized by a unique point $\pi_\nu(p_\nu) \in \partial D_\nu$; that is

$$\delta_\nu = d(p_\nu, \partial D_\nu) = |p_\nu - \pi_\nu(p_\nu)|.$$

For each such ν, we can apply a translation τ_ν followed by a rotation σ_ν and thus transform the domain D_ν into a new domain $\theta_\nu(D)$, where $\theta_\nu = \sigma_\nu \circ \tau_\nu$, such that $\pi_\nu(p_\nu) \in \partial D_\nu$ corresponds to $0 \in \partial\theta_\nu(D_\nu)$ and the normal to $\partial\theta_\nu(D_\nu)$ is along the $\Re z_n$-axis. Note that the point $p_\nu \in D_\nu$ now corresponds to the point $(0, \dots, 0, -\delta_\nu) \in \theta_\nu(D_\nu)$. It is also evident that the sequence $\{\theta_\nu(D_\nu)\}$ converges in the C^∞-topology to D. Therefore, again by the invariance of the Λ-metric under translations and unitary rotations, we will assume without loss of generality that $0 \in \partial D_\nu$, that the normal to ∂D_ν at 0 is along the $\Re z_n$-axis, and that $p_\nu = (0, \dots, 0, -\delta_\nu)$.

With this normalization, we have

$$\begin{aligned} R_\nu(p_\nu, v_{N\nu}(p_\nu)) &= R_\nu\big(p_\nu, (0, \dots, 0, *)\big) \\ &= \frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2}\left(-\frac{\partial^2 g_{\nu n\bar{n}}}{\partial z_n \partial \bar{z}_n}(p_\nu) + \sum_{\alpha,\beta=1}^{n} g_\nu^{\beta\bar{\alpha}}(p_\nu)\frac{\partial g_{\nu n\bar{\alpha}}}{\partial z_n}(p_\nu)\frac{\partial g_{\nu\beta\bar{n}}}{\partial \bar{z}_n}\right), \end{aligned} \tag{1.1}$$

where

$$g_{\nu\alpha\bar{\beta}} = \frac{\partial^2 \log(-\Lambda_\nu)}{\partial z_\alpha \partial \bar{z}_\beta} \tag{1.2}$$

are the components of the Λ-metric ds_ν^2 on D_ν and $g_\nu^{\alpha\bar{\beta}}$ are the entries of the matrix $(g_{\nu\alpha\bar{\beta}})^{-1}$. To compute the limit of the right-hand side of (1.1) as $\nu \to \infty$, we must find the asymptotics of the metric components $g_{\nu\alpha\bar{\beta}}$ and their derivatives along the sequence $\{p_\nu\}$. From (1.2), it is natural to hope that this can be achieved by computing the asymptotics of Λ_ν and their derivatives

$$D^{A\bar{B}}\Lambda_\nu = \frac{\partial^{|A|+|B|}\Lambda_\nu}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}} \quad \text{for } A = (\alpha_1, \dots, \alpha_n) \text{ and } B = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$$

along $\{p_\nu\}$. In this regard, we prove the following theorem.

THEOREM 1.3. *Let $\{D_\nu\}$ be a sequence of C^∞-smoothly bounded domains in $\mathbf{C}^n$ that converges in the C^∞-topology to a C^∞-smoothly bounded domain D in $\mathbf{C}^n$. Choose C^∞-smooth defining functions ψ_ν for D_ν and ψ for D such that $\{\psi_\nu\}$ converges in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to ψ. Let $p_\nu \in D_\nu$ be such that $\{p_\nu\}$ converges to $p_0 \in \partial D$. Define the half-space*

$$\mathcal{H} = \left\{ w \in \mathbf{C}^n : 2\Re\left(\sum_{\alpha=1}^{n} \psi_\alpha(p_0) w_\alpha\right) - 1 < 0 \right\},$$

and let $\Lambda_{\mathcal{H}}$ denote the Robin function for $\mathcal{H}$. Then

$$(-1)^{|A|+|B|}D^{A\bar{B}}\Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2+|A|+|B|} \to D^{A\bar{B}}\Lambda_H(p_0)$$

as $\nu \to \infty$.

We emphasize that $\psi_\alpha = \partial\psi/\partial z_\alpha$ in this theorem and should not be confused with the function ψ_ν. We will show in Section 6 that the asymptotics obtained in this theorem suffice to calculate the limit of the first term of (1.1). However, it turns out that the second term remains indeterminate by these asymptotics. Hence calculating this term requires finer asymptotics of Λ_ν and their derivatives. A similar situation was handled in [1] by using the following result of Levenberg and Yamaguchi [7]. The function λ defined by

$$\lambda(p) = \begin{cases} \Lambda(p)(\psi(p))^{2n-2} & \text{if } p \in D, \\ -|\partial\psi(p)|^{2n-2} & \text{if } p \in \partial D \end{cases} \tag{1.3}$$

is C^2 up to $\bar{D}$. We will call λ the *normalized* Robin function associated to (D, ψ). Thus it is expected that finer asymptotics of Λ_ν and their derivatives along $\{p_\nu\}$ could be obtained if the functions $\lambda_\nu = \Lambda_\nu\psi_\nu^{2n-2}$ and their derivatives along $\{p_\nu\}$ were bounded. Theorem 1.2 shows that $\lambda_\nu(p_\nu)$ converges to $\lambda(p_0)$, and in Theorem 1.4 we establish the convergence of first and second derivatives of λ_ν along $\{p_\nu\}$.

THEOREM 1.4. *Under the hypotheses of Theorem 1.2, we have*

(1) $\displaystyle\lim_{\nu\to\infty} \frac{\partial\lambda_\nu}{\partial p_\alpha}(p_\nu) = \frac{\partial\lambda}{\partial p_\alpha}(p_0)$ *and*

(2) $\displaystyle\lim_{\nu\to\infty} \frac{\partial^2\lambda_\nu}{\partial p_\alpha\partial\bar{p}_\beta}(p_\nu) = \frac{\partial^2\lambda}{\partial p_\alpha\partial\bar{p}_\beta}(p_0)$.

Here λ_ν *and* λ *are the normalized Robin functions associated to* (D_ν, ψ_ν) *and* (D, ψ), *respectively.*

We remark that—unlike the Bergman, Carathéodory, and Kobayashi metrics—the Λ-metric is not invariant under biholomorphisms in general (see e.g. [1]). The only information we have on this score is that any biholomorphism between two C^∞-smoothly bounded strongly pseudoconvex domains is Lipschitz with respect to the Λ-metric (this follows from [1, Thm. 1.4]). Despite that drawback, our exploration of this metric is devoted to identifying which of its various properties are analogous to those possessed by these invariant metrics.

Another goal of this paper is to study the existence of closed geodesics for the Λ-metric of a given homotopy type. In [6] Herbort proved that, on a C^∞-smoothly bounded strongly pseudoconvex domain D in $\mathbf{C}^n$ that is not simply connected, every nontrivial homotopy class in $\pi_1(D)$ contains a closed geodesic for the Bergman metric. By studying the boundary behavior of the Λ-metric, we prove the following analogue for the Λ-metric.

THEOREM 1.5. *Let D be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$ that is not simply connected. Then every nontrivial homotopy class in $\pi_1(D)$ contains a closed geodesic for the Λ-metric.*

Let D be C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$. Donnelly and Fefferman [4] proved that D does not admit any square-integrable harmonic (p,q)-form relative to the Bergman metric except when $p+q=n$, in which case the space of such forms is infinite dimensional. A more transparent and elementary proof of the infinite dimensionality of the L^2-cohomology of the middle dimension was given by Ohsawa [9]. In [3], Donnelly gave an alternative proof of the vanishing of the L^2-cohomology outside the middle dimension via the following observation of Gromov [5]. If M is a complete Kähler manifold of complex dimension n such that the Kähler form ω of M can be written as $\omega = d\eta$, where η is bounded in supremum norm, then M does not admit any square-integrable harmonic i form for $i \neq n$. Finally, we observe that these ideas can be applied to the Λ-metric to prove the following result.

THEOREM 1.6. *Let D be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$, and let $\mathcal{H}_2^{p,q}(D)$ be the space of square-integrable harmonic (p,q)-forms relative to the Λ-metric. Then*

$$\dim \mathcal{H}_2^{p,q}(D) = \begin{cases} 0 & \text{if } p+q \neq n, \\ \infty & \text{if } p+q = n. \end{cases}$$

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2. Properties of λ

Let D be a C^∞-smoothly bounded domain in $\mathbf{C}^n$ with a C^∞-smooth defining function ψ defined on all of $\mathbf{C}^n$. In this section, we recall some basic properties of the normalized Robin function λ associated to (D,ψ). We start by describing the geometric meaning of $\lambda(p)$. Given $p \in D$, let

$$T\colon D \times \mathbf{C}^n \to \mathbf{C}^n$$

be the map defined by

$$T(p,z) = \frac{z-p}{-\psi(p)}. \tag{2.1}$$

Set

$$D(p) = \begin{cases} T(p,D) & \text{if } p \in D, \\ \left\{ w \in \mathbf{C}^n : 2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(p) w_\alpha\right) - 1 < 0 \right\} & \text{if } p \in \partial D. \end{cases} \tag{2.2}$$

Thus $\{D(p) : p \in \bar{D}\}$ is a family of domains in $\mathbf{C}^n$ each containing the origin. When $p \in D$, we have that $D(p)$ is the image of D under the affine transformation $T(p,\cdot)$ and hence by [10, Prop. 5.1] that

$$\Lambda_{D(p)}(0) = \Lambda(p)(\psi(p))^{2n-2} = \lambda(p).$$

If $p \in \partial D$ then $D(p)$ is a half-space for which we have the explicit formula

$$\Lambda_{D(p)}(0) = -|\partial\psi(p)|^{2n-2} = \lambda(p)$$

(cf. [1, (1.4)]). Thus, for each $p \in \bar{D}$, $\lambda(p)$ is the Robin constant for $D(p)$ at the origin. We will denote the Green function for $D(p)$ with pole at p by $g(p, w)$.

To discuss the regularity of the function $\lambda(p)$ on $\bar{D}$, we set

$$\mathcal{D} = \bigcup_{p \in D} (p, D(p)) = \{(p, w) : p \in D,\ w \in D(p)\}.$$

The set $\mathcal{D}$ can be considered as a variation of domains in $\mathbf{C}^n$ with parameter space D—in other words, as a map

$$\mathcal{D} \colon p \to D(p)$$

that associates to each $p \in D$ a domain $D(p) \subset \mathbf{C}^n$. We call $\mathcal{D} \colon p \to D(p)$ the *variation associated to* (D, ψ). The function

$$f(p, w) = 2\Re\left\{\sum_{\alpha=1}^{n} \int_0^1 \big(w_\alpha \psi_\alpha(p - \psi(p)tw)\big)\, dt\right\} - 1, \tag{2.3}$$

which was constructed in [7], is jointly smooth on $\mathbf{C}^n \times \mathbf{C}^n$. If we take $\tilde{\mathcal{D}} = D \times \mathbf{C}^n$, then the following statements hold.

(i) $\mathcal{D} = \{(p, w) \in \tilde{\mathcal{D}} : f(p, w) < 0\}$, $\partial\mathcal{D} := \{(p, w) : p \in D, w \in \partial D(p)\} = \{(p, w) \in \tilde{\mathcal{D}} : f(p, w) = 0\}$, and $\operatorname{Grad}_{(p,w)} f \neq 0$ on $\partial\mathcal{D}$.

(ii) For each $p \in D$ we have $D(p) = \{w \in \mathbf{C}^n : f(p, w) < 0\}$, $\partial D(p) = \{w \in \mathbf{C}^n : f(p, w) = 0\}$, and $\operatorname{Grad}_w f(p, w) \neq 0$ on $\partial D(p)$.

Therefore, we say that the variation $\mathcal{D} \colon p \to D(p)$ is smooth and is defined by $f(p, w)$. It is evident that the variation

$$\mathcal{D} \cup \partial\mathcal{D} \colon p \to D(p) \cup \partial D(p) = \bar{D}(p)$$

is diffeomorphically equivalent to the trivial variation $D \times \bar{D}$. It follows that $g(p, w)$ has a C^4 extension to a neighborhood of $\mathcal{D} \setminus D \times \{0\}$. Now fix a point $p_0 \in D$ and let $\bar{B}(0, r) \subset D(p_0)$. Then there exists a neighborhood U of p_0 in D such that $\bar{B}(0, r) \subset D(p)$ for all $p \in U$. Because $g(p, w) - |w|^{-2n+2}$ is a harmonic function of $w \in D(p)$ and is equal to $\lambda(p)$ when $w = 0$, we can use the mean value property of harmonic functions to obtain

$$\begin{aligned}\lambda(p) &= \frac{1}{r^{2n-1}\sigma_{2n}} \int_{\partial B(0,r)} (g(p, w) - |w|^{-2n+2})\, dS_w \\ &= -\frac{1}{r^{2n-2}} + \frac{1}{r^{2n-1}\sigma_{2n}} \int_{\partial B(0,r)} g(p, w)\, dS_w,\end{aligned} \tag{2.4}$$

where by dS we denote the surface area measure on a smooth surface in $\mathbf{R}^{2n}$ and by σ_{2n} the surface area of $\partial B(0, 1)$. It follows that $\lambda(p)$ is smooth on U and thus on D.

Now let $1 \leq \gamma \leq n$. Observe that, for each $p \in D$, the functions

$$\frac{\partial g}{\partial p_\gamma}(p, w),\ \ \frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p, w)$$

are harmonic in all of $D(p)$ and that

$$\frac{\partial g}{\partial p_\gamma}(p,0) = \frac{\partial \lambda}{\partial p_\gamma}(p), \qquad \frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\beta}(p,0) = \frac{\partial^2 \lambda}{\partial p_\gamma \partial \bar{p}_\gamma}.$$

To find the boundary values of these functions in terms of f, consider the quantities k_1^γ and k_2^γ:

$$\begin{aligned} k_1^\gamma(p,w) &= \frac{\partial f}{\partial p_\gamma}(p,w)|\partial_w f(p,w)|^{-1}, \\ k_2^\gamma(p,w) &= \mathcal{L}^\gamma f(p,w)|\partial_w f(p,w)|^{-3}; \end{aligned} \tag{2.5}$$

here

$$\mathcal{L}^\gamma f = \frac{\partial^2 f}{\partial p_\gamma \partial \bar{p}_\gamma}|\partial_w f|^2 - 2\Re\left(\frac{\partial f}{\partial p_\gamma}\sum_{\alpha=1}^n \frac{\partial f}{\partial \bar{w}_\alpha}\frac{\partial^2 f}{\partial w_\alpha \partial \bar{p}_\gamma}\right) + \left|\frac{\partial f}{\partial p_\gamma}\right|^2 \Delta_w f \tag{2.6}$$

is defined wherever $\partial_w f(p,w) \neq 0$ and thus in particular on

$$\partial\mathcal{D} = \bigcup_{p\in D}(p, \partial D(p)).$$

Observe that, on $\partial\mathcal{D}$, the quantities k_1^γ and k_2^γ are independent of the defining function f for $\mathcal{D}$. Since $g(p,w) > 0$ on $\mathcal{D}$, $g(p,w) = 0$ on $\partial\mathcal{D}$, and

$$|\partial_w g(p,w)| = -\frac{1}{2}\frac{\partial g}{\partial n_w}(p,w) > 0 \text{ on } \partial\mathcal{D},$$

it follows that we can use $-g(p,w)$ as a defining function for $\mathcal{D}$. Thus, for all $(p,w) \in \partial\mathcal{D}$, we have

$$\frac{\partial g}{\partial p_\gamma}(p,w) = -k_1^\gamma(p,w)|\partial_w g(p,w)|$$

and

$$\mathcal{L}^\gamma g(p,w) = -k_2^\gamma(p,w)|\partial_w g(p,w)|^3.$$

Since $g(p,w)$ is of class C^4 up to $\partial D(p)$, we have $\Delta_w g(p,w) = 0$ for $w \in \partial D(p)$ and hence, by (2.6),

$$\begin{aligned} \frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma} &= -k_2^\gamma|\partial_w g| + 2\Re\left(\frac{\partial g/\partial p_\gamma}{|\partial_w g|}\sum_{\alpha=1}^n \frac{\partial g/\partial \bar{w}_\alpha}{|\partial_w g|}\frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}\right) \\ &= -k_2^\gamma|\partial_w g| - 2\Re\left(k_1^\gamma \sum_{i=1}^n \frac{\partial g/\partial \bar{w}_\alpha}{|\partial_w g|}\frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}\right) \end{aligned}$$

for $w \in \partial D(p)$. We summarize this result as follows.

PROPOSITION 2.1. *The function $g(p,w)$ is smooth up to $\mathcal{D} \cup \partial\mathcal{D} = \{(p,w) : p \in D, w \in \bar{D}(p)\}$. If $1 \le \gamma \le n$ and $p \in D$, then*:

(1) $(\partial g/\partial p_\gamma)(p)$ *is a harmonic function of $w \in D(p)$ with*

$$\frac{\partial g}{\partial p_\gamma}(p,0) = \frac{\partial \lambda}{\partial p_\gamma}(p)$$

and with boundary values

$$\frac{\partial g}{\partial p_\gamma}(p,w) = -k_1(p,w)|\partial_w g(p,w)|, \quad w \in \partial D(p);$$

(2) $(\partial^2 g/\partial p_\gamma \partial \bar{p}_\gamma)(p)$ *is a harmonic function of* $w \in D(p)$ *with*

$$\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p,0) = \frac{\partial^2 \lambda}{\partial p_\gamma \partial \bar{p}_\gamma}(p)$$

and with boundary values

$$\begin{aligned} &\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p,w) \\ &\quad = -k_2^\gamma(p,w)|\partial_w g(p,w)| \\ &\qquad - 2\Re\left(k_1^\gamma(p,w)\sum_{\alpha=1}^{n}\frac{(\partial g/\partial \bar{w}_\alpha)(p,w)}{|\partial_w g(p,w)|}\frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p,w)\right), \quad w \in \partial D(p). \end{aligned}$$

Toward this end, it was proved in [7] that $g(p,w)$ is C^2 up to $\{(p,w) : p \in \bar{D}, w \in \bar{D}(p)\}$ by deriving the following estimates. There exists a constant C, independent of $p \in \partial D$, such that

$$\begin{aligned} |k_1^\gamma(p,w)| &\le C|w|^2, \\ |k_2^\gamma(p,w)| &\le C|w|^3, \\ |\partial_w g(p,w)| &\le C|w|^{-2n+1}, \\ |\partial^2 g/\partial \bar{w}_\alpha \partial p_\gamma| &\le C|w|^{-2n+2} \end{aligned} \tag{2.7}$$

for all $w \in \partial D$ with $|w| \ge 1$. Moreover, the derivatives $\partial g/\partial p_\gamma$ and $\partial^2 g/\partial p_\gamma \partial \bar{p}_\gamma$ are given by the following proposition.

PROPOSITION 2.2. *Let* $1 \le \gamma \le n$. *Then, for* $p \in \bar{D}$ *and* $a \in D(p)$,

$$\frac{\partial g}{\partial p_\gamma}(p,a) = \frac{1}{2(n-1)\sigma_{2n}}\int_{\partial D(p)} k_1^\gamma(p,w)|\partial_w g(p,w)|\frac{\partial g_a(p,w)}{\partial n_w}\,dS_w \tag{2.8}$$

and

$$\begin{aligned} &\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p,a) \\ &\quad = \frac{1}{2(n-1)\sigma_{2n}}\int_{\partial D(p)} k_2^\gamma(p,w)|\partial_w g(p,w)|\frac{\partial g_a(p,w)}{\partial n_w}\,dS_w \\ &\qquad + \frac{1}{(n-1)\sigma_{2n}}\Re\sum_{\alpha=1}^{n} \\ &\qquad\quad \int_{\partial D(p)} k_1^\gamma(p,w)\frac{(\partial g/\partial \bar{w}_\alpha)(p,w)}{|\partial_w g(p,w)|}\frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p,w)\frac{\partial g}{\partial n_w}(w)\,dS_w. \end{aligned} \tag{2.9}$$

Here $g_a(p,w)$ *is the Green function for* $D(p)$ *with pole at* a.

We note that, for $p \in D$, the preceding formulas are consequences of Proposition 2.1. For $p \in \partial D$, these formulas were obtained in [7] by finding

$$\lim_{D \ni q \to p} \frac{\partial g}{\partial p_\gamma}(q,a) \quad \text{and} \quad \lim_{D \ni q \to p} \frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(q,a).$$

A particular case of this proposition is the following.

PROPOSITION 2.3. *Let $1 \le \gamma \le n$ and $p \in \bar{D}$. Then*

$$\frac{\partial \lambda}{\partial p_\gamma}(p) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_1^\gamma(p,\zeta) |\partial_w g(p,\zeta)| \frac{\partial g(p,w)}{\partial n_w}\, dS_w \tag{2.10}$$

and

$$\begin{aligned} &\frac{\partial^2 \lambda}{\partial p_\gamma \partial \bar{p}_\gamma}(p) \\ &\quad = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_2^\gamma(p,w) |\partial_w g(p,\zeta)|^2\, dS_w \\ &\qquad + \frac{1}{(n-1)\sigma_{2n}} \Re \sum_{\alpha=1}^{n} \\ &\qquad\quad \int_{\partial D(p)} k_1^\gamma(p,w) \frac{(\partial g/\partial \bar{w}_\alpha)(p,w)}{|\partial_w g(p,w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p,w) \frac{\partial g}{\partial n_w}(p,w)\, dS_w. \end{aligned} \tag{2.11}$$

We now consider a sequence $\{D_\nu\}$ of C^∞-smoothly bounded domains in $\mathbf{C}^n$ that converges in C^∞-topology to D. We choose C^∞-smooth defining functions ψ_ν for the domains D_ν such that $\{\psi_\nu\}$ converges in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to ψ. This implies, in particular, that D_ν converges in the Hausdorff sense to D. For each $\nu \ge 1$, consider the scaling map $T_\nu\colon D_\nu \times \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$T_\nu(p,z) = \frac{z-p}{-\psi_\nu(p)}$$

and the family of domains $\{D_\nu(p) : p \in \bar{D}_\nu\}$ defined by

$$D_\nu(p) = \begin{cases} T_\nu(p, D_\nu) & \text{if } p \in D_\nu, \\ \left\{ w \in \mathbf{C}^n : 2\Re\left(\sum_{i=1}^n \psi_{\nu i}(p) w_i\right) - 1 < 0 \right\} & \text{if } p \in \partial D_\nu. \end{cases}$$

The normalized Robin function $\lambda_\nu(p)$ for (D_ν, ψ_ν) is then the Robin constant for $D_\nu(p)$ at 0. We will denote the Green function for D_ν with pole at 0 by $g_\nu(p,w)$. Also, let

$$\mathcal{D}_\nu = \bigcup_{p \in D_\nu} (p, D_\nu(p)) = \{(p,w) : p \in D_\nu,\ w \in D_\nu(p)\}$$

be the variation associated to (D_ν, ψ_ν) and let

$$f_\nu(p,w) = 2\Re\left\{ \sum_{\alpha=1}^{n} \int_0^1 \left(w_\alpha (\psi_\nu)_\alpha (p - \psi_\nu(p) t w) \right) dt \right\} - 1. \tag{2.12}$$

Then $f_\nu(p,w)$ is a smooth function on $\mathbf{C}^n \times \mathbf{C}^n$ that defines the variation $\mathcal{D}_\nu$. It is evident that the functions $f_\nu(p,w)$ converge in the C^∞-topology on compact subsets of $\mathbf{C}^n \times \mathbf{C}^n$ to the function

$$f(p,w) = 2\Re\left\{\sum_{\alpha=1}^{n}\int_0^1 \bigl(w_\alpha\psi_\alpha(p-\psi(p)tw)\bigr)\,dt\right\} - 1,$$

which defines the variation $\mathcal{D}$ associated to (D,ψ).

Now let $p_\nu \in D_\nu$ be such that $\{p_\nu\}$ converges to $p_0 \in \partial D$. For brevity, we let

$$\begin{aligned} T^\nu(z) &= T_\nu(p_\nu, z) = \frac{z-p_\nu}{-\psi_\nu(p_\nu)},\\ D^\nu &= D_\nu(p_\nu) = T^\nu(D_\nu), \quad \text{and} \\ g^\nu(w) &= g_\nu(p_\nu, w). \end{aligned} \tag{2.13}$$

Thus $g^\nu(w)$ is the Green function for D^ν with pole at 0. Let $1 \le \gamma \le n$. By Proposition 2.1, $(\partial g_\nu/\partial p_\gamma)(p_\nu,w)$ is a harmonic function of $w \in D^\nu$ with boundary values

$$-k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)|; \tag{2.14}$$

here

$$k_1^{\nu\gamma}(w) = k_{1\nu}^{\gamma}(w) = \frac{\partial f_\nu}{\partial p_\gamma}(p_\nu,w)|\partial_w f_\nu(p_\nu,w)|^{-1}. \tag{2.15}$$

Similarly, $(\partial^2 g_\nu/\partial p_\gamma\partial\bar{p}_\gamma)(p_\nu,w)$ is a harmonic function of $w \in D^\nu$ with boundary values

$$\begin{aligned} &\frac{\partial^2 g_\nu}{\partial p_\gamma\partial\bar{p}_\gamma}(p_\nu,w)\\ &\quad = -k_2^{\nu\gamma}(w)|\partial_w g^\nu(w)|\\ &\qquad - 2\Re\left(k_1^{\nu\gamma}(w)\sum_{\alpha=1}^{n}\frac{(\partial g^\nu/\partial\bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|}\,\frac{\partial^2 g_\nu}{\partial w_\alpha\partial\bar{p}_\gamma}(p_\nu,w)\right), \quad w \in \partial D^\nu, \end{aligned} \tag{2.16}$$

where

$$k_2^{\nu\gamma}(w) = \mathcal{L}^\gamma f_\nu(p_\nu,w)|\partial_w f_\nu(p_\nu,w)|^{-3} \tag{2.17}$$

and $\mathcal{L}^\gamma$ is defined by (2.6).

We shall conclude this section by finding uniform bounds for the functions $k_1^{\nu\gamma}(w)$ and $k_2^{\nu\gamma}(w)$ near the boundary of ∂D^ν, which will be required to estimate the boundary values (2.14) and (2.16) in Sections 4 and 5. For $0 < r < 1$, let $\mathcal{E}^\nu(r)$ be the collar about ∂D^ν defined by

$$\mathcal{E}^\nu(r) = \bigcup_{w_0\in\partial D^\nu}\{w \in D^\nu : |w-w_0| < r|w_0|\}.$$

Note that $\mathcal{E}^\nu(r)$ lies in D^ν and that $\bar{\mathcal{E}}^\nu(r)$ does not contain the origin. Similarly, let $\mathcal{E}_\nu(r)$ be the collar around ∂D_ν defined by

$$\mathcal{E}_\nu(r) = \bigcup_{z_0\in\partial D_\nu}\{z \in D_\nu : |z-z_0| < r|z_0-p_\nu|\}.$$

Here $\mathcal{E}_\nu(r)$ lies in D_ν and does not contain the point p_ν. Note also that

$$\mathcal{E}_\nu(r) = (T^\nu)^{-1}(\mathcal{E}^\nu(r)). \tag{2.18}$$

LEMMA 2.4. *There exist a constant $m > 0$, a number $0 < r < 1$, and an integer I such that*

$$|\partial_w f_\nu(p_\nu, w)| > m$$

for all $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$.

Proof. Choose a δ-neighborhood U of ∂D, that is,

$$U = \{z \in \mathbf{C}^n : d(z, \partial D) < \delta\},$$

and a constant $m > 0$ such that $|\partial\psi(p)| > 2m$ for $p \in U$. Since $\partial\psi_\nu$ converges uniformly on $\bar{U}$ to $\partial\psi$, there exists an integer I such that

$$|\partial\psi_\nu(p)| > m \tag{2.19}$$

for $\nu \geq I$ and $p \in U$. Modify the integer I so that $\partial D_\nu \subset N(\delta/2)$ for all $\nu \geq I$. Since $p_\nu \to p_0 \in \partial D$, we can assume without loss of generality that $p_\nu \in U$ for all $\nu \geq I$. Now define

$$r = \frac{\delta}{3\delta + 2\,\mathrm{diam}(D)}.$$

Then it is evident that

$$\mathcal{E}_\nu(r) \subset U \tag{2.20}$$

for $\nu \geq I$. Now fix $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$. If we define $z = T_\nu^{-1}w = p_\nu - \psi_\nu(p_\nu)w$ then, by (2.18),

$$z \in \mathcal{E}_\nu(r) \subset U.$$

From (2.12) it follows that

$$|\partial_w f_\nu(p_\nu, w)| = |\partial\psi_\nu(z)| > m$$

by (2.19). □

We now modify Step 4 of [7, Chap. 4] to obtain the following estimates.

LEMMA 2.5. *Let r and I be as in Lemma 2.4. Then there exists a constant $M > 0$ such that*

(i) $|(\partial f_\nu/\partial w_\alpha)(p_\nu, w)| < M$,
(ii) $|(\partial f_\nu/\partial p_\gamma)(p_\nu, w)| < M(1 + |w|^{-1})|w|^2$,
(iii) $|(\partial^2 f_\nu/\partial w_\alpha \partial w_\beta)(p_\nu, w)| < M|w|^{-1}$,
(iv) $|(\partial^2 f_\nu/\partial p_\gamma \partial w_\alpha)(p_\nu, w)| < M(1 + |w|^{-1})|w|$, *and*
(v) $|(\partial^2 f_\nu/\partial p_\gamma \partial p_\mu)(p_\nu, w)| < M(1 + |w|^{-1} + |w|^{-2})|w|^3$

for all $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$.

Proof. Let U be as in the proof of Lemma 2.4, and choose $R > 0$ such that $U \subset B(0, R)$. Since $\{\psi_\nu\}$ converges in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to ψ, we can find a constant $M_1 > 0$ such that ψ, ψ_ν $(\nu \geq 1)$, and their derivatives of order ≤ 2 are bounded in absolute value by M_1 on $\bar{B}(0, R)$.

Now let $\nu \geq I$ and let $w \in \mathcal{E}^\nu(r)$. Then

$$p_\nu - \psi_\nu(p_\nu)tw \in B(0, R), \quad 0 \leq t \leq 1. \tag{2.21}$$

Before proving this, note that it implies in particular that ψ_ν and its derivatives of order ≤ 2 are bounded in absolute value by M_1 at the points $p_\nu - \psi_\nu(p_\nu)tw$ for all $0 \leq t \leq 1$. Now to prove (2.21), we let $0 \leq t \leq 1$. Set

$$z = T_\nu^{-1} w = p_\nu - \psi_\nu(p_\nu)w.$$

Then $z \in \mathcal{E}_\nu(r)$ by (2.18) and hence $z \in U$ by (2.20). Now

$$p_\nu - \psi_\nu(p_\nu)tw = p_\nu + t(z - p_\nu) = (1-t)p_\nu + tz \in B(0, R)$$

since $p_\nu, z \in U \subset B(0, R)$.

(i) Differentiating (2.3) with respect to w_α under the integral sign, we have

$$\frac{\partial f}{\partial w_\alpha}(p, w) = \psi_\alpha(p - \psi(p)w), \quad p, w \in \mathbf{C}^n.$$

Hence, for $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$,

$$\left| \frac{\partial f_\nu}{\partial w_\alpha}(p_\nu, w) \right| = |\psi_{\nu\alpha}(p_\nu - \psi_\nu(p_\nu)w)| \leq M_1.$$

(ii) Differentiating (2.3) with respect to p_γ under the integral sign, we have

$$\begin{aligned} \frac{\partial f}{\partial p_\gamma}(p, w) = \sum_{\alpha=1}^n \int_0^1 & \frac{\partial}{\partial p_\gamma}\big(w_\alpha \psi_\alpha(p - \psi(p)tw)\big) \\ & + \frac{\partial}{\partial p_\gamma}\big(\bar{w}_\alpha \psi_{\bar{\alpha}}(p - \psi(p)tw)\big)\, dt, \quad p, w \in \mathbf{C}^n. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial p_\gamma}\big(w_\alpha \psi_\alpha(p - \psi(p)tw)\big) = {} & w_\alpha \psi_{\gamma\alpha}(p - \psi(p)tw) \\ & - 2t\psi_\gamma(p) \Re \sum_{i=1}^n w_i w_\alpha \psi_{i\alpha}(p - \psi(p)tw). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial f}{\partial p_\gamma}(p, w) = {} & \sum_{\alpha=1}^n \int_0^1 \big(w_\alpha \psi_{\gamma\alpha}(p - \psi(p)tw) + \bar{w}_\alpha \psi_{\gamma\bar{\alpha}}(p - \psi(p)tw)\big)\, dt \\ & - 2\psi_\gamma(p) \Re \sum_{i,\alpha=1}^n \int_0^1 \big(w_i w_\alpha \psi_{i\alpha}(p - \psi(p)tw) \\ & \qquad + w_i \bar{w}_\alpha \psi_{i\bar{\alpha}}(p - \psi(p)tw)\big) t\, dt. \end{aligned} \tag{2.22}$$

Hence, for $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$,

$$\begin{aligned}
&\left|\frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w)\right| \\
&\quad\le \sum_{\alpha=1}^{n}\int_0^1 |w_\alpha||\psi_{\nu\gamma\alpha}(p_\nu-\psi_\nu(p_\nu)tw)| + |\bar{w}_\alpha||\psi_{\nu\gamma\bar{\alpha}}(p_\nu-\psi_\nu(p_\nu)tw)|\,dt \\
&\qquad + 2|\psi_{\nu\gamma}(p_\nu)|\sum_{i,\alpha=1}^{n}\int_0^1 |w_i||w_\alpha||\psi_{\nu i\alpha}(p_\nu-\psi_\nu(p_\nu)tw)| \\
&\qquad + |w_i||\bar{w}_\alpha||\psi_{\nu i\bar{\alpha}}(p_\nu-\psi_\nu(p_\nu)tw)|t\,dt \\
&\quad\le \int_0^1 2|w|\sqrt{n}M_1\,dt + 2M_1\sum_{i=1}^{n}\int_0^1 2|w_i||w|\sqrt{n}M_1 t\,dt \\
&\quad\le 2\sqrt{n}M_1|w| + 2n^{3/2}(M_1)^2|w|^2 \\
&\quad\le M_2(1+|w|^{-1})|w|^2,
\end{aligned}$$

where $M_2 = 2n^{3/2}(M_1)^2$.

(iii) Differentiating (2.3) with respect to w_α under the integral sign, we have

$$\frac{\partial f}{\partial w_\alpha}(p,w) = \psi_\alpha(p-\psi(p)w), \quad p, w \in \mathbf{C}^n.$$

Differentiating this equation with respect to w_β yields

$$\frac{\partial^2 f}{\partial w_\beta \partial w_\alpha}(p,w) = (-\psi(p))\psi_{\alpha\beta}(p-\psi(p)w), \quad p, w \in \mathbf{C}^n.$$

Let $\nu \ge I$ and $w \in \mathcal{E}^\nu(r)$. Let

$$z = T_\nu^{-1}w = p_\nu - \psi_\nu(p_\nu)w.$$

Then, by (2.21), $z \in B(0,R)$. Now

$$\left|\frac{\partial^2 f_\nu}{\partial w_\beta \partial w_\alpha}(p_\nu, w)\right| \le \frac{|z-p_\nu|}{|w|}|\psi_{\nu\alpha\beta}(z)| \le 2RM_1|w|^{-1} = M_3|w|^{-1},$$

where $M_3 = 2RM_1$. Finally, by differentiating (2.22) we obtain (iv) and (v). $\square$

PROPOSITION 2.6. *There exist* $0 < r < 1$, *a constant* C, *and an integer* I *such that*

(1) $|k_1^{\nu\gamma}(w)| \le C(1+|w|^{-1})|w|^2$ *and*

(2) $|k_2^{\nu\gamma}(w)| \le C(1+|w|^{-1}+|w|^{-2})|w|^3$

for all $\nu \ge I$ *and* $w \in \bar{\mathcal{E}}^\nu(r)$.

Proof. Let $0 < r < 1$, let $m > 0$, and let I be as in Lemma 2.4. Choose M as in Lemma 2.5. Then, by (2.15),

$$|k_1^\nu(w)| = \left|\frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w)\right| |\partial_w f_\nu(p_\nu, w)|^{-1} < \frac{M}{m}(1+|w|^{-1})|w|^2$$

for $\nu \ge I$ and $w \in \mathcal{E}^\nu(r)$. Also, since $0 \notin \bar{\mathcal{E}}^\nu(r)$, the function

$$|k_1^\nu(w)|(1+|w|^{-1})^{-1}|w|^{-2}$$

is continuous up to $\bar{\mathcal{E}}^\nu(r)$ and hence (1) follows.

Similarly, from (2.17) it follows that

$$\begin{aligned}|k_2^\nu(w)| &< \frac{1}{m^3}\big(M(1+|w|^{-1}+|w|^{-2})|w|^3M^2\\ &\qquad + 2nM(1+|w|^{-1})|w|^2MM(1+|w|^{-1})|w|\\ &\qquad + (M(1+|w|^{-1})|w|^2)^2nM|w|^{-1}\big)\\ &\le C(1+|w|^{-1}+|w|^{-2})|w|^3\end{aligned}$$

for some constant C whenever $\nu \ge I$ and $w \in \mathcal{E}^\nu(r)$. Again the function

$$|k_2^\nu(w)|(1+|w|^{-1}+|w|^{-2})^{-1}|w|^{-3}$$

is continuous up to $\bar{\mathcal{E}}^\nu(r)$ and so (2) follows. $\square$

3. Asymptotics of Λ_ν

In this section we prove Theorem 1.3. First we recall the following stability result from [1].

PROPOSITION 3.1. *Let D be a domain in $\mathbf{C}^n$ with C^2-smooth boundary, and let $\{D_j\}$ be a C^2-perturbation of D. Let $G(z,p)$ be the Green function for D with pole at p, and let $\Lambda(p)$ be the Robin function for D. Similarly, let $G_j(z,p)$ be the Green function for D_j with pole at p and let $\Lambda_j(p)$ the Robin function for D_j. Then*

$$\lim_{j\to\infty} G_j(z,p) = G(z,p)$$

uniformly on compact subsets of $D \setminus \{p\}$, and

$$\lim_{j\to\infty} D^{A\bar{B}}\Lambda_j(p) = D^{A\bar{B}}\Lambda(p)$$

uniformly on compact subsets of D.

For a proof see [1, Prop. 7.1, Prop. 7.2]. Proposition 3.1, together with [7, Prop. 5.1], yields the following boundary behavior of the functions $G_j(z,p)$.

COROLLARY 3.2. *Let D be a domain in $\mathbf{C}^n$ with C^∞-smooth boundary, and let $\{D_j\}$ be a C^∞-perturbation of D. Let $z_j \in \bar{D}_j$ be such that $\{z_j\}$ converges to a point $z_0 \in \partial D$. Then, for any $p \in D$,*

$$\lim_{j\to\infty} G_j(z_j,p) = G(z_0,p);$$

identifying $z=(z_1,\dots,z_n)\in\mathbf{C}^n$ with $x=(x_1,\dots,x_{2n})\in\mathbf{R}^{2n}$, we have

$$\lim_{j\to\infty}\frac{\partial G_j}{\partial x_k}(z_j,p) = \frac{\partial G}{\partial x_k}(z_0,p)$$

for $1\le k\le 2n$.

Proof. Since the Green function is invariant under translation and rotation, we assume without loss of generality that $z_0 = 0$ and that the normal to ∂D at z_0 is along the x_{2n}-axis. By the implicit function theorem, we can find a ball $B(0, r)$, a C^∞-smooth function ϕ defined on $B(0', r) \subset \mathbf{R}^{2n-1}$, and a sequence $\{\phi_j\}$ of C^∞-smooth functions defined on $B(0', r)$ that converges in C^∞-topology on compact subsets of $B(0', r)$ to ϕ such that

$$\begin{aligned} B(0,r) \cap \partial D &= \{(x', \phi(x')) : x' \in B(0', r)\}, \\ B(0,r) \cap \partial D_j &= \{(x', \phi_j(x')) : x' \in B(0', r)\}. \end{aligned} \tag{3.1}$$

Now let $p \in D$. Shrinking r if necessary, let us assume that $2r < |p|$. Then, for $z \in B(0, r) \cap D_j$,

$$G_j(z, p) < |z - p|^{-2n+2} < r^{-2n+2}. \tag{3.2}$$

Consider the dilation

$$Z = Sz = \frac{z}{r}$$

and set

$$\Omega = S(B(0,r) \cap D), \qquad \Omega_j = S(B(0,r) \cap D_j).$$

Define

$$u(Z) = r^{2n-2} G(z, p), \quad Z \in \Omega,$$

and

$$u_j(Z) = r^{2n-2} G_j(z, p), \quad Z \in \Omega_j.$$

Then, by (3.1) and (3.2) and in view of Proposition 3.1, the sequence $\{u_j\}$ on $\{\Omega_j\}$ satisfies the hypothesis of [7, Prop. 5.1]. Therefore,

$$\lim_{j \to \infty} u_j(Z_j) = u(0),$$

$$\lim_{j \to \infty} \frac{\partial u_j}{\partial \tilde{x}_k}(Z_j) = \frac{\partial u}{\partial \tilde{x}_k}(0),$$

where $Z_j = Sz_j$. This implies that

$$\lim_{j \to \infty} G_j(z_j, p) = G(0, p),$$

$$\lim_{j \to \infty} \frac{\partial G_j}{\partial x_k}(z_j, p) = \frac{\partial G}{\partial x_k}(0, p). \qquad \square$$

Proof of Theorem 1.3. Consider the affine maps $T^\nu : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$T^\nu(z) = \frac{z - p_\nu}{-\psi_\nu(p_\nu)}$$

as well as the scaled domains $D^\nu = T^\nu(D_\nu)$. Recall from Section 2 that a defining function for D^ν is given by

$$f_\nu(p_\nu, w) = 2\Re\left\{ \sum_{\alpha=1}^{n} \int_0^1 \big(w_\alpha \psi_{\nu\alpha}(p_\nu - \psi_\nu(p_\nu) t w) \big)\, dt \right\} - 1.$$

It is clear that $\{f_\nu(p_\nu, \cdot)\}$ converges in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to

$$f(p_0, w) = 2\Re\left(\sum_{\alpha=1}^{n} \psi_\alpha(p_0) w_\alpha\right) - 1,$$

which implies that $\{D^\nu\}$ is a C^∞-perturbation of the half-space

$$\mathcal{H} = \left\{w : 2\Re\left(\sum_{\alpha=1}^{n} \psi_\alpha(p_0) w_\alpha\right) - 1 < 0\right\}.$$

Therefore, by Proposition 3.1,

$$\lim_{\nu\to\infty} D^{A\bar{B}} \Lambda_{D^\nu}(0) = D^{A\bar{B}} \Lambda_{\mathcal{H}}(0). \tag{3.3}$$

Now, by [1, (1.1)], we have

$$\Lambda_{D^\nu}(p) = \Lambda_\nu(p_\nu - p\psi_\nu(p_\nu))(\psi_\nu(p))^{2n-2}.$$

Differentiating this expression yields

$$D^{A\bar{B}} \Lambda_{D^\nu}(0) = (-1)^{|A|+|B|} D^{A\bar{B}} \Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2+|A|+|B|}.$$

Hence from (3.3) it follows that

$$\lim_{\nu\to\infty} D^{A\bar{B}}(-1)^{|A|+|B|} D^{A\bar{B}} \Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2+|A|+|B|} = D^{A\bar{B}} \Lambda_{\mathcal{H}}(0),$$

which completes the proof. □

4. Estimates on the First Derivatives

Let $1 \le \gamma \le n$. By Proposition 2.1, $(\partial g_\nu / \partial p_\gamma)(p_\nu, w)$ is a harmonic function of $w \in D^\nu$,

$$\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, 0) = \frac{\partial \lambda_\nu}{\partial p_\gamma}(p_\nu),$$

and

$$\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) = -k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)|, \quad w \in \partial D^\nu. \tag{4.1}$$

Therefore,

$$\frac{\partial \lambda_\nu}{\partial p_\gamma}(p_\nu) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w. \tag{4.2}$$

So to find the limit of these integrals, we must estimate the boundary values (4.1). For this we modify Step 3 of [7, Chap. 4].

LEMMA 4.1. *There exist a number $0 < \rho < 1$ and an integer I such that, for $\nu \ge I$ and $w_0 \in \partial D^\nu$, we can find a ball of radius $\rho|w_0|$ that is externally tangent to ∂D^ν at w_0.*

Proof. Since D is bounded, we can find a ball $B(0, R)$ that contains D. Since $\{D_\nu\}$ converges in C^2-topology to D, there exists an integer I such that $D_\nu \subset B(0, R)$ for all $\nu \ge I$. By the implicit function theorem there exists a number $\tilde{\rho}$ such that, by modifying I, we can find a ball of radius $\tilde{\rho}$ that is externally tangent to ∂D_ν at

z_0 for each $\nu \geq I$ and $z_0 \in \partial D_\nu$. Now let $\nu \geq I$ and $w_0 \in \partial D^\nu$. Since D^ν is obtained from D_ν by means of a translation followed by dilation of factor $-\psi_\nu(p_\nu)$, we can find a ball of radius $\tilde{\rho}/(-\psi_\nu(p_\nu))$ that is externally tangent to ∂D^ν at w_0. Furthermore, there exists a $z_0 \in \partial D_\nu$ such that

$$w_0 = \frac{z_0 - p_\nu}{-\psi_\nu(p_\nu)};$$

this implies that

$$\frac{\tilde{\rho}}{-\psi_\nu(p_\nu)} = \frac{\tilde{\rho}|w_0|}{|z_0 - p_\nu|} \geq \frac{\tilde{\rho}}{2R}|w_0|.$$

Thus, by taking $\rho = \tilde{\rho}/2R$, we can find a ball of radius $\rho|w_0|$ that is tangent to ∂D^ν at w_0. □

PROPOSITION 4.2. *There exist an integer I and a constant $C > 0$ such that*

$$|\partial_w g^\nu(w)| \leq C|w|^{-2n+1}$$

for all $\nu \geq I$ and $w \in \partial D^\nu$.

Proof. Choose $0 < \rho < 1$ along with an integer I and a constant C as in Lemma 4.1. Let $\nu \geq I$ and $w_0 \in \partial D^\nu$. Let B be the ball of radius $\rho|w_0|$ that is externally tangent to ∂D^ν at w_0, and let E be the ball centred at w_0 and of radius $\rho|w_0|$. Then $w \in E$ implies that

$$|w| > |w_0| - \rho|w_0| = (1-\rho)|w_0|.$$

Hence, for $w \in E \cap D^\nu$,

$$0 < g^\nu(w) \leq |w|^{-2n+2} < ((1-\rho)|w_0|)^{-2n+2}.$$

By Step 2 of [7, Chap. 4], we have

$$|\partial_w g^\nu(w_0)| \leq c((1-\rho)|w_0|)^{-2n+2}(\rho|w_0|)^{-1},$$

where c does not depend on $g^\nu(w)$ or D^ν. Therefore,

$$|\partial_w g^\nu(w_0)| \leq C|w_0|^{-2n+1},$$

where $C = c\rho^{-1}(1-\rho)^{-2n+2}$ is independent of ν and $w_0 \in \partial D^\nu$. □

PROPOSITION 4.3. *There exist a constant $C > 0$ and an integer I such that*

$$\left|\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)\right| = |k_1^{\nu\gamma}(w)||\partial_w g^\nu(w)| \leq C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial D^\nu,$$

for all $\nu \geq I$.

Proof. By Proposition 2.6, there exist a constant C and an integer I such that

$$|k_1^{\nu\gamma}(w)| \leq C(1+|w|^{-1})|w|^2, \quad w \in \partial D^\nu,$$

for all $\nu \geq I$. In view of Proposition 4.2, we can modify the constant C and the integer I so that

$$|\partial_w g^\nu(w)| \leq C|w|^{-2n+1}, \quad w \in \partial D^\nu,$$

for all $\nu \geq I$. Thus, from (4.1) it follows that

$$\left|\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)\right| = |k_1^{\nu\gamma}(w)||\partial_w g^\nu(w)| \leq C^2(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial D^\nu,$$

for all $\nu \geq I$. □

PROPOSITION 4.4.

$$\lim_{\nu\to\infty} \frac{\partial \lambda_\nu}{\partial p_\gamma}(p_\nu) = \frac{\partial \lambda}{\partial p_\gamma}(p_0).$$

Proof. In view of Proposition 2.3, we have to prove that

$$\lim_{\nu\to\infty} \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial \mathcal{H}} k_1^{\gamma}(p_0, w)|\partial g(p_0, w)| \frac{\partial g}{\partial n_w}(p_0, w)\, dS_w, \quad (4.3)$$

where $\mathcal{H} = D(p_0)$. Let $R > 1$. Then the boundary surfaces $B(0, R) \cap \partial D^\nu$ converge to $B(0, R) \cap \mathcal{H}$ continuously in the sense that the unit normal vectors

$$\frac{\partial_w g^\nu(w)}{|\partial_w g^\nu(w)|} \to \frac{\partial g(p_0, w)}{|\partial_w g(p_0, w)|}$$

uniformly on compact sets, except at the corners $B(0, R) \cap \partial D^\nu$. Also, if $w^\nu \in \partial D^\nu$ and $\{w^\nu\}$ converges to $w^0 \in \partial\mathcal{H}$, then by definition we have

$$\lim_{\nu\to\infty} k_1^{\nu\gamma}(w^\nu) = k_1^\gamma(p_0, w^0) \quad (4.4)$$

and, by Corollary 3.2,

$$\lim_{\nu\to\infty} \frac{\partial g^\nu}{\partial w_\alpha}(w^\nu) = \frac{\partial g}{\partial w_\alpha}(p_0, w^0) \quad (4.5)$$

for $1 \leq \alpha \leq n$. Hence

$$\lim_{\nu\to\infty} \frac{1}{2(n-1)\sigma_{2n}} \int_{B(0,R)\cap\partial D^\nu} k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w = \frac{1}{2(n-1)\sigma_{2n}} \int_{B(0,R)\cap\partial \mathcal{H}} k_1^{\gamma}(p_0, w)|\partial g(p_0, w)| \frac{\partial g}{\partial n_w}(p_0, w)\, dS_w. \quad (4.6)$$

To estimate these integrals outside the ball $B(0, R)$, note that by Proposition 4.3 there exist a constant C and an integer I such that

$$|k_1^{\nu\gamma}(w)||\partial_w g^\nu(w)| \leq C|w|^{-2n+3}, \quad w \in \partial D^\nu,\ |w| > 1,$$

for all $\nu \geq I$. Therefore,

$$\left|\frac{1}{2(n-1)\sigma_{2n}} \int_{B^c(0,R)\cap\partial D^\nu} k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w\right| \leq CR^{-2n+3} \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial B^c(0,R)\cap\partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_\zeta}(w)\right) dS_w \quad (4.7)$$

for all $\nu \geq I$. Since

$$\int_{\partial B^c(0,R)\cap\partial D^\nu}\left(-\frac{\partial g^\nu}{\partial n_\zeta}(w)\right)dS_w \le \int_{\partial D^\nu}\left(-\frac{\partial g^\nu}{\partial n_w}(w)\right)dS_w = (2n-2)\sigma_{2n},$$

it follows from (4.7) that

$$\left|\frac{1}{2(n-1)\sigma_{2n}}\int_{B^c(0,R)\cap\partial D^\nu} k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)|\frac{\partial g^\nu}{\partial n_w}(w)\,dS_w\right| = O(R^{-2n+3}) \qquad (4.8)$$

uniformly for all $\nu \ge I$. By (2.7), we can modify the constant C so that

$$|k_1^\gamma(p_0,w)||\partial_w g(p_0,w)| \le C|w|^{-2n+3}, \quad w\in\partial\mathcal{H},\ |w|>1;$$

then, much as before, we obtain

$$\left|\frac{1}{2(n-1)\sigma_{2n}}\int_{B^c(0,R)\cap\partial\mathcal{H}} k_1^{\gamma}(p_0,w)|\partial_w g(p_0,w)|\frac{\partial g}{\partial n_w}(w)\,dS_w\right|$$
$$= O(R^{-2n+3}). \quad (4.9)$$

Now (4.3) follows from (4.6), (4.8), and (4.9). □

REMARK 4.5. The arguments of this section also imply that, for any $a\in\mathcal{H}$,

$$\begin{aligned}\lim_{\nu\to\infty}\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu,a) &= \lim_{\nu\to\infty}\frac{1}{2(n-1)\sigma_{2n}}\int_{\partial D^\nu} k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)|\frac{\partial g_{\nu a}}{\partial n_w}(p_\nu,w)\,dS_w\\ &= \frac{1}{2(n-1)\sigma_{2n}}\int_{\partial\mathcal{H}} k_1^{\gamma}(w)|\partial_w g^0(w)|\frac{\partial g_a}{\partial n_w}(p_\nu,w)\,dS_w\\ &= \frac{\partial g}{\partial p_\gamma}(p_0,a).\end{aligned}$$

Moreover, by Proposition 4.3, the functions $(\partial g_\nu/\partial p_\gamma)(p_\nu,w)$ are uniformly bounded on compact subsets of $\mathcal{H}$ for all large ν. Indeed, let $\bar{B}(0,r)\subset\mathcal{H}$. Then $\bar{B}(0,r)\subset D^\nu$ for all large ν. It follows that

$$\left|\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu,w)\right| \le Cr^{-2n+3}(1+r^{-1})$$

for $w\in\partial D^\nu$ and hence, by the maximum principle, for $w\in D^\nu$. We may thus conclude that $\{(\partial g_\nu/\partial p_\gamma)(p_\nu,a)\}$ converges uniformly on compact subsets of $\mathcal{H}$ to $(\partial g/\partial p_\gamma)(p_0,a)$.

5. Estimates on the Second Derivatives

By Proposition 2.1, $(\partial^2 g_\nu/\partial p_\gamma\partial\bar{p}_\gamma)(p_\nu,w)$ is a harmonic function of $w\in D^\nu$,

$$\frac{\partial^2 g_\nu}{\partial p_\gamma\partial\bar{p}_\gamma}(p_\nu,0) = \frac{\partial^2\lambda_\nu}{\partial p_\gamma\partial\bar{p}_\gamma}(p_\nu),$$

and

$$\begin{aligned}\frac{\partial^2 g_\nu}{\partial p_\gamma \partial \bar{p}_\gamma}&(p_\nu, w)\\ &= -k_2^{\nu\gamma}(w)|\partial_w g^\nu(w)|\\ &\quad - 2\Re\biggl(k_1^{\nu\gamma}(w)\sum_{\alpha=1}^{n}\frac{(\partial g^\nu/\partial \bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|}\frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w)\biggr), \quad w \in \partial D^\nu. \end{aligned} \tag{5.1}$$

Therefore,

$$\begin{aligned}\frac{\partial^2 \lambda_\nu}{\partial p_\gamma \partial \bar{p}_\gamma}&(p_\nu)\\ = \frac{1}{2(n-1)\sigma_{2n}}&\int_{\partial D^\nu} k_2^\nu(w)|\partial_w g^\nu(\zeta)|\frac{\partial g^\nu}{\partial n_w}(w)\, dS_w + \frac{1}{(n-1)\sigma_{2n}}\Re\sum_{\alpha=1}^{n}\\ &\int_{\partial D^\nu} k_1^{\nu\gamma}(w)\frac{(\partial g^\nu/\partial \bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|}\frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w)\frac{\partial g^\nu}{\partial n_w}(w)\, dS_w. \end{aligned} \tag{5.2}$$

Using arguments similar to those in the previous section, we obtain

$$\begin{aligned}\lim_{\nu\to\infty}\frac{1}{2(n-1)\sigma_{2n}}&\int_{\partial D^\nu} k_2^\nu(w)|\partial_w g^\nu(\zeta)|\frac{\partial g^\nu}{\partial n_w}(w)\, dS_w\\ &= \frac{1}{2(n-1)\sigma_{2n}}\int_{\partial \mathcal{H}} k_2(p_0, w)|\partial_w g(p_0, w)|\frac{\partial g}{\partial n_w}(w)\, dS_w, \end{aligned} \tag{5.3}$$

where $\mathcal{H} = D(p_0)$. Hence we need only find the limit of the second integrals, which requires that we estimate the functions

$$\frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w) \tag{5.4}$$

on ∂D^ν. Since $(\partial g_\nu/\partial p_\gamma)(p_\nu, w)$ is a harmonic function of $w \in D^\nu$ with boundary values

$$F^\nu(w) = -k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)| = -\frac{(\partial f_\nu/\partial p_\gamma)(p_\nu, w)}{|\partial_w f_\nu(p_\nu, w)|}|\partial_w g^\nu(w)|, \tag{5.5}$$

it follows that estimating (5.4) requires that we estimate the derivatives of $F^\nu(w)$. This will be done by modifying Steps 2 and 3 of [7, Chap. 5].

In what follows we will identify the point $z = (z_1, \dots, z_n)$ in $\mathbf{C}^n$ with the point $x = (x_1, \dots, x_{2n})$ in $\mathbf{R}^{2n}$. Similarly, $w = (w_1, \dots, w_n)$ and $W = (W_1, \dots, W_n)$ in $\mathbf{C}^n$ will be identified with $y = (y_1, \dots, y_{2n})$ and $Y = (Y_1, \dots, Y_{2n})$ in $\mathbf{R}^{2n}$, respectively. We begin by giving a version of a tubular neighborhood theorem.

PROPOSITION 5.1. *There exist $0 < r < 1$ and $M > 1$ and an integer I such that, for $\nu \geq I$ and any $z_0 = (x_0', x_{02n})$ in the neighborhood*

$$\bigcup_{z\in\partial D_\nu}\{z + tn_z : -r < t < r\}$$

of ∂D_ν, $B(z_0, r) \cap \partial D_\nu$ can be represented—after a rotation and translation of coordinates—in the form $x_{2n} = \phi(x')$, where:

(a) *$\phi(x')$ is smooth in $B(x'_0, r) \subset \mathbf{R}^{2n-1}$ with $\phi(x'_0) = x_{02n} - t$, where t is such that $z_0 = z_0^* + tn_{z_0^*}$ for some $z_0^* \in \partial D^\nu$; and*
(b) *all partial derivatives of ϕ of order ≤ 6 are bounded in absolute value on $B(x'_0, r)$ by M.*

Now fix r, M, and I as in Proposition 5.1. Modifying the integer I if necessary, we may assume that

$$d(p_\nu, \partial D) < r$$

and

$$\partial D_\nu \subset \{z : d(z, \partial D) < r\}$$

for all $\nu \ge I$. This implies that

$$|\tilde{z}_\nu - p_\nu| < \operatorname{diam}(D) + 2r \tag{5.6}$$

for $\nu \ge I$ and $\tilde{z}_\nu \in \partial D_\nu$. Now choose $0 < \eta < 1$ such that

$$\frac{\eta}{1-\eta}(\operatorname{diam}(D) + 2r) < r. \tag{5.7}$$

LEMMA 5.2. *Let $\nu \ge I$, and let $w^\nu \in D^\nu \setminus \{0\}$ be such that*

$$\{w \in \mathbf{C}^n : |w - w^\nu| < \eta |w^\nu|\} \cap \partial D^\nu \ne \emptyset.$$

Let $S^\nu \colon \mathbf{C}^n \to \mathbf{C}^n$ be the affine map defined by

$$W = S^\nu(w) = \frac{w - w^\nu}{\eta |w^\nu|},$$

and set

$$\Omega^\nu = S^\nu(\{w \in \mathbf{C}^n : |w - w^\nu| < \eta |w^\nu|\} \cap D^\nu) = \{|W| < 1\} \cap S^\nu(D^\nu).$$

Then we can find a $\Phi^\nu \in C^\infty(\{Y' : |Y'| < 1\})$ with
(1) *$\{|W| < 1\} \cap \partial \Omega^\nu = \{Y_{2n} = \Phi^\nu(Y')\}$ and*
(2) *$|\partial^\alpha \Phi^\nu / \partial Y^\alpha| \le M$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| \le 6$ if $|Y'| < 1$.*

Proof. Let

$$z_\nu = (T^\nu)^{-1}(w^\nu) = p_\nu - \psi_\nu(p_\nu) w_\nu,$$

and let

$$b_\nu = (T^\nu)^{-1}(\{w : |w - w^\nu| < \eta |w_\nu|\}) = \{z \in \mathbf{C}^n : |z - z_\nu| < \eta |z_\nu - p_\nu|\}.$$

Then $b_\nu \cap \partial D_\nu \ne \emptyset$, so there is a point $\tilde{z}_\nu \in \partial D_\nu$ such that

$$|\tilde{z}_\nu - z_\nu| < \eta |z_\nu - p_\nu| \le \eta(|z_\nu - \tilde{z}_\nu| + |\tilde{z}_\nu - p_\nu|).$$

Therefore, by (5.6) and (5.7),

$$|\tilde{z}_\nu - z_\nu| < \frac{\eta}{1-\eta} |\tilde{z}_\nu - p_\nu| \le \frac{\eta}{1-\eta}(\operatorname{diam}(D) + 2r) < r; \tag{5.8}$$

hence

$$z_\nu \in \bigcup_{z \in \partial D_\nu} \{z + t n_z : -r < t < r\}.$$

By Proposition 5.1, $B(z_\nu, r) \cap \partial D_\nu$ can be represented—after a rotation and translation of coordinates—in the form $x_{2n} = \phi_\nu(x')$, where $\phi_\nu(x')$ is C^∞ on $B(x'_\nu, r)$,

$$\phi_\nu(x'_\nu) = x_{\nu 0} - t_\nu \tag{5.9}$$

for

$$-t_\nu = d(z_\nu, \partial D_\nu) < \eta |z_\nu - p_\nu|, \tag{5.10}$$

and all partial derivatives of ϕ_ν of order ≤ 6 are bounded in absolute value by M. The surface

$$\{(x', x_{2n}) : x_{2n} = \phi_\nu(x'),\ |x' - x'_\nu| < r\}$$

is mapped by $S^\nu \circ T^\nu$ onto the surface

$$\{(Y', Y_{2n}) : Y_{2n} = \Phi^\nu(Y'),\ |Y'| < R^\nu\},$$

where

$$\Phi^\nu(Y') = \frac{\phi_\nu(p'_\nu - \psi_\nu(p_\nu)y^{\nu\prime} - \psi_\nu(p_\nu)\eta|w^\nu|Y')}{-\psi_\nu(p_\nu)\eta|w^\nu|} + \frac{\psi_\nu(p_\nu)y^\nu_{2n} - p_{\nu 2n}}{-\psi_\nu(p_\nu)\eta|w^\nu|}$$

and

$$R^\nu = \frac{r}{-\psi_\nu(p_\nu)\eta|w^\nu|} = \frac{r}{\eta|z_\nu - p_\nu|}$$

for $w^\nu = (y^{\nu\prime}, y^\nu_{2n})$ and $p_\nu = (p'_\nu, p_{\nu 2n})$.

Yet from (5.8) we have

$$\begin{aligned}\eta|z_\nu - p_\nu| &\leq \eta(|z_\nu - \tilde{z}_\nu| + |\tilde{z}_\nu - p_\nu|)\\ &\leq \eta\left(\frac{\eta}{1-\eta}|\tilde{z}_\nu - p_\nu| + |\tilde{z}_\nu - p_\nu|\right) = \frac{\eta}{1-\eta}|\tilde{z}_\nu - p_\nu| < r\end{aligned}$$

and so $R^\nu > 1$. This implies that

$$\{|W| < 1\} \cap \partial\Omega^\nu \subset \{(Y', Y_{2n}) : Y_{2n} = \Phi^\nu(Y'),\ |Y'| < R^\nu\}.$$

From the properties of ϕ_ν and the explicit formula for Φ^ν just given, it follows that

$$\{|W| < 1\} \cap \partial\Omega^\nu = \{Y_{2n} = \Phi^\nu(Y')\}.$$

Here $\Phi^\nu \in C^\infty(\{Y' : |Y'| < 1\})$, and Φ^ν also satisfies:

(a) $0 < \Phi^\nu(0) < 1$, by (5.9) and (5.10); and
(b) $|\partial^\alpha \Phi^\nu / \partial Y^\alpha| < M$ for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq 6$ if $|Y'| < 1$. □

Now we modify Step 2 of [7, Chap. 5] to obtain the following uniform estimates.

PROPOSITION 5.3. *There exist a constant $C > 0$ and an integer I such that, for $1 \leq i, j, k \leq 2n$,*

(1) $|(\partial g^\nu / \partial y_i)(w)| \leq C|w|^{-2n+1}$,
(2) $|(\partial^2 g^\nu / \partial y_i \partial y_j)(w)| \leq C|w|^{-2n}$, *and*
(3) $|(\partial^3 g^\nu / \partial y_i \partial y_j \partial y_k)(w)| \leq C|w|^{-2n-1}$

for all $\nu \geq I$ and $w \in \bar{D}^\nu \setminus \{0\}$.

Proof. The proofs of (1), (2), and (3) are similar, so we prove only (1). Fix $1 \le i \le 2n$. Suppose that (1) is not true. Then there exists a sequence $\{w^\nu\}$ such that $w^\nu \in D^\nu \setminus \{0\}$ and

$$\lim_{\nu\to\infty}\left|\frac{\partial g^\nu}{\partial y_i}(w^\nu)\right||w^\nu|^{2n-1} = \infty. \tag{5.11}$$

We claim that, for all but finitely many ν,

$$B(w^\nu) = \{w \in \mathbf{C}^n : |w - w^\nu| < \eta|w^\nu|\}$$

intersects ∂D^ν. Indeed, suppose that $B(w^\nu) \cap \partial D^\nu = \emptyset$ for some ν. Then $B(w^\nu) \subset D^\nu$ and so

$$g^\nu(w) \le |w|^{-2n+2} \le (1-\eta)^{-2n+2}|w^\nu|^{-2n+2}, \quad w \in \partial B(w^\nu).$$

Now, by the Poisson integral formula, there exists a constant $c_n > 0$ independent of ν such that

$$\left|\frac{\partial g^\nu}{\partial y_i}(w^\nu)\right| \le \frac{c_n}{(1-\eta)^{2n-2}\eta}|w^\nu|^{-2n+1}.$$

Yet by (5.11) this can be true for only finitely many ν, from which the claim follows. Hence if

$$\Omega^\nu = S^\nu(B(w^\nu) \cap D^\nu) = \{|W| < 1\} \cap S^\nu(D^\nu)$$

then, by Lemma 5.2, for all large ν we can find functions $\Phi^\nu \in C^\infty(\{Y' : |Y'| < 1\})$ such that

$$\Omega^\nu = \{|W| < 1\} \cap \{Y = (Y', Y_{2n}) : |Y'| < 1,\ Y_{2n} < \Phi^\nu(Y')\}$$

and

$$\left|\frac{\partial^\alpha \Phi^\nu}{\partial Y^\alpha}\right| < M \ \text{ for all } |\alpha| \le N \quad \text{if } |Y'| < 1.$$

Since M is independent of ν, it follows from the Arzela–Ascoli theorem that, after passing to a subsequence if necessary, $\{\Phi^\nu\}$ together with all partial derivatives of order ≤ 6 converge uniformly on compact subsets of $\{Y' : |Y'| < 1\}$ to a function $\Phi \in C^6(\{Y' : |Y'| < 1\})$. Set

$$\Omega = \{|W| < 1\} \cap \{Y = (Y', Y_{2n}) : |Y'| < 1,\ Y_{2n} < \Phi(Y')\}.$$

Now define the function u^ν on Ω^ν by

$$u^\nu(W) = |w_\nu|^{2n-2}(1-\eta)^{2n-2}g^\nu(w)$$

for $W = (w - w_\nu)/(\eta|w_\nu|)$. Then u^ν is harmonic on Ω^ν and continuous up to $\partial\Omega^\nu$, and $u_\nu(W) = 0$ on $\{|W| < 1\} \cap \partial\Omega^\nu$. Since

$$0 < g^\nu(w) < |w|^{-2n+2} < (1-\eta)^{-2n+2}|w^\nu|^{-2n+2}, \quad w \in B(w^\nu) \cap D^\nu,$$

we have

$$0 < u^\nu(W) < 1, \quad W \in \Omega^\nu.$$

By Harnack's theorem (and passing to a subsequence), $\{u^\nu\}$ converges uniformly on compact subsets of Ω to a harmonic function u on Ω. From [7, Prop. 5.1] it follows that

$$\lim_{\nu\to\infty}\left|\frac{\partial u^\nu}{\partial y_i}(0)\right| = \frac{\partial u}{\partial y_i}(0),$$

which is finite. So by the definition of u^ν we have

$$\lim_{\nu\to\infty}\left|\frac{\partial g^\nu}{\partial y_i}(w^\nu)\right||w^\nu|^{2n-1} < \infty,$$

which is a contradiction. Therefore, (1) must hold. □

We now want to modify Step 3 of [7, Chap. 4]. Recall that

$$\mathcal{E}^\nu(r) = \bigcup_{w_0\in\partial D^\nu}\{w \in D^\nu : |w - w_0| < r|w_0|\}$$

is a collar about ∂D^ν lying in D^ν whose closure does not contain the origin. Similarly,

$$\mathcal{E}_\nu(r) = (T^\nu)^{-1}(\mathcal{E}^\nu(r)) = \bigcup_{z_0\in\partial D_\nu}\{z \in D_\nu : |z - z_0| < r_0|z_0 - p_\nu|\}$$

is a collar about ∂D_ν lying in D_ν whose closure does not contain the point p_ν.

LEMMA 5.4. *There exist* $0 < r_0 < 1$, *a constant* $C > 0$, *and an integer* I *such that*

$$\left|\frac{\partial^2 g^\nu}{\partial y_i\partial y_j}(w)\right||\partial_w g^\nu(w)|^{-1} \le C|w|^{-1}, \quad w \in \mathcal{E}^\nu(r_0), \tag{5.12}$$

for all $\nu \ge I$.

Proof. By the relations

$$g^\nu(w) = \psi_\nu(p_\nu)^{2n-2}G_\nu(z, p_\nu) \quad \text{and} \quad z = p_\nu - \psi_\nu(p_\nu)w,$$

we observe that (5.12) is equivalent to

$$\left|\frac{\partial^2 G_\nu}{\partial x_i\partial x_j}(z, p_\nu)\right||\partial_z G_\nu(z, p_\nu)|^{-1} \le C|z - p_\nu|^{-1}, \quad z \in \mathcal{E}_\nu(r_0). \tag{5.13}$$

We shall prove (5.13) by contradiction. So suppose there do *not* exist $0 < r_0 < 1$, $C > 0$, and integer I such that (5.13) holds for all $\nu \ge I$. Then there exist a sequence $\{z_{0\nu}\}$ with $z_{0\nu} \in \partial D_\nu$ and also a sequence $\{z_\nu\}$ with

$$z_\nu \in D_\nu \ \text{ and } \ |z_\nu - z_{0\nu}| < \frac{1}{\nu}|z_{0\nu} - p_\nu|, \quad \nu \ge 1, \tag{5.14}$$

such that

$$\left|\frac{\partial^2 G_\nu}{\partial x_i\partial x_j}(z_\nu, p_\nu)\right||\partial_z G_\nu(z_\nu, p_\nu)|^{-1} \ge \nu|z_\nu - p_\nu|^{-1}, \quad \nu \ge 1. \tag{5.15}$$

By passing to a subsequence if necessary, we may assume that

$$\lim_{\nu\to\infty} z_{0\nu} = z_0 \in \partial D.$$

Then, by (5.14),

$$\lim_{\nu\to\infty} z_\nu = z_0.$$

Next we claim that $p_0 = z_0$. Suppose that this is not true. Then we can find an $\varepsilon > 0$ such that $B(p_0, 2\varepsilon) \cap B(z_0, \varepsilon) = \emptyset$. Taking ε sufficiently small and ν sufficiently large, by the implicit function theorem we can find a C^∞-smooth function ϕ on $B(x_0', \varepsilon)$ and a sequence $\{\phi_\nu\}$ of C^∞-smooth functions on $B(x_0', \varepsilon)$ that converges in C^∞-topology on compact subsets of $B(x_0', \varepsilon)$ to ϕ such that

$$\begin{aligned} B(z_0, \varepsilon) \cap \partial D &= \{(x', \phi(x')) : x' \in B(x_0', \varepsilon)\}, \\ B(z_0, \varepsilon) \cap \partial D_\nu &= \{(x', \phi_\nu(x')) : x' \in B(x_0', \varepsilon)\}. \end{aligned} \tag{5.16}$$

We assume without loss of generality that all p_ν lie in $B(p_0, \varepsilon)$. Then

$$G_\nu(z, p_\nu) \le |z - p_\nu|^{-2n+2} < \varepsilon^{-2n+2}, \quad z \in B(z_0, \varepsilon) \cap D_\nu. \tag{5.17}$$

Now consider the affine map

$$Z = Sz = \frac{z - z_0}{\varepsilon},$$

and set

$$\Omega = S(B(z_0, \varepsilon/2) \cap D), \qquad \Omega_\nu = S(B(z_0, \varepsilon/2) \cap D_\nu).$$

Define

$$h_\nu(Z) = \varepsilon^{2n-2} G(z, p_\nu), \quad Z \in \Omega_\nu.$$

Then h_ν is harmonic on Ω_ν; $h_\nu = 0$ on $B(0,1) \cap \partial\Omega_\nu$; and, by (5.17),

$$0 < h_\nu(Z) \le 1, \quad Z \in \Omega_\nu.$$

Therefore, by Harnack's principle (and after passing to a subsequence if necessary), $\{h_\nu\}$ converges uniformly on compact subsets of Ω to a positive harmonic function h. In view of (5.16), the sequence $\{h_\nu\}$ on $\{\Omega_\nu\}$ satisfies the hypothesis of [7, Prop. 5.1] and so

$$\begin{aligned} \lim_{\nu\to\infty} |\partial_Z h_\nu(Z_\nu)| &= |\partial_Z h(0)|, \\ \lim_{\nu\to\infty} \left| \frac{\partial^2 h_\nu}{\partial \tilde{X}_i \partial \tilde{X}_j}(Z_\nu) \right| &= \left| \frac{\partial^2 h}{\partial \tilde{X}_i \partial \tilde{X}_j}(0) \right| < \infty; \end{aligned} \tag{5.18}$$

here $Z_\nu = Sz_\nu$. By the Hopf lemma,

$$|\partial_Z h(0)| > 0.$$

Therefore,

$$\begin{aligned} \lim_{\nu\to\infty} \frac{|(\partial^2 G_\nu/\partial x_i \partial x_j)(z_\nu, p_\nu)|}{|\partial_z G_\nu(z_\nu, p_\nu)|} |z_\nu - p_\nu| &= \varepsilon \lim_{\nu\to\infty} \frac{|(\partial^2 h_\nu/\partial X_i \partial X_j)(Z_\nu)|}{|\partial_Z h_\nu(Z_\nu)|} |z_\nu - p_\nu| \\ &= \frac{|(\partial^2 h/\partial X_i \partial X_j)(0)|}{|\partial_Z h(0)|} |z_0 - p_0| < \infty, \end{aligned}$$

which contradicts (5.15). Hence we must have $p_0 = z_0$, and the claim follows.

Now we define

$$k_\nu = |p_\nu - z_{0\nu}|.$$

Consider the affine maps $S_\nu : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$\tilde{z} = S_\nu(z) = \frac{z - p_\nu}{k_\nu},$$

and let $\tilde{D}_\nu = S_\nu(D_\nu)$. A defining function for $\tilde{D}_\nu$ is given by

$$\begin{aligned}\psi_\nu \circ S_\nu^{-1}(\tilde{z}) &= \psi_\nu(p_\nu + k_\nu \tilde{z}) \\ &= \psi_\nu(p_\nu) + 2k_\nu \Re\left(\sum_{\alpha=1}^{n} (\psi_\nu)_\alpha(p_\nu)\tilde{z}_\alpha\right) + k_\nu^2 O(1)\end{aligned}$$

for $\tilde{z}$ on a compact subset of $\mathbf{C}^n$. Since $\{\psi_\nu\}$ converges in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to ψ, we note that $O(1)$ is independent of ν. Now

$$\tilde{\psi}_\nu(\tilde{z}) = \frac{\psi_\nu \circ S_\nu^{-1}(\tilde{z})}{k_\nu} = \frac{\psi_\nu(p_\nu)}{k_\nu} + 2\Re\left(\sum_{\alpha=1}^{n} (\psi_\nu)_\alpha(p_\nu)\tilde{z}_\alpha\right) + k_\nu O(1)$$

is again a defining function for $\tilde{D}_\nu$. Note that we can find a ball B, centered at p_0, as well as positive smooth functions ϕ_ν on B such that

$$-\psi_\nu(p) = \phi_\nu(p) d(p, \partial D_\nu), \quad p \in B.$$

Differentiating this relation shows that, for all large ν, the functions ϕ_ν are uniformly bounded above by a constant $c > 0$ on a possibly smaller ball B' that is also centered at p_0. This implies that, for all large ν,

$$\left|\frac{\psi_\nu(p_\nu)}{k_\nu}\right| \le \frac{c d_\nu(p_\nu, \partial D_\nu)}{|p_\nu - z_{0\nu}|} \le c;$$

therefore, after passing to a subsequence, $\{\psi_\nu(p_\nu)/k_\nu\}$ converges to a number $\tilde{c} \le 0$. Thus the functions $\tilde{\psi}_\nu$ converge in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to the function

$$\tilde{\psi}(\tilde{z}) = \tilde{c} + 2\Re\left(\sum_{\alpha=1}^{n} \psi_\alpha(p_0)\tilde{z}_\alpha\right).$$

This implies that the domains $\tilde{D}_\nu$ are C^∞-perturbation of the half-space

$$\tilde{H} = \left\{\tilde{z} \in \mathbf{C}^n : \tilde{c} + 2\Re\left(\sum_{\alpha=1}^{n} \psi_\alpha(p_0)\tilde{z}_\alpha\right) < 0\right\}.$$

Since $\tilde{c} \le 0$, it is evident that

$$0 \in \bar{\tilde{\mathcal{H}}}. \tag{5.19}$$

We will now derive a contradiction by proving that (5.19) is false. First, observe that $0 = S_\nu(p_\nu) \in \tilde{D}_\nu$. Let $\tilde{g}_\nu(\tilde{z})$ be the Green function for $\tilde{D}_\nu$ with pole at 0. Then

$$\tilde{g}_\nu(\tilde{z}) = G(z, p_\nu) k_\nu^{2n-2}. \tag{5.20}$$

Now let $\tilde{z}_{0\nu} = S_\nu(z_{0\nu})$. Then $\tilde{z}_{0\nu} \in \partial\tilde{D}_\nu$ and

$$|\tilde{z}_{0\nu}| = \left|\frac{z_{0\nu} - p_\nu}{k_\nu}\right| = 1.$$

Therefore, after passing to a subsequence, $\{\tilde{z}_{0\nu}\}$ converges to a point $\tilde{z}_0$ with

$$|\tilde{z}_0| = 1.$$

Evidently, $\tilde{z}_0 \in \partial\tilde{H}$. If we let $\tilde{z}_\nu = S_\nu(z_\nu)$, then

$$|\tilde{z}_\nu - \tilde{z}_{0\nu}| = \left|\frac{z_\nu - z_{0\nu}}{k_\nu}\right| < \frac{1}{\nu}$$

by (5.14). Therefore,

$$\lim_{\nu\to\infty} \tilde{z}_\nu = \tilde{z}_0.$$

Now we derive the contradiction by considering two cases as follows.

Case I: $0 \in \tilde{H}$. Let $\tilde{g}(\tilde{z})$ be the Green function for $\tilde{H}$ with pole at 0. Then, by Corollary 3.2,

$$\lim_{\nu\to\infty} |\partial_{\tilde{z}}\tilde{g}_\nu(\tilde{z}_\nu)| = |\partial_{\tilde{z}}\tilde{g}(\tilde{z}_0)| > 0,$$
$$\lim_{\nu\to\infty} \frac{\partial^2\tilde{g}_\nu}{\partial\tilde{x}_k\partial\tilde{x}_l}(\tilde{z}_\nu) = \frac{\partial^2\tilde{g}}{\partial\tilde{x}_k\partial\tilde{x}_l}(\tilde{z}_0) \neq \infty.$$

By (5.20), we have

$$\begin{aligned}\lim_{\nu\to\infty}&\left|\frac{\partial^2 G}{\partial x_i\partial x_j}(z_\nu, p_\nu)\right||\partial_z G(z_\nu, p_\nu)|^{-1}|z_\nu - p_\nu| \\ &= \lim_{\nu\to\infty}\left|\frac{\partial^2\tilde{g}_\nu}{\partial\tilde{x}_i\partial\tilde{x}_j}(\tilde{z}_\nu)\right||\partial_{\tilde{z}}\tilde{g}_\nu(\tilde{z}_\nu)|^{-1}|\tilde{z}_\nu| < \infty,\end{aligned}$$

which contradicts (5.15) and so $0 \notin \tilde{\mathcal{H}}$.

Case II: $0 \in \partial\tilde{H}$. By the implicit function theorem, we can find a ball $B(\tilde{z}_0, \varepsilon)$, a C^∞-smooth function ϕ on $B(\tilde{x}'_0, \varepsilon)$, and a sequence $\{\phi_\nu\}$ of C^∞-smooth functions on $B(\tilde{x}'_0, \varepsilon)$ that converges in the C^∞-topology on compact subsets of $B(\tilde{x}'_0, \varepsilon)$ to ϕ such that

$$\begin{aligned} B(\tilde{x}_0, \varepsilon) \cap \partial\tilde{\mathcal{H}} &= \{(\tilde{x}', \phi(\tilde{x}')) : \tilde{x}' \in B(\tilde{x}'_0, \varepsilon)\}, \\ B(\tilde{x}_0, \varepsilon) \cap \partial\tilde{D}_\nu &= \{(\tilde{x}', \phi_\nu(\tilde{x}')) : \tilde{x}' \in B(\tilde{x}'_0, \varepsilon)\}. \end{aligned} \tag{5.21}$$

We assume without loss of generality that $\varepsilon < 1/2$. Then, since $|\tilde{z}_0| = 1$,

$$g_\nu(\tilde{z}) < |\tilde{z}|^{-2n+2} < 2^{2n-2}, \quad \tilde{z} \in B(\tilde{z}_0, \varepsilon) \cap \tilde{D}_\nu. \tag{5.22}$$

Now consider the affine map

$$\tilde{Z} = S\tilde{z} = \frac{\tilde{z} - \tilde{z}_0}{\varepsilon},$$

and set

$$\Omega = S(B(\tilde{z}_0, \varepsilon) \cap \tilde{H}), \qquad \Omega_\nu = S(B(\tilde{z}_0, \varepsilon) \cap \tilde{D}_\nu).$$

Define

$$h(\tilde{Z}) = 2^{-2n+2} g(\tilde{z}), \quad \tilde{Z} \in \Omega, \tag{5.23}$$

$$h_\nu(\tilde{Z}) = 2^{-2n+2} g_\nu(\tilde{z}), \quad \tilde{Z} \in \Omega_\nu. \tag{5.24}$$

Then h_ν is a positive harmonic funtion on Ω_ν and satisfies $h_\nu = 0$ on $B(0,1)\cap\partial\Omega_\nu$. Moreover, by (5.22),

$$0 < h_\nu(\tilde{Z}) < 1, \quad \tilde{Z} \in \Omega_\nu.$$

After passing to a subsequence if necessary, it follows from Harnack's principle that $\{h_\nu\}$ converges uniformly on compact subsets of Ω to a positive harmonic function h satisfying $h = 0$ on $B(0,1)\cap\partial\Omega$. In view of (5.21), the sequence $\{h_\nu\}$ satisfies the hypothesis of [7, Prop. 5.1]; therefore, by (5.20) and (5.24),

$$\begin{aligned}\lim_{\nu\to\infty}&\left|\frac{\partial^2 G}{\partial x_i \partial x_j}(z_\nu, p_\nu)\right| |\partial_z G(z_\nu, p_\nu)|^{-1}|z_\nu - p_\nu| \\ &= \varepsilon \lim_{\nu\to\infty}\left|\frac{\partial^2 h_\nu}{\partial \tilde{X}_i \partial \tilde{X}_j}(\tilde{Z}_\nu)\right| |\partial_{\tilde{Z}} h_\nu(\tilde{Z}_\nu)|^{-1}|\tilde{z}_\nu| = \varepsilon\left|\frac{\partial^2 h}{\partial \tilde{X}_i \partial \tilde{X}_j}(0)\right| |\partial_{\tilde{Z}} h(0)|^{-1},\end{aligned}$$

where $\tilde{Z}_\nu = S\tilde{z}_\nu$. Now, by the reflection principle, h extends as a harmonic function to a neighborhood of 0; hence the quantity on the extreme right of the previously displayed equation is finite. This contradicts (5.15), so it follows that $0 \notin \partial\tilde{\mathcal{H}}$.

By Case I and Case II we have $0 \notin \bar{\tilde{\mathcal{H}}}$, which contradicts (5.19). Thus (5.13) holds, completing the proof of Lemma 5.4. □

Recall that if $r > 0$ and I are as in Lemma 2.4, then the function $F^\nu(w)$ is defined and smooth on the collar $\mathcal{E}^\nu(r)$.

PROPOSITION 5.5. *There exist* $0 < r < 1$, *a constant* $C > 0$, *and an integer* I *such that*

(1) $|F^\nu(w)| < C(1 + |w|^{-1})|w|^{-2n+3}$,
(2) $|(\partial F^\nu/\partial y_i)(w)| < C(1 + |w|^{-1})|w|^{-2n+2}$, *and*
(3) $|(\partial^2 F^\nu/\partial y_i \partial y_j)(w)| < C(1 + |w|^{-1})|w|^{-2n+1}$

for all $\nu \geq I$ *and* $w \in \mathcal{E}^\nu(r)$.

Proof. Choose $m > 0$, $0 < r < 1$, and I as in Lemma 2.4. Choose $M > 0$ as in Lemma 2.5. Modify I and choose a constant C so that Proposition 5.3 holds. Modify r and I so that Lemma 5.4 holds. Now fix $\nu \geq I$.

(1) Let $w \in \mathcal{E}^\nu(r)$, $|w| > 1$. Then, by Lemma 2.4, Lemma 2.5, and Proposition 5.3,

$$\begin{aligned}|F^\nu(w)| &= \frac{|(\partial f_\nu/\partial p_\gamma)(p_\nu, w)|}{|\partial_w f_\nu(p_\nu, w)|}|\partial_w g^\nu(w)| \\ &\leq \frac{M(1+|w|^{-1})|w|^2}{m} C|w|^{-2n+1} = C_2(1+|w|^{-1})|w|^{-2n+3},\end{aligned}$$

where $C_1 = MC/m$ is independent of ν and w.

(2) Differentiating $F^\nu(w)$ with respect to y_i yields

$$\begin{aligned}\frac{\partial F^\nu}{\partial y_i} = \frac{-\partial^2 f_\nu/\partial p_\gamma \partial y_i}{|\partial_w f_\nu|}|\partial_w g^\nu| &+ \frac{1}{4}\frac{\partial f_\nu}{\partial p_\gamma}\frac{\sum_{k=1}^{2n}(\partial f_\nu/\partial y_k)(\partial^2 f_\nu/\partial y_k \partial y_i)}{|\partial_w f_\nu|^3}|\partial_w g^\nu| \\ &- \frac{1}{4}\frac{\partial f_\nu}{\partial p_\gamma}\frac{1}{|\partial_w f_\nu|}\frac{\sum_{k=1}^{2n}(\partial g_\nu/\partial y_k)(\partial^2 g_\nu/\partial y_k \partial y_i)}{|\partial_w g^\nu|}.\end{aligned} \tag{5.25}$$

Thus, for $w \in \mathcal{E}^\nu(r)$ with $|w| > 1$, it follows from Lemma 2.4, Lemma 2.5, Proposition 5.3, and the relation

$$\frac{\partial g^\nu/\partial y_k}{|\partial_w g^\nu|} \leq 2$$

that

$$\begin{aligned}
\left|\frac{\partial F^\nu}{\partial y_i}(w)\right| &\leq \frac{M(1+|w|^{-1})|w|}{m} C|w|^{-2n+1} \\
&\quad + \frac{1}{4}M(1+|w|^{-1})|w|^2 \frac{2nMM|w|^{-1}}{m^3} C|w|^{-2n+1} \\
&\quad + \frac{1}{4}M(1+|w|^{-1})|w|^2 \frac{1}{m} 2n2C|w|^{-2n} \\
&\leq C_2(1+|w|^{-1})|w|^{-2n+2}.
\end{aligned}$$

(3) In order to prove this estimate, we differentiate (5.25) with respect to y_j and then estimate as before. All terms, except for those of the form

$$\frac{\partial f_\nu/\partial p_\gamma}{|\partial_w f_\nu|} \frac{(\partial^2 g_\nu/\partial y_k \partial y_i)(\partial^2 g_\nu/\partial y_l \partial y_i)}{|\partial_w g_\nu|}$$

or

$$\frac{\partial f_\nu/\partial p_\nu}{|\partial_w f_\nu|} \frac{(\partial g_\nu/\partial y_k)(\partial g_\nu/\partial y_l)(\partial^2 g_\nu/\partial y_k \partial y_i)(\partial^2 g_\nu/\partial y_l \partial y_j)}{|\partial_w g_\nu|^3},$$

are bounded by a constant times $(1+|w|^{-1})|w|^{-2n+1}$ for $w \in \mathcal{E}^\nu(r)$. Also by Lemma 5.4, the above terms are bounded by a constant times $(1+|w|^{-1})|w|^{-2n+1}$ for $w \in \mathcal{E}^\nu(r_0)$. □

We now modify Steps 4 and 5 of [7, Chap. 5] to derive an upper bound for $(\partial^2 g_\nu/\partial \bar{w}_\alpha \partial p_\gamma)(p_\nu, w)$.

Proposition 5.6. *There exist $0 < r < 1$ and an integer I such that, for $\nu \geq I$ and $w_0 \in \partial D^\nu$, we can find a function $F^*(w)$ (depending on the parameters ν and w_0) of class C^2 on*

$$E = \{w \in D^\nu : |w - w_0| < r|w_0|\}$$

such that

$$H_E F^*(w) = \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w), \quad w \in E.$$

Moreover, there exists a constant $C > 0$, independent of ν, and a $w_0 \in \partial D^\nu$ such that:

(1) $|F^*(w)| < C(1+|w_0|^{-1})|w_0|^{-2n+3}$ *in E;*
(2) $|(\partial F^*/\partial y_i)(w_0)| < C(1+|w_0|^{-1})|w_0|^{-2n+2}$, $i = 1, \dots, n$;
(3) $|\Delta_w F^*(w)| < C(1+|w_0|^{-1})|w_0|^{-2n+1}$ *in E.*

Proof. Choose $0 < r < 1$, a constant C, and an integer I as in Proposition 5.5. Now fix $\nu \geq I$ and $w_0 \in \partial D^\nu$ and let

$$B = \{w : |w - w_0| < r|w_0|\}.$$

Then $E = B \cap D^\nu$. Since

$$\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) = H_{D^\nu} F^\nu(w)$$

on D^ν, the function $(\partial g_\nu/\partial p_\gamma)(p_\nu, w)$ is harmonic on E with boundary values

$$\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) = \begin{cases} F^\nu(w) & \text{if } w \in B \cap \partial D^\nu, \\ H_{D^\nu} F^\nu(w) & \text{if } w \in \partial B \cap D^\nu. \end{cases} \tag{5.26}$$

Let u be the harmonic function on E with boundary values

$$u(w) = \begin{cases} 0 & \text{if } w \in B \cap \partial D^\nu, \\ H_{D^\nu} F^\nu - F^\nu(w) & \text{if } w \in \partial B \cap D^\nu, \end{cases}$$

and set

$$F^*(w) = F^\nu(w) + u(w), \quad w \in E.$$

Then

$$H_E F^* = H_E F^\nu + u$$

is a harmonic function on E with boundary values (5.26), whence

$$H_E F^*(w) = \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)$$

on E. This proves the first part of the proposition.

We prove the second part by observing that Proposition 5.5, together with the continuity of the function

$$|F^\nu(w)|(1 + |w|^{-1})^{-1}|w|^{2n-3}$$

up to $\bar{\mathcal{E}}^\nu(r)$, implies that

$$|F^\nu(w)| \le C(1 + |w|^{-1})|w|^{-2n+3}, \quad w \in \bar{\mathcal{E}}^\nu(r).$$

In particular, this expression holds for $w \in \bar{E}$. Also, since $(1 + |w|^{-1})|w|^{-2n+3}$ is superharmonic on $\mathbf{C}^n$, it also implies that

$$|H_{D^\nu} F^\nu(w)| \le C(1 + |w|^{-1})|w|^{-2n+3}, \quad w \in \bar{D}^\nu. \tag{5.27}$$

Therefore,

$$|H_{D^\nu} F^\nu(w) - F^\nu(w)| \le 2C(1 + |w|^{-1})|w|^{-2n+3}, \quad w \in \partial B \cap D^\nu,$$

which implies that

$$|u(w)| \le 2C(1 + |w|^{-1})|w|^{-2n+3}, \quad w \in \bar{E}.$$

Since $E \subset \{w : |w - w_0| < r|w_0|\}$, it follows that

$$\begin{aligned} |F^*(w)| \le |F_\nu(w)| + |u(w)| &\le 3C(1 + |w|^{-1})|w|^{-2n+3} \\ &\le 3C(1-r)^{-2n+2}(1 + |w_0|^{-1})|w_0|^{-2n+3}, \quad w \in E. \end{aligned}$$

This proves (1).

To prove (2), observe that the preceding calculation yields

$$|u(w)| \le 2C(1-r)^{-2n+2}(1 + |w_0|^{-1})|w_0|^{-2n+3}, \quad w \in E.$$

Also, $u(w) = 0$ for $w \in B \cap \partial D^\nu$. By Lemma 4.1, we can modify the integer I if necessary to find a $\rho > 0$, independent of ν and w_0, such that there exists a ball of radius $\rho|w_0|$ that is externally tangent to ∂D^ν at w_0. Hence, by taking $R = \min(\rho|w_0|, r|w_0|)$ in Step 2 of [7, Chap. 4], we can find a constant c independent of D^ν and u such that

$$
\begin{aligned}
|\partial_w u(w_0)| &< \frac{2cC(1-r)^{-2n+2}(1+|w_0|^{-1})|w_0|^{-2n+3}}{\min(r|w_0|, \rho|w_0|)} \\
&= \tilde{C}(1+|w_0|^{-1})|w_0|^{-2n+2};
\end{aligned}
$$

here $\tilde{C}$ is independent of ν, and $w_0 \in \partial D^\nu$. This, together with Proposition 5.5, implies that

$$
\left|\frac{\partial F^*}{\partial y_i}(w_0)\right| \le \left|\frac{\partial F_\nu}{\partial y_i}(w_0)\right| + \left|\frac{\partial u}{\partial y_i}(w_0)\right| \le (C+\tilde{C})(1+|w_0|^{-1})|w_0|^{-2n+2},
$$

which proves (2).

Finally, since u is harmonic we obtain from Proposition 5.5 that

$$
\begin{aligned}
|\Delta_w F^*(w)| = |\Delta_w F^\nu(w)| &\le nC(1+|w|^{-1})|w|^{-2n+1} \\
&\le nC(1-r)^{-2n}(1+|w_0|^{-1})|w_0|^{-2n+1}, \quad w \in E,
\end{aligned}
$$

and this proves (3). $\square$

PROPOSITION 5.7. *There exist a constant $C > 0$ and an integer I such that*

$$
\left|\frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p_\nu, w)\right| < C(1+|w|^{-1})|w|^{-2n+2} \tag{5.28}
$$

for all $\nu \ge I$ and $w \in \bar{D}^\nu$.

Proof. Let $0 < r < 1$, $C > 0$, and I be as in Proposition 5.6, and fix $\nu \ge I$. By the maximum principle, it suffices to prove (5.28) for $w_0 \in \partial D^\nu$. Given such w_0, we let F^* be a C^2-smooth function on

$$
E = \{w \in D^\nu : |w - w_0| < r|w_0|\}
$$

satisfying the estimates of Proposition 5.6. Now consider the affine map

$$
W = S(w) = \frac{w - w_0}{r|w_0|},
$$

and let $\Omega = S(E)$. Define the functions u and h on Ω by setting

$$
u(W) = \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) \quad \text{and} \quad h(W) = F^*(w).
$$

Then $u = H_\Omega h$ on Ω and, by Proposition 5.6:

(1) $|h(W)| < C(1+|w_0|^{-1})|w_0|^{-2n+3}$ in Ω;

(2) $\displaystyle \left|\frac{\partial h}{\partial Y_i}(0)\right| = \left|\frac{\partial F^*}{\partial y_i}(w_0)\right| r|w_0| < Cr(1+|w_0|^{-1})|w_0|^{-2n+3}$;

(3) $|\Delta_W h(W)| = |\Delta_w F^*(w)| r^2 |w_0|^2 \leq Cr^2(1+|w_0|^{-1})|w_0|^{-2n+3} \leq Cr(1+|w_0|^{-1})|w_0|^{-2n+3}$ in Ω.

By Lemma 4.1, we can modify the integer I to find a $\rho > 0$ that is independent of ν and w_0 such that there exists a ball B of radius $\rho|w_0|$ that is externally tangent to ∂D^ν at w_0. Setting $T(B) = \tilde{B}$, we see that the ball $\tilde{B} \subset \mathbf{C}^n \setminus \Omega$ has radius ρ/r and is tangent to $\partial\Omega$ at 0. Let $\tilde{B}_2$ be the ball with the same center as $\tilde{B}$ but with radius $\rho/r + 2$. Hence, by [7, Lemma 5.1′, p. 60], there exists a constant M depending only on ρ/r such that

$$|\partial_{\overline{W}} u(0)| \leq MC(1+|w_0|^{-1})|w_0|^{-2n+3}.$$

Since

$$\frac{\partial u}{\partial \overline{W}_\alpha}(0) = \frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p, w_0) r |w_0|$$

we have

$$\left| \frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p_\nu, w_0) \right| \leq \frac{MC}{r}(1+|w_0|^{-1})|w_0|^{-2n+2},$$

which proves the proposition. □

PROPOSITION 5.8. *Let $w^\nu \in \partial D^\nu$ be such that $\{w^\nu\}$ converges to $w^0 \in \partial\mathcal{H} = \partial D(p_0)$. Then*

$$\lim_{\nu\to\infty} \frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p_\nu, w^\nu) = \frac{\partial^2 g}{\partial \bar{w}_\alpha \partial p_\gamma}(p_0, w^0).$$

Proof. This follows from standard boundary elliptic regularity arguments and from the fact that D^ν is C^∞-close to D. □

PROPOSITION 5.9.

$$\lim_{\nu\to\infty} \frac{\partial^2 \lambda_\nu}{\partial p_\gamma \partial \bar{p}_\gamma}(p_\nu) = \frac{\partial^2 \lambda}{\partial p_\gamma \partial \bar{p}_\gamma}(p_0).$$

Proof. By Proposition 2.3 and (5.3), we need only prove that

$$\lim_{\nu\to\infty} \int_{\partial D^\nu} k_1^{\nu\gamma}(w) \frac{(\partial g^\nu/\partial \bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w) \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w$$
$$= \int_{\partial\mathcal{H}} k_1^\gamma(p_0, w) \frac{(\partial g/\partial \bar{w}_\alpha)(p_0, w)}{|\partial_w g(p_0, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p_0, w) \frac{\partial g}{\partial n_w}(p_0, w)\, dS_w. \quad (5.29)$$

Let $R > 1$. Then, by Proposition 5.8 and the arguments in the proof of Proposition 4.4, we have

$$\lim_{\nu\to\infty} \int_{B(0,R)\cap\partial D^\nu} k_1^{\nu\gamma}(w) \frac{(\partial g^\nu/\partial \bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w) \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w$$
$$= \int_{B(0,R)\cap\partial\mathcal{H}} k_1^\gamma(p_0, w) \frac{(\partial g/\partial \bar{w}_\alpha)(p_0, w)}{|\partial_w g(p_0, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p_0, w) \frac{\partial g}{\partial n_w}(p_0, w)\, dS_w. \quad (5.30)$$

To estimate these integrals outside $B(0, R)$ we note that, by Corollary 2.6, there exist a constant C and an integer I such that

$$|k_1^{\nu\gamma}(w)| \leq C|w|^2, \quad w \in \partial D^\nu, \ |w| > 1,$$

for $\nu \geq I$. In view of Proposition 5.7, we can modify C and I so that

$$\left|\frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p_\nu, w)\right| \leq C|w|^{-2n+2}, \quad w \in \partial D^\nu, \ |w| > 1,$$

for $\nu \geq I$. Therefore,

$$\left|\int_{B^c(0,R)\cap\partial D^\nu} k_1^\nu(w) \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w) \frac{(\partial g^\nu/\partial \bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|} \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w\right| \\ \leq C^2 R^{-2n+4} \int_{B^c(0,R)\cap\partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w)\right) dS_w \quad (5.31)$$

for $\nu \geq I$. Again we have

$$\int_{B^c(0,R)\cap\partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w)\right) dS_w \leq \int_{\partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w)\right) dS_w = 2(n-1)\sigma_{2n}$$

and hence, by (5.31),

$$\left|\int_{B^c(0,R)\cap\partial D^\nu} k_1^\nu(w) \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w) \frac{(\partial g^\nu/\partial \bar{w}_\alpha)(w)}{|\partial_w g^\nu(w)|} \frac{\partial g^\nu}{\partial n_w}(w)\, dS_w\right| \\ = O(R^{-2n+4}) \quad (5.32)$$

uniformly for all $\nu \geq I$. Also, by (2.7) we can modify the constant C so that

$$|k_1^\gamma(p_0, w)| \leq C|w|^2 \quad \text{and} \quad \left|\frac{\partial^2 g}{\partial \bar{w}_\alpha \partial p_\gamma}(p_0, w)\right| \leq C|w|^{-2n+2}$$

for $w \in \partial\mathcal{H}$ with $|w| > 1$. As before, we obtain

$$\left|\int_{B^c(0,R)\cap\partial\mathcal{H}} k_1^\gamma(p_0, w) \frac{(\partial g/\partial \bar{w}_\alpha)(p_0, w)}{|\partial_w g(p_0, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p_0, w) \frac{\partial g}{\partial n_w}(p_0, w)\, dS_w\right| \\ = O(R^{-2n+4}). \quad (5.33)$$

From (5.30), (5.32), and (5.33) it now follows that (5.29) holds. □

Proof of Theorem 1.4. Given Proposition 4.4, we only need to prove that

$$\lim_{\nu\to\infty} \frac{\partial^2 \lambda_\nu}{\partial p_\alpha \partial \bar{p}_\beta}(p_\nu) = \frac{\partial^2 \lambda}{\partial p_\alpha \partial \bar{p}_\beta}(p_0).$$

However, this equality follows from Proposition 5.9 by a unitary change of coordinates. □

6. Holomorphic Sectional Curvature

In this section we prove Theorem 1.1 under the normalization described in the Introduction; that is, we will compute the right-hand side of equation (1.1). To avoid confusion, we first recall the following notation.

(a) $\{D_\nu\}$ is sequence of smoothly bounded strongly pseudoconvex domains such that $0 \in D_\nu$ for each $\nu \geq 1$ and the normal to ∂D_ν at 0 is along the $\Re z_n$-axis.
(b) $\{D_\nu\}$ converges in the C^∞-topology to a C^∞-smoothly bounded strongly pseudoconvex domain D; thus $0 \in \partial D$, and the normal to ∂D at 0 is along the $\Re z_n$-axis.
(c) $p_\nu = (0, \ldots, 0, \delta_\nu) \in D_\nu$, where $\delta_\nu = d(p_\nu, \partial D_\nu)$ and $p_\nu \to p_0 = 0 \in \partial D$.
(d) $(g_\nu)_{\alpha\beta}$ and $g_{\alpha\beta}$ are the components of the Λ-metrics on D_ν and D, respectively.
(e) ψ_ν and ψ are C^∞-smooth defining functions for D_ν and D, respectively, such that $\{\psi_\nu\}$ converges in the C^∞-topology on compact subsets of $\mathbf{C}^n$ to ψ; we further assume that $\partial\psi_\nu(0) = \partial\psi(0) = (0, \ldots, 0, 1)$.

LEMMA 6.1. *We have*

(i) $\lim_{\nu\to\infty} (g_\nu)_{\alpha\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^2 = (2n-2)\psi_\alpha(0)\psi_{\bar\beta}(0)$,
(ii) $\lim_{\nu\to\infty} (\partial(g_\nu)_{\alpha\bar\beta}/\partial z_\gamma)(p_\nu)(\psi_\nu(p_\nu))^3 = -2(2n-2)\psi_\alpha(0)\psi_{\bar\beta}(0)\psi_\gamma(0)$, *and*
(iii) $\lim_{\nu\to\infty} (\partial^2(g_\nu)_{\alpha\bar\beta}/\partial z_\gamma \partial z_{\bar\delta})(p_\nu)(\psi_\nu(p_\nu))^4 = 6(2n-2)\psi_\alpha(0)\psi_{\bar\beta}(0)\psi_\delta(0)$.

Proof. Let $\mathcal{H}$ be the half-space

$$\mathcal{H} = \left\{ z \in \mathbf{C}^n : 2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(0) z_\alpha\right) - 1 < 0 \right\} = \{z \in \mathbf{C}^n : 2\Re z_n - 1 < 0\}.$$

From [1, (1.4)], the Robin function for $\mathcal{H}$ is given by

$$\begin{aligned}\Lambda_{\mathcal{H}}(z) &= -\left(\frac{|\partial\psi(0)|}{2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(0) z_\alpha\right) - 1}\right)^{2n-2} \\ &= -\left(2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(0) z_\alpha\right) - 1\right)^{-2n+2}\end{aligned}$$

so that

- $\Lambda_{\mathcal{H}}(0) = -1$,
- $(\Lambda_{\mathcal{H}})_a(0) = -(2n-2)\psi_a(0)$,
- $(\Lambda_{\mathcal{H}})_{ab}(0) = -(2n-2)(2n-1)\psi_a(0)\psi_b(0)$,
- $(\Lambda_{\mathcal{H}})_{abc}(0) = -(2n-2)(2n-1)(2n)\psi_a(0)\psi_b(0)\psi_c(0)$, and
- $(\Lambda_{\mathcal{H}})_{abcd}(0) = -(2n-2)(2n-1)(2n)(2n+1)\psi_a(0)\psi_b(0)\psi_c(0)\psi_d(0)$;

here the indices a, b, c, d refer to either holomorphic or conjugate holomorphic derivatives. Hence, by Theorem 1.2, we have

- $\Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2} \to -1$,
- $\Lambda_{\nu a}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} \to (2n-2)\psi_a(0)$,
- $\Lambda_{\nu ab}(p_\nu)(\psi_\nu(p_\nu))^{2n} \to -(2n-2)(2n-1)\psi_a(0)\psi_b(0)$,
- $\Lambda_{\nu abc}(p_\nu)(\psi_\nu(p_\nu))^{2n+1} \to (2n-2)(2n-1)(2n)\psi_a(0)\psi_b(0)\psi_c(0)$, and
- $\Lambda_{\nu abcd}(p_\nu)(\psi_\nu(p_\nu))^{2n+2} \to$ $-(2n-2)(2n-1)(2n)(2n+1)\psi_a(0)\psi_b(0)\psi_c(0)\psi_d(0)$.

Now

$$g_{\alpha\bar\beta} = \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar z_\beta} = \frac{\Lambda_{\alpha\bar\beta}}{\Lambda} - \frac{\Lambda_\alpha \Lambda_{\bar\beta}}{\Lambda^2}. \tag{6.1}$$

Multiplying both sides of this equation by ψ^2 yields

$$g_{\alpha\bar\beta}\psi^2 = \frac{\Lambda_{\alpha\bar\beta}\psi^{2n}}{\Lambda\psi^{2n-2}} - \frac{(\Lambda_\alpha\psi^{2n-1})(\Lambda_{\bar\beta}\psi^{2n-1})}{(\Lambda\psi^{2n-2})^2}.$$

It follows that

$$\lim_{\nu\to\infty} g_{\nu\alpha\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^2 = (2n-2)\psi_\alpha(0)\psi_{\bar\beta}(0),$$

which is (i).

Differentiating (6.1) with respect to z_γ, we obtain

$$\frac{\partial g_{\alpha\bar\beta}}{\partial z_\gamma} = \frac{\Lambda_{\alpha\bar\beta\gamma}}{\Lambda} - \left(\frac{\Lambda_{\alpha\bar\beta}\Lambda_\gamma}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma}\Lambda_{\bar\beta}}{\Lambda^2} + \frac{\Lambda_{\bar\beta\gamma}\Lambda_\alpha}{\Lambda^2}\right) + \frac{2\Lambda_\alpha\Lambda_{\bar\beta}\Lambda_\gamma}{\Lambda^3}. \tag{6.2}$$

Multiplying both sides of this equation by ψ^3, we get

$$\begin{aligned}
&\frac{\partial g_{\alpha\bar\beta}}{\partial z_\gamma}\psi^3 \\
&= \frac{\Lambda_{\alpha\bar\beta\gamma}\psi^{2n+1}}{\Lambda\psi^{2n-2}} \\
&\quad - \left(\frac{(\Lambda_{\alpha\bar\beta}\psi^{2n})(\Lambda_\gamma\psi^{2n-1})}{(\Lambda\psi^{2n-2})^2} + \frac{(\Lambda_{\alpha\gamma}\psi^{2n})(\Lambda_{\bar\beta}\psi^{2n-1})}{(\Lambda\psi^{2n-2})^2}\right. \\
&\qquad \left. + \frac{(\Lambda_{\bar\beta\gamma}\psi^{2n})(\Lambda_\alpha\psi^{2n-1})}{(\Lambda\psi^{2n-2})^2}\right) \\
&\quad + \frac{2(\Lambda_\alpha\psi^{2n-1})(\Lambda_{\bar\beta}\psi^{2n-1})(\Lambda_\gamma\psi^{2n-1})}{(\Lambda\psi^{2n-2})^3}.
\end{aligned}$$

Hence

$$\lim_{\nu\to\infty} \frac{\partial g_{\nu\alpha\bar\beta}}{\partial z_\gamma}(p_\nu)\psi_\nu(p_\nu)^3 = -2(2n-2)\psi_\alpha(0)\psi_{\bar\beta}(p)\psi_\gamma(0),$$

which is (ii).

Differentiating (6.2) with respect to $\bar z_\delta$ yields

$$\begin{aligned}
\frac{\partial^2 g_{\alpha\bar\beta}}{\partial z_\gamma \partial \bar z_{\bar\delta}} &= \frac{\Lambda_{\alpha\bar\beta\gamma\bar\delta}}{\Lambda} - \left(\frac{\Lambda_{\alpha\bar\beta\gamma}\Lambda_{\bar\delta}}{\Lambda^2} + \frac{\Lambda_{\alpha\bar\beta\bar\delta}\Lambda_\gamma}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma\bar\delta}\Lambda_{\bar\beta}}{\Lambda^2} + \frac{\Lambda_{\bar\beta\gamma\bar\delta}\Lambda_\alpha}{\Lambda^2}\right) \\
&\quad - \left(\frac{\Lambda_{\alpha\bar\beta}\Lambda_{\gamma\bar\delta}}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma}\Lambda_{\bar\beta\bar\delta}}{\Lambda^2} + \frac{\Lambda_{\alpha\bar\delta}\Lambda_{\bar\beta\gamma}}{\Lambda^2}\right) \\
&\quad + 2\left(\frac{\Lambda_{\alpha\bar\beta}\Lambda_\gamma\Lambda_{\bar\delta}}{\Lambda^3} + \frac{\Lambda_{\alpha\gamma}\Lambda_{\bar\beta}\Lambda_{\bar\delta}}{\Lambda^3} + \frac{\Lambda_{\bar\beta\gamma}\Lambda_\alpha\Lambda_{\bar\delta}}{\Lambda^3} + \frac{\Lambda_{\alpha\bar\delta}\Lambda_{\bar\beta}\Lambda_\gamma}{\Lambda^3}\right. \\
&\qquad \left. + \frac{\Lambda_{\bar\beta\bar\delta}\Lambda_\alpha\Lambda_\gamma}{\Lambda^3} + \frac{\Lambda_{\gamma\bar\delta}\Lambda_\alpha\Lambda_{\bar\beta}}{\Lambda^3}\right) - \frac{6\Lambda_\alpha\Lambda_{\bar\beta}\Lambda_\gamma\Lambda_{\bar\delta}}{\Lambda^4}.
\end{aligned}$$

If we multiply both sides by ψ^4, then this equation can be written in a form where Λ is multiplied by ψ^{2n-2} and the first-, second-, third-, and fourth-order derivatives of Λ are multiplied by ψ^{2n-1}, ψ^{2n}, ψ^{2n+1}, and ψ^{2n+2}, respectively. It follows that

$$\lim_{\nu\to\infty} \frac{\partial^2 g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma \partial \bar{z}_{\bar{\delta}}}(p_\nu)(\psi_\nu(p_\nu))^4 = 6(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)\psi_\gamma(0)\psi_{\bar{\delta}}(0),$$

which is (iii). □

To obtain finer asymptotics of the derivatives of Λ_ν along $\{p_\nu\}$, we need the following lemma.

LEMMA 6.2. *Let* $1 \le \alpha \le n-1$. *Then*

$$\lim_{\nu\to\infty} \frac{(\psi_\nu)_\alpha(p_\nu)}{\psi_\nu(p_\nu)} = \frac{1}{2}(\psi_{\alpha n}(0) + \psi_{\alpha\bar{n}}(0)).$$

Proof. Fix a ν and define the function f on $[0,1]$ by

$$f(t) = \psi_\nu(tp_\nu) = \psi_\nu(0,\dots,0,-\delta_\nu t). \tag{6.3}$$

From Taylor's theorem it follows that

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(s)$$

for some $s \in (0,1)$. Therefore, by successive application of the chain rule to (6.3), we obtain

$$\begin{aligned}\psi_\nu(p_\nu) = &-\delta_\nu((\psi_\nu)_n(0) + (\psi_\nu)_{\bar{n}}(0))\\ &+ \frac{\delta_\nu^2}{2}((\psi_\nu)_{nn}(\zeta_\nu) + 2(\psi_\nu)_{n\bar{n}}(\zeta_\nu) + (\psi_\nu)_{\bar{n}\bar{n}}(\zeta_\nu)),\end{aligned} \tag{6.4}$$

where $\zeta_\nu = sp_\nu$.

Now fix $1 \le \alpha \le n-1$ and define the function g on $[0,1]$ by

$$g(t) = (\psi_\nu)_\alpha(tp_\nu) = (\psi_\nu)_\alpha(0,\dots,0,-\delta_\nu t). \tag{6.5}$$

By Taylor's theorem, we have

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(s)$$

for some $s' \in (0,1)$. Therefore, by successive application of the chain rule to (6.5),

$$\begin{aligned}(\psi_\nu)_\alpha(p_\nu) = &-\delta_\nu((\psi_\nu)_{\alpha n}(0) + (\psi_\nu)_{\alpha\bar{n}}(0))\\ &+ \frac{\delta_\nu^2}{2}((\psi_\nu)_{\alpha nn}(\eta_\nu) + 2(\psi_\nu)_{\alpha n\bar{n}}(\eta_\nu) + (\psi_\nu)_{\alpha\bar{n}\bar{n}}(\eta_\nu)),\end{aligned} \tag{6.6}$$

where $\eta_\nu = s'p_\nu$. It is now evident from (6.4) and (6.6) that

$$\lim_{\nu\to\infty} \frac{(\psi_\nu)_\alpha(p_\nu)}{\psi_\nu(p_\nu)} = \frac{1}{2}(\psi_{\alpha n}(0) + \psi_{\alpha\bar{n}}(0)),$$

so the lemma is proved. □

Using Lemma 6.2 and Theorem 1.3, we obtain the following finer asymptotics of the first- and second-order derivatives of Λ_ν along $\{p_\nu\}$.

LEMMA 6.3. *Let* $1 \le \alpha \le n-1$ *and* $1 \le \beta \le n$. *Then*:

(i) $\lim_{\nu\to\infty} \Lambda_{\nu\alpha}(p_\nu)(\psi_\nu(p_\nu))^{2n-2} = \lambda_\alpha(0) + (2n-2)C_\alpha$;

(ii) $\lim_{\nu\to\infty} \Lambda_{\alpha\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} =$
$-(2n-2)\lambda_\alpha(0)\psi_{\bar\beta}(0) - (2n-2)(2n-1)\psi_{\bar\beta}(0)C_\alpha + (2n-2)\psi_{\alpha\bar\beta}(0)$,

where $C_\alpha = \frac{1}{2}(\psi_{\alpha n}(0) + \psi_{\alpha\bar n}(0))$.

Proof. The normalized Robin function

$$\lambda(z) = \begin{cases} \Lambda(z)(\psi(z))^{2n-2} & \text{if } z \in D, \\ -|\partial\psi(z)|^{2n-2} & \text{if } z \in \partial D \end{cases} \tag{6.7}$$

associated to (D, ψ) is C^2 on $\bar D$. In particular, $\lambda(0) = -1$. Differentiating λ with respect to z_α, we obtain

$$\Lambda_\alpha \psi^{2n-2} = \lambda_\alpha - (2n-2)\lambda\psi^{-1}\psi_\alpha.$$

Hence, by Theorems 1.2 and 1.3 and Lemma 6.2,

$$\lim_{\nu\to\infty} \Lambda_{\nu\alpha}(p_\nu)(\psi_\nu(p_\nu))^{2n-2} = \lambda_\alpha(0) + (2n-2)C_\alpha,$$

which is (i). Similarly, differentiating (6.7) with respect to z_α followed by $\bar z_\beta$ yields

$$\begin{aligned}\Lambda_{\alpha\bar\beta}\psi^{2n-1} &= \lambda_{\alpha\bar\beta}\psi - (2n-2)(\lambda_\alpha\psi_{\bar\beta} + \lambda_{\bar\beta}\psi_\alpha) \\ &\quad + (2n-2)(2n-1)\lambda\psi^{-1}\psi_\alpha\psi_{\bar\beta} - (2n-2)\lambda\psi_{\alpha\bar\beta}.\end{aligned}$$

Again by Theorems 1.2 and 1.3 and Lemma 6.2, we have

$$\begin{aligned}\lim_{\nu\to\infty} \Lambda_{\alpha\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} &= -(2n-2)\lambda_\alpha(0)\psi_{\bar\beta}(0) \\ &\quad - (2n-2)(2n-1)\psi_{\bar\beta}(0)C_\alpha + (2n-2)\psi_{\alpha\bar\beta}(0),\end{aligned}$$

which is (ii). □

LEMMA 6.4. *Let* $1 \le \alpha \le n-1$ *and* $1 \le \beta \le n$. *Then*

$$\lim_{\nu\to\infty} g_{\nu\alpha\bar\beta}(p_\nu)(\psi_\nu(p_\nu)) = (2n-2)\left(\frac{1}{2}\{\psi_{\alpha n}(0) + \psi_{\alpha\bar n}(0)\}\psi_{\bar\beta}(0) - \psi_{\alpha\bar\beta}(0)\right).$$

Proof. We have

$$g_{\alpha\bar\beta} = \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar z_\beta} = \frac{\Lambda_{\alpha\bar\beta}}{\Lambda} - \frac{\Lambda_\alpha\Lambda_{\bar\beta}}{\Lambda^2}.$$

Multiplying both sides of this equation by ψ yields

$$g_{\alpha\bar\beta}\psi = \frac{\Lambda_{\alpha\bar\beta}\psi^{2n-1}}{\Lambda\psi^{2n-2}} - \frac{(\Lambda_\alpha\psi^{2n-2})(\Lambda_{\bar\beta}\psi^{2n-1})}{(\Lambda\psi^{2n-2})^2}. \tag{6.8}$$

By the proof of Lemma 6.1,

$$\Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2} \to -1$$

and

$$\Lambda_{\nu\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} \to (2n-2)\psi_{\bar\beta}(0).$$

We can now use Lemma 6.3 and (6.8) to obtain

$$\begin{aligned}\lim_{\nu\to\infty} & g_{\nu\alpha\bar\beta}(p_\nu)\psi_\nu(p_\nu)\\ &= (2n-2)\lambda_\alpha(0)\psi_{\bar\beta}(0) + (2n-2)(2n-1)\psi_{\bar\beta}(0)C\\ &\quad - (2n-2)\psi_{\alpha\bar\beta}(0) - \{\lambda_\alpha(0) + (2n-2)C_\alpha\}\{(2n-2)\psi_{\bar\beta}(0)\}.\end{aligned}$$

Simplifying the right-hand side, we have

$$\begin{aligned}\lim_{\nu\to\infty} g_{\nu\alpha\bar\beta}(p_\nu)\psi_\nu(p_\nu) &= (2n-2)(\psi_{\bar\beta}(0)C_\alpha - \psi_{\alpha\bar\beta}(0))\\ &= (2n-2)\Big(\frac{1}{2}\{\psi_{\alpha n}(0) + \psi_{\alpha\bar n}(0)\}\psi_{\bar\beta}(0) - \psi_{\alpha\bar\beta}(0)\Big). \qquad \square\end{aligned}$$

Because we have no information about the third-order derivatives of $\lambda(p) = \psi^{2n-2}\Lambda(p)$ near the boundary of D, the method just described fails to give finer asymptotics of $\Lambda_{\nu\alpha\bar\beta\gamma}$. By Proposition 2.1, however, the function

$$g(p,w) = \psi(p)^{2n-2}G(p,z) \tag{6.9}$$

for $w = (z-p)/(-\psi(p))$ is C^2 up to $\mathcal{D} \cup \partial\mathcal{D}$. Also, for each $p \in D$, both $(\partial g/\partial p_\alpha)(p)$ and $(\partial^2 g/\partial p_\alpha \partial\bar p_\beta)(p)$ are harmonic functions of $w \in \bar D(p)$ and hence can be differentiated infinitely often with respect to w. Moreover,

$$\frac{\partial g}{\partial p_\alpha}(p,0) = \frac{\partial\lambda}{\partial p_\alpha}(p) \quad \text{and} \quad \frac{\partial^2 g}{\partial p_\alpha \partial\bar p_\beta}(p,0) = \frac{\partial^2\lambda}{\partial p_\alpha\partial\bar p_\beta}. \tag{6.10}$$

In what follows, we exploit these properties to calculate finer asymptotics of $\Lambda_{\nu\alpha\bar\beta\gamma}$ by expressing it in terms of mixed derivatives of g_ν.

By [7, Prop. 6.1], the functions

$$\begin{aligned} G_\alpha(p,z) &= \Big(\frac{\partial G}{\partial p_\alpha} + \frac{\partial G}{\partial z_\alpha}\Big)(p,z),\\ G_{\alpha\bar\beta}(p,z) &= \Big(\frac{\partial G_\alpha}{\partial p_{\bar\beta}} + \frac{\partial G_\alpha}{\partial \bar z_\beta}\Big)(p,z)\end{aligned} \tag{6.11}$$

are real-analytic and symmetric functions in $D \times D$ and are harmonic in z and in p. By [7, 6.14],

$$\Lambda_{\alpha\bar\beta\gamma}(p) = 2\frac{\partial G_{\alpha\bar\beta}}{\partial z_\gamma}(p,p). \tag{6.12}$$

By [7, Prop. 6.2], the functions

$$\begin{aligned} g_0(p,w) &= g(p,w) + \frac{1}{n-1}\sum_{i=1}^n w_i \frac{\partial g}{\partial w_i},\\ g_\alpha(p,w) &= \psi(p)\frac{\partial g}{\partial p_\alpha}(p,w) - (n-1)\psi_\alpha(p)\big(g_0(p,w) + \overline{g_0(p,w)}\big)\end{aligned} \tag{6.13}$$

are harmonic functions of $w \in D(p)$ for each $p \in \bar{D}$. From [7, p. 83] it follows that

$$\begin{aligned}&\frac{\partial G_{\alpha\bar{\beta}}}{\partial z_\gamma}(p,p)\\&\quad = -(\psi(p))^{-2n-1}\left\{-2n\psi_{\bar{\beta}}(p)\frac{\partial g_\alpha}{\partial w_\gamma}(p,0) + \psi(p)\frac{\partial^2 g_\alpha}{\partial w_\gamma \partial \bar{p}_\beta}(p,0)\right\}. \qquad (6.14)\end{aligned}$$

Combining (6.12) and (6.14) now yields

$$\Lambda_{\alpha\bar{\beta}\gamma}(p)(\psi(p))^{2n} = 4n\frac{\psi_{\bar{\beta}}(p)}{\psi(p)}\frac{\partial g_\alpha}{\partial w_\gamma}(p,0) - \frac{\partial^2 g_\alpha}{\partial w_\gamma \partial \bar{p}_\beta}(p,0). \qquad (6.15)$$

LEMMA 6.5. *Let* $1 \le \alpha, \gamma \le n$ *and* $1 \le \beta \le n-1$. *Then*

$$\lim_{\nu\to\infty} \Lambda_{\nu\alpha\bar{\beta}\gamma}(p_\nu)(\psi_\nu(p_\nu))^{2n}$$

exists and is finite.

Proof. By (6.15) and Lemma 6.2, we need only prove that

$$\lim_{\nu\to\infty}\frac{\partial g_{\nu\alpha}}{\partial w_\gamma}(p_\nu,0) \quad \text{and} \quad \lim_{\nu\to\infty}\frac{\partial^2 g_{\nu\alpha}}{\partial w_\gamma \partial \bar{p}_\beta}(p_\nu,0)$$

exist and are finite.

We know that $g_{\nu\alpha}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$. To estimate the boundary values of these functions, note that the first term of $g_{\nu 0}(p_\nu, w)$ (i.e., $g_\nu(p_\nu, w)$) is bounded by $|w|^{-2n+2}$ for all ν; by Proposition 5.3, the second term is bounded by $C|w|^{-2n+2}$ for all large ν. Therefore, by (6.13),

$$|g_{\nu 0}(p_\nu, w)| \le C|w|^{-2n+2}, \quad w \in \partial D^\nu, \qquad (6.16)$$

for all large ν. From Proposition 4.3 it follows that $|(\partial g_\nu/\partial p_\alpha)(p_\nu, w)|$ is bounded by $C(1+|w|^{-1})|w|^{-2n+3}$ for all large ν; in addition, $\psi_\nu(p_\nu)$ and $\psi_{\nu\alpha}(p_\nu)$ are bounded by a constant C for all large ν. Hence, using (6.13) and (6.16) we obtain

$$|g_{\nu\alpha}(p_\nu, w)| \le C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial D^\nu, \qquad (6.17)$$

for all large ν.

Choose $r > 0$ such that $\bar{B}(0,r) \subset \mathcal{H}$. Since D^ν converges in the Hausdorff sense to $\mathcal{H}$, there exists an integer I such that $\bar{B}(0,r) \subset D^\nu$ for all $\nu \ge I$. Consequently,

$$|w| > r \qquad (6.18)$$

for all $\nu \ge I$ and $w \in \partial D^\nu$. Hence by (6.17) we have

$$|g_{\nu\alpha}(p_\nu, w)| \le Cr^{-2n+3}(1+r^{-1}), \quad w \in \partial D^\nu,$$

for all large ν and so $g_{\nu\alpha}(p_\nu, w)$ is uniformly bounded on $B(0,r)$ for all large ν. Moreover, by [7, Prop. 6.2] and the equality $(\partial g_\nu/\partial p_\alpha)p_\nu, 0) = (\partial\lambda_\nu/\partial p_\alpha)(p_\nu)$,

$$g_{\nu\alpha}(p_\nu, 0) = \psi_\nu(p_\nu)\frac{\partial\lambda_\nu}{\partial p_\alpha}(p_\nu) - (2n-2)\psi_{\nu\alpha}(p_\nu)\lambda(p_\nu), \qquad (6.19)$$

which converges. It follows from Harnack's principle that

$$\lim_{\nu\to\infty} \frac{\partial g_{\nu\alpha}}{\partial w_\gamma}(p_\nu, 0)$$

exists.

Now differentiating (6.13) with respect to $\bar{p}_\beta$, we obtain

$$\frac{\partial g_0}{\partial \bar{p}_\beta}(p,w) = \frac{\partial g}{\partial \bar{p}_\beta} + \frac{1}{n-1}\sum_{i=1}^n w_i \frac{\partial^2 g}{\partial \bar{p}_\beta \partial w_i} \tag{6.20}$$

and

$$\begin{aligned}\frac{\partial g_\alpha}{\partial \bar{p}_\beta}(p,w) &= \psi(p)\frac{\partial^2 g}{\partial p_\alpha \partial \bar{p}_\beta}(p,w) + \psi_{\bar{\beta}}(p)\frac{\partial g}{\partial p_\alpha}(p,w)\\ &\quad - (n-1)\psi_\alpha(p)\left(\frac{\partial g_0}{\partial \bar{p}_\beta}(p,w) + \overline{\frac{\partial g_0}{\partial p_\beta}(p,w)}\right)\\ &\quad - (n-1)\psi_{\alpha\bar{\beta}}(p)\big(g_0(p,w) + \overline{g_0(p,w)}\big),\end{aligned} \tag{6.21}$$

which are harmonic functions of $w \in D$. As before, $|\partial g_\nu/\partial \bar{p}_\beta|$ is bounded by $C(1+|w|^{-1})|w|^{-2n+3}$ for all large ν; and by Proposition 5.7, $|\partial^2 g_\nu/\partial \bar{p}_\beta \partial w_i|$ is bounded by $C(1+|w|^{-1})|w|^{-2n+2}$ for all large ν. It follows that

$$\left|\frac{\partial g_{\nu 0}}{\partial \bar{p}_\beta}(p_\nu, w)\right| \le C|w|^{-2n+3}, \quad w \in \partial D^\nu, \tag{6.22}$$

for all large ν. By Proposition 2.1, for $1 \le \gamma \le n$ and $p \in D$ we have

$$\left|\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p,w)\right| \le |k_2^\gamma(p,w)||\partial_w g(p,w)| + 2|k_1^\gamma| \sum_{i=1}^n \left|\frac{\partial^2 g}{\partial w_i \partial \bar{p}_\gamma}\right|, \quad w \in \partial D(p).$$

It follows that

$$\left|\frac{\partial^2 g_\nu}{\partial p_\gamma \partial \bar{p}_\gamma}(p_\nu, w)\right| \le C(1+|w|^{-1}+|w|^{-2})|w|^{-2n+4}, \quad w \in \partial D^\nu,$$

and hence—by a unitary change of coordinates—that

$$\left|\frac{\partial^2 g_\nu}{\partial p_\alpha \partial \bar{p}_\beta}(p_\nu, w)\right| \le C(1+|w|^{-1}+|w|^{-2})|w|^{-2n+4}, \quad w \in \partial D^\nu,$$

for all large ν. Thus

$$\begin{aligned}\left|\frac{\partial g_{\nu\alpha}}{\partial \bar{p}_\beta}(p_\nu, w)\right| &\le C(1+|w|^{-1}+|w|^{-2})|w|^{-2n+4}\\ &\le Cr^{-2n+4}(1+r^{-1}+r^{-2}), \quad w \in \partial D^\nu,\end{aligned}$$

for all large ν. Therefore, the sequence $\{(\partial g_{\nu\alpha}/\partial \bar{p}_\beta)(p_\nu, w)\}$ is uniformly bounded on $B(0,r)$. Moreover,

$$\begin{aligned}\frac{\partial g_{\nu\alpha}}{\partial \bar{p}_\beta}(p_\nu, 0) &= \psi_\nu(p_\nu)\frac{\partial^2 \lambda_\nu}{\partial p_\alpha \partial \bar{p}_\beta}(p_\nu) + \psi_{\nu\bar{\beta}}(p_\nu)\frac{\partial \lambda_\nu}{\partial p_\alpha}(p_\nu)\\ &\quad - (2n-2)\psi_{\nu\alpha}(p_\nu)\frac{\partial \lambda_\nu}{\partial \bar{p}_\beta}(p_\nu) - (2n-2)\psi_{\nu\alpha\bar{\beta}}(p_\nu)\lambda_\nu(p_\nu),\end{aligned}$$

which converges. It now follows from Harnack's principle that

$$\lim_{\nu\to\infty} \frac{\partial^2 g_{\nu\alpha}}{\partial w_\gamma \partial \bar{p}_\beta}(p_\nu, 0)$$

exists. $\square$

LEMMA 6.6. *Let* $1 \le \alpha, \gamma \le n$ *and* $1 \le \beta \le n-1$. *Then*

$$\lim_{\nu\to\infty} \frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma}(p_\nu)(\psi(p_\nu))^2$$

exists and is finite.

Proof. From (6.2), we obtain

$$\begin{aligned}
\frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma}\psi_\nu^2 = {} & \frac{\Lambda_{\nu\alpha\bar{\beta}\gamma}\psi_\nu^{2n}}{\Lambda_\nu \psi_\nu^{2n-2}} \\
& - \bigg(\frac{(\Lambda_{\nu\alpha\bar{\beta}}\psi_\nu^{2n-1})(\Lambda_{\nu\gamma}\psi_\nu^{2n-1})}{(\Lambda_\nu\psi_\nu^{2n-2})^2} + \frac{(\Lambda_{\nu\alpha\gamma}\psi_\nu^{2n})(\Lambda_{\nu\bar{\beta}}\psi_\nu^{2n-2})}{(\Lambda_\nu\psi_\nu^{2n-2})^2} \\
& \quad + \frac{(\Lambda_{\nu\bar{\beta}\gamma}\psi_\nu^{2n-1})(\Lambda_{\nu\alpha}\psi_\nu^{2n-1})}{(\Lambda_\nu\psi_\nu^{2n-2})^2} \bigg) \\
& + \frac{2(\Lambda_{\nu\alpha}\psi_\nu^{2n-1})(\Lambda_{\nu\bar{\beta}}\psi_\nu^{2n-2})(\Lambda_{\nu\gamma}\psi_\nu^{2n-1})}{(\Lambda_\nu\psi_\nu^{2n-2})^3}.
\end{aligned}$$

Given Theorem 1.2 and Lemma 6.3, we can see that the second and third terms have finite limits along $\{p_\nu\}$; by Lemma 6.5, the first term has a finite limit along $\{p_\nu\}$. $\square$

LEMMA 6.7. *The limit*

$$\lim_{\nu\to\infty} \det(g_{\nu\alpha\bar{\beta}}(p_\nu))(\psi_\nu(p_\nu))^{n+1}$$

exists and is nonzero.

Proof. Let $(\Delta_{\alpha\bar{\beta}})$ be the cofactor matrix of $(g_{\alpha\bar{\beta}})$. Then expanding by the nth row yields

$$\det(g_{\alpha\bar{\beta}}) = g_{n\bar{1}}\Delta_{n\bar{1}} + \cdots + g_{n\bar{n}}\Delta_{n\bar{n}}.$$

Therefore,

$$\det(g_{\alpha\bar{\beta}})\psi^{n+1} = (g_{n\bar{1}}\psi^2)(\Delta_{n\bar{1}}\psi^{n-1}) + \cdots + (g_{n\bar{n}}\psi^2)(\Delta_{n\bar{n}}\psi^{n-1}). \tag{6.23}$$

Note that

$$\begin{aligned}
\Delta_{n\bar{\alpha}}\psi^{n-1} &= \psi^{n-1}(-1)^{n+\alpha}\det\begin{pmatrix} g_{1\bar{1}} & \cdots & g_{1\overline{\alpha-1}} & g_{1\overline{\alpha+1}} & \cdots & g_{1\bar{n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-1\bar{1}} & \cdots & g_{n-1\overline{\alpha-1}} & g_{n-1\overline{\alpha+1}} & \cdots & g_{n-1\bar{n}} \end{pmatrix} \\
&= (-1)^{n+\alpha}\det\begin{pmatrix} g_{1\bar{1}}\psi & \cdots & g_{1\overline{\alpha-1}}\psi & g_{1\overline{\alpha+1}}\psi & \cdots & g_{1\bar{n}}\psi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-1\bar{1}}\psi & \cdots & g_{n-1\overline{\alpha-1}}\psi & g_{n-1\overline{\alpha+1}}\psi & \cdots & g_{n-1\bar{n}}\psi \end{pmatrix}.
\end{aligned}$$

By Lemma 6.4, if $1 \le \alpha \le n-1$ and $1 \le \beta \le n$ then the term $g_{\nu\alpha\bar{\beta}}(p_\nu)\psi_\nu(p_\nu)$ converges to a finite quantity. It follows that if $1 \le \alpha \le n-1$ then

$$\lim_{\nu\to\infty} \Delta_{\nu n\bar{\alpha}}(p_\nu)(\psi_\nu(p_\nu))^{n-1}$$

exists and is finite. Also, if $1 \le \alpha, \beta \le n-1$ then $g_{\nu\alpha\bar{\beta}}(p_\nu)\psi_\nu(p_\nu)$ converges to $-(2n-2)\psi_{\alpha\bar{\beta}}(0)$. Hence

$$\lim_{\nu\to\infty} \Delta_{\nu n\bar{n}}(p_\nu)(\psi_\nu(p_\nu))^{n-1} = (-1)^n(2n-2)^n \det(\psi_{\alpha\bar{\beta}}(0))_{1\le\alpha,\beta\le n-1}.$$

Finally, by Lemma 6.1, if $1 \le \alpha, \beta \le n$ then $g_{\nu\alpha\bar{\beta}}(p_\nu)(\psi_\nu(p_\nu))^2$ converges to $(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)$. Now it follows from (6.23) that

$$\lim_{\nu\to\infty} \det(g_{\nu\alpha\bar{\beta}}(p_\nu))(\psi_\nu(p_\nu))^{n+1} = (-1)^n(2n-2)^{n+1} \det(\psi_{\alpha\bar{\beta}}(0))_{1\le\alpha,\beta\le n-1} \neq 0$$

because D is strongly pseudoconvex at 0. □

Proof of Theorem 1.1. We have

$$-\frac{1}{(g_{n\bar{n}}(z))^2}\frac{\partial^2 g_{n\bar{n}}}{\partial z_n \partial \bar{z}_n}(z) = -\frac{1}{\big(g_{n\bar{n}}(z)(\psi(z))^2\big)^2}\frac{\partial^2 g_{n\bar{n}}}{\partial z_n \partial \bar{z}_n}(z)(\psi(z))^4.$$

By Lemma 6.1,

$$\begin{aligned}
&-\frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2}\frac{\partial^2 g_{\nu n\bar{n}}}{\partial z_n \partial \bar{z}_n}(p_\nu)\\
&\qquad \to -\frac{1}{\{(2n-2)\psi_n(0)\psi_{\bar{n}}(0)\}^2}\{6(2n-2)\psi_n(0)\psi_{\bar{n}}(0)\psi_n(0)\psi_{\bar{n}}(0)\}\\
&\qquad = -\frac{3}{n-1}.
\end{aligned}$$

To compute the limit of the second term, note that $g^{\beta\bar{\alpha}} = \Delta_{\alpha\bar{\beta}}/\det(g_{\alpha\bar{\beta}})$. There are various cases to be considered depending on α and β.

Case 1: $\alpha \neq n$ and $\beta \neq n$. Here

$$\frac{1}{g_{n\bar{\alpha}}^2} g^{\beta\bar{\alpha}} \frac{\partial g_{n\bar{\alpha}}}{\partial z_n}\frac{\partial g_{\beta\bar{n}}}{\partial \bar{z}_n} = \frac{1}{(g_{n\bar{n}}\psi^2)^2(\det(g_{i\bar{j}})\psi^{n+1})}(\Delta_{\alpha\bar{\beta}}\psi^n)\bigg(\frac{\partial g_{n\bar{\alpha}}}{\partial z_n}\psi^2\bigg)\bigg(\frac{\partial g_{\beta\bar{n}}}{\partial \bar{z}_n}\psi^3\bigg).$$

By Lemma 6.1,

$$g_{\nu n\bar{n}}(p_\nu)(\psi_\nu(p_\nu))^2 \to (2n-2).$$

By Lemma 6.7, $\det(g_{i\bar{j}}(p_\nu))(\psi_\nu p_\nu))^{n+1}$ converges to a nonzero finite quantity. Also,

$$\Delta_{\alpha\bar{\beta}} = \sum_\sigma (-1)^{\operatorname{sgn}(\sigma)} g_{1\overline{\sigma(1)}} g_{2\overline{\sigma(2)}} \cdots g_{n\overline{\sigma(n)}},$$

where the summation runs over all permutations

$$\sigma : \{1, \dots, \alpha-1, \alpha+1, \dots, n\} \to \{1, \dots, \beta-1, \beta+1, \dots, n\}.$$

Hence

$$\Delta_{\alpha\bar{\beta}}\psi^n = \sum_\sigma (-1)^{\operatorname{sgn}(\sigma)} (g_{1\overline{\sigma(1)}}\psi)(g_{2\overline{\sigma(2)}}\psi)\cdots(g_{n\overline{\sigma(n)}}\psi^2).$$

According to Lemma 6.4, if $1 \le i \le n-1$ then $g_{\nu i\overline{\sigma(i)}}(p_\nu)(\psi_\nu(p_\nu))$ converges to a finite quantity. Also,

$$g_{\nu n\overline{\sigma(n)}}(p_\nu)(\psi(p_\nu))^2 \to (2n-2)\psi_n(0)\psi_{\overline{\sigma(n)}}(0)$$

by Lemma 6.1. Thus $\Delta_{\nu\alpha\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^n$ converges to a finite quantity.

By Lemma 6.6, $(\partial g_{\nu n\bar\alpha}/\partial z_n)(p_\nu)(\psi_\nu(p_\nu))^2$ converges to a finite quantity; and by Lemma 6.1,

$$\frac{\partial g_{\nu\beta\bar n}}{\partial \bar z_n}(p_\nu)(\psi_\nu(p_\nu))^3 = \overline{\left(\frac{\partial g_{\nu n\bar\beta}}{\partial z_n}(p_\nu)(\psi_\nu(p_\nu))^3\right)}$$
$$\to -2(2n-2)\overline{(\psi_{\nu n}(0))}\,\overline{(\psi_{\bar\beta}(0))}\,\overline{(\psi_n(0))} = 0.$$

Hence

$$\lim_{\nu\to\infty}\frac{1}{(g_{\nu n\bar n}(p_\nu))^2}g_\nu^{\beta\bar\alpha}(p_\nu)\frac{\partial g_{\nu n\bar\alpha}}{\partial z_n}(p_\nu)\frac{\partial g_{\nu\beta\bar n}}{\partial \bar z_n}(p_\nu) = 0.$$

Case 2: $\alpha = n$ and $\beta \ne n$. Here

$$\frac{1}{g_{n\bar n}^2}g^{\beta\bar n}\frac{\partial g_{n\bar n}}{\partial z_n}\frac{\partial g_{\beta\bar n}}{\partial \bar z_n}$$
$$= \frac{1}{(g_{n\bar n}\psi^2)^2(\det(g_{i\bar j})\psi^{n+1})}(\Delta_{n\bar\beta}\psi^{n-1})\left(\frac{\partial g_{n\bar n}}{\partial z_n}\psi^3\right)\left(\frac{\partial g_{\beta\bar n}}{\partial \bar z_n}\psi^3\right).$$

By Lemma 6.1,

$$g_{\nu n\bar n}(p_\nu)(\psi_\nu(p_\nu))^2 \to (2n-2);$$

also, $\det(g_{\nu\alpha\bar\beta}(p_\nu))(\psi(p_\nu))^{n+1}$ has a nonzero limit and $\Delta_{\nu n\bar\beta}(p_\nu)(\psi_\nu(p_\nu))^{n-1}$ converges to a finite quantity (these claims follow from Lemma 6.7). Now Lemma 6.1 implies that

$$\frac{\partial g_{\nu n\bar n}}{\partial z_n}(p_\nu)(\psi_\nu(p_\nu))^3 \to -2(2n-2)\psi_n(0)\psi_{\bar n}(0)\psi_n(0) = -2(2n-2)$$

and

$$\frac{\partial g_{\nu\beta\bar n}}{\partial \bar z_n}(p_\nu)(\psi_\nu(p_\nu))^3 = \overline{\left(\frac{\partial g_{\nu n\bar\beta}}{\partial z_n}(p_\nu)(\psi_\nu(p_\nu))^3\right)}$$
$$\to -2(2n-2)\overline{(\psi_n(0))}\,\overline{(\psi_{\bar\beta}(0))}\,\overline{(\psi_n(0))} = 0.$$

Therefore,

$$\lim_{\nu\to\infty}\frac{1}{(g_{\nu n\bar n}(p_\nu))^2}g_\nu^{\beta\bar n}(p_\nu)\frac{\partial g_{\nu n\bar n}}{\partial z_n}(p_\nu)\frac{\partial g_{\nu\beta\bar n}}{\partial \bar z_n}(p_\nu) = 0.$$

Case 3: $\alpha \ne n$ and $\beta = n$. This case is similar to Case 2, and we have

$$\lim_{\nu\to\infty}\frac{1}{(g_{\nu n\bar n}(p_\nu))^2}g_\nu^{n\bar\alpha}(p_\nu)\frac{\partial g_{\nu n\bar\alpha}}{\partial z_n}(p_\nu)\frac{\partial g_{\nu n\bar n}}{\partial \bar z_n}(p_\nu) = 0.$$

Case 4: $\alpha = n$ and $\beta = n$. In this case, we have

$$\frac{1}{g_{n\bar n}^2}g^{n\bar n}\frac{\partial g_{n\bar n}}{\partial z_n}\frac{\partial g_{n\bar n}}{\partial \bar z_n}$$
$$= \frac{1}{(g_{n\bar n}\psi^2)^2(\det(g_{i\bar j})\psi^{n+1})}(\Delta_{n\bar n}\psi^{n-1})\left(\frac{\partial g_{n\bar n}}{\partial z_n}\psi^3\right)\left(\frac{\partial g_{n\bar n}}{\partial \bar z_n}\psi^3\right).$$

By Lemma 6.1,

$$g_{\nu n\bar{n}}(p_\nu)(\psi_\nu(p_\nu))^2 \to (2n-2)$$

and both

$$\frac{\partial g_{\nu n\bar{n}}}{\partial z_n}(p_\nu)(\psi_\nu(p_\nu))^3,\ \frac{\partial g_{n\bar{n}}}{\partial \bar{z}_n}(p_\nu)(\psi_\nu(p_\nu))^3 \to -2(2n-2);$$

by Lemma 6.7,

$$\Delta_{\nu n\bar{n}}(p_\nu)(\psi_\nu(p_\nu))^{n-1} \to (-1)^n(2n-2)^n \det(\psi_{i\bar{j}}(0))_{1\le i,j\le n-1}$$

and

$$\det(g_{\nu i\bar{j}}(p_\nu))(\psi_\nu(p_\nu))^{n+1} \to (-1)^n(2n-2)^{n+1} \det(\psi_{i\bar{j}}(0))_{1\le i,j\le n-1}.$$

Hence

$$\lim_{\nu\to\infty} \frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2} g_\nu^{n\bar{n}}(p_\nu)\frac{\partial g_{\nu n\bar{n}}}{\partial z_n}(p_\nu)\frac{\partial g_{\nu n\bar{n}}}{\partial \bar{z}_n}(p_\nu) = \frac{2}{n-1}.$$

From the various cases we finally obtain

$$\lim_{\nu\to\infty} R(z_\nu, v_N(z_\nu)) = \frac{-3}{n-1} + \frac{2}{n-1} = \frac{-1}{n-1}. \qquad \square$$

7. Existence of Closed Geodesics

In this section we prove Theorem 1.5. The main tool that we will use is the following theorem of Herbort [6].

THEOREM 7.1. *Let G be a bounded domain in $\mathbf{R}^k$ such that $\pi_1(G)$ is nontrivial. Assume that the following conditions are satisfied.*

(i) *For each $p \in \bar{G}$, there is an open neighborhood $U \subset \mathbf{R}^k$ such that the set $G \cap U$ is simply connected.*

(ii) *G is equipped with a complete Riemannian metric g that possesses the following property.*

   (P) *For each $S > 0$ there is a $\delta > 0$ such that, for every point $p \in G$ with $d(p, \partial D) < \delta$ and every $X \in \mathbf{R}^k$, $g(p, X) \ge S|X|^2$.*

Then every nontrivial homotopy class in $\pi_1(G)$ contains a closed geodesic for g.

It is evident that a C^∞-smoothly bounded domain D satisfies part (i) of the theorem. To see whether the Λ-metric satisfies property (P), consider a C^∞-smoothly bounded pseudoconvex domain D in $\mathbf{C}^n$ and suppose that ψ is C^∞-smooth defining function for D. Then, differentiating the relation

$$\lambda = \Lambda\psi^{2n-2}$$

with respect to z_α, we obtain

$$\frac{\partial \log(-\Lambda)}{\partial z_\alpha} = \lambda^{-1}\lambda_\alpha - 2(n-1)\psi^{-1}\psi_\alpha. \tag{7.1}$$

Now differentiating this with respect to $\bar{z}_\beta$ yields

$$\frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} = \lambda^{-1}\lambda_{\alpha\bar{\beta}} - \lambda^{-2}\lambda_\alpha\lambda_{\bar{\beta}} + 2(n-1)\psi^{-2}\psi_\alpha\psi_{\bar{\beta}} - 2(n-1)\psi^{-1}\psi_{\alpha\bar{\beta}}. \tag{7.2}$$

Hence, for $v \in \mathbf{C}^n$,

$$\begin{aligned} ds^2(v,v) &= \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} v^\alpha \bar{v}^\beta \\ &= \lambda^{-1}\mathcal{L}_\lambda(z,v) - \lambda^{-2}|\langle v, \bar{\partial}\lambda\rangle|^2 + 2(n-1)\psi^{-2}|\langle v, \bar{\partial}\psi\rangle|^2 \\ &\quad - 2(n-1)\psi^{-1}\mathcal{L}_\psi(z,v). \end{aligned} \tag{7.3}$$

Lemma 7.2. *Let D be a C^∞-smoothly bounded pseudoconvex domain in $\mathbf{C}^n$, and let ψ be a C^∞-smooth defining function for D. Let $z_0 \in \partial D$ and $v \in \mathbf{C}^n$ with $|v| = 1$. Then*

$$\lim_{z\to z_0}(-\psi(z))^2\, ds_z^2(v,v) = 2(n-1)|\langle v, \bar{\partial}\psi(z_0)\rangle|^2.$$

Also, if $\langle v, \bar{\partial}\psi(z_0)\rangle = 0$ then

$$\lim_{z\to z_0}(-\psi(z))\, ds_z^2(v,v) = 2(n-1)\mathcal{L}_\psi(z_0,v).$$

Finally, the limits just given are uniform in z_0 and v.

Proof. Since λ is C^2-smooth up to $\bar{D}$ and since ψ is C^∞-smooth, it follows that the terms

$$\langle v, \partial\bar{\lambda}(z)\rangle,\ \langle v, \partial\bar{\psi}(z)\rangle,\ \mathcal{L}_\lambda(z,v),\ \text{and } \mathcal{L}_\psi(z,v)$$

are uniformly bounded for all $z \in \bar{D}$ and all $v \in \mathbf{C}^n$ with $|v| = 1$. Also, since $\lambda = -|\partial\psi|^{2n-2}$ on ∂D, it is evident that λ^{-1} is bounded near ∂D.

By the foregoing observation it is clear from (7.3) that

$$\lim_{z\to z_0}(\psi(z))^2\, ds_z^2(v,v) = 2(n-1)|\langle v, \bar{\partial}\psi(z_0)\rangle|^2$$

uniformly for $z_0 \in \partial D$ and unit vectors v. This proves the first part of the lemma.

To prove the second part, observe that if $\langle v, \bar{\partial}\psi(z_0)\rangle = 0$ then

$$\langle v, \bar{\partial}\psi(z)\rangle = \langle v, \bar{\partial}\psi(z)\rangle - \langle v, \bar{\partial}\psi(z_0)\rangle = \langle v, \bar{\partial}\psi(z) - \bar{\partial}\psi(z_0)\rangle.$$

Since

$$|\bar{\partial}\psi(z) - \bar{\partial}\psi(z_0)| \lesssim (-\psi(z))$$

uniformly for z near z_0, it follows that

$$|\langle v, \bar{\partial}\psi(z)\rangle| \lesssim (-\psi(z))$$

uniformly for z near z_0 and for unit vectors v satisfying $\langle v, \bar{\partial}\psi(z_0)\rangle = 0$. Combining this with our previous observation, it now follows from (7.3) that

$$\lim_{z\to z_0}(-\psi(z))\, ds_z^2(v,v) = 2(n-1)\mathcal{L}_\psi(z_0,v)$$

uniformly for $z_0 \in \partial D$ and for unit vectors v satisfying $\langle v, \bar{\partial}\psi(z_0)\rangle = 0$. Thus the lemma is proved. □

Proposition 7.3. *Let D be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$, and let ψ be a C^∞-smooth defining function for D. Then there exist a neighborhood U of ∂D and a constant $K > 0$, depending only on D, such that*

$$ds_z^2(v,v) \geq K\frac{|v|^2}{-\psi(z)}, \quad z \in U \cap D,\ v \in \mathbf{C}^n.$$

Proof. Let $z_0 \in \partial D$ and $v_0 \in \mathbf{C}^n$ with $|v_0| = 1$. If $\langle v_0, \bar\partial\psi(z_0)\rangle \neq 0$ then, by Lemma 7.2,

$$(-\psi(z))^2\, ds_z^2(v_0, v_0) \geq (n-1)|\langle v_0, \bar\partial\psi(z_0)\rangle|^2$$

for z sufficiently close z_0. From this it follows that

$$(-\psi(z))^2\, ds_z^2(v, v) \geq \frac{(n-1)}{2}|\langle v_0, \bar\partial\psi(z_0)\rangle|^2$$

for z sufficiently close z_0 and for unit vectors v sufficiently close to v_0. Hence

$$\begin{aligned} ds_z^2(v, v) &\geq \frac{(n-1)}{2}|\langle v_0, \bar\partial\psi(z_0)\rangle|^2 \frac{1}{(-\psi(z))^2} \\ &\geq \frac{(n-1)}{2}|\langle v_0, \bar\partial\psi(z_0)\rangle|^2 \frac{1}{(-\psi(z))} \end{aligned} \tag{7.4}$$

for z sufficiently close z_0 and for unit vectors v sufficiently close to v_0.

If $\langle v_0, \bar\partial\psi(z_0)\rangle = 0$ then $\mathcal{L}_\psi(z_0, v_0) > 0$ and, again by Lemma 7.2,

$$-\psi(z)\, ds_z^2(v_0, v_0) \geq (n-1)\mathcal{L}_\psi(z_0, v_0)$$

for z sufficiently close z_0; thus

$$-\psi(z)\, ds_z^2(v, v) \geq \frac{(n-1)}{2}\mathcal{L}_\psi(z_0, v_0)$$

for z sufficiently close z_0 and for unit vectors v sufficiently close to v_0. Then

$$ds_z^2(v, v) \geq \frac{(n-1)}{2}\mathcal{L}_\psi(z_0, v_0)\frac{1}{-\psi(z)} \tag{7.5}$$

for z sufficiently close z_0 and for unit vectors v sufficiently close to v_0. Since ∂D and $\{v \in \mathbf{C}^n : |v| = 1\}$ are compact, (7.4) and (7.5) together imply that there exists a constant $K > 0$ such that

$$ds_z^2(v, v) \geq K\frac{|v|^2}{-\psi(z)}$$

for z near ∂D and for unit vectors v. The proof of the proposition now follows from the homogeneity of $ds_z^2(v, v)$ in the vector variable. □

Proof of Theorem 1.5. By Proposition 7.3, the Λ-metric is complete on D and satisfies property (P) of Theorem 7.1, from which the proof follows. □

8. L^2-Cohomology of the Λ-Metric

In this section we prove Theorem 1.6. Let us first recall the definition of L^2-cohomology. Let M be a complete Kähler manifold of complex dimension n. Let Ω_2^i be the space of square-integrable i-forms on M. Then the (reduced) L^2-cohomology of the complex

$$\Omega_2^0(M) \xrightarrow{d_0} \Omega_2^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{2n-1}} \Omega_2^{2n}(M) \xrightarrow{d_{2n}} 0$$

is defined by

$$H_2^i(M) = \frac{\ker d_i}{\overline{\operatorname{Im} d_{i-1}}},$$

where the closure is taken in L^2. Now, let $\mathcal{H}_2^i(M)$ be the space of square-integrable harmonic i-forms on M. Then the completeness of the metric implies that

$$H_2^i(M) \cong \mathcal{H}_2^i(M).$$

We have the following result from [3] on the vanishing of the L^2-cohomology outside the middle dimension.

THEOREM 8.1. *Let M be a complete Kähler manifold of complex dimension n. Suppose that the Kähler form ω of M can be written as $\omega = d\eta$, where η is bounded in supremum norm. Then $\mathcal{H}_2^i(M) = 0$ for $i \neq n$.*

We also have the following result from [9] on the infinite dimensionality of the L^2-cohomology of the middle dimension.

THEOREM 8.2. *Let D be a domain in a connected complex manifold of dimension n, and let ds^2 be a Hermitian metric on D. Suppose there exists a nondegenerate regular boundary point $z_0 \in \partial D$. Also, suppose there exist a neighborhood U of z_0, a local defining function ϕ for D defined on U, and a Hermitian metric ds_U^2 defined on U such that*

$$C^{-1}\, ds^2 < (-\phi)^{-a}\, ds_U^2 + (-\phi)^{-b} \partial\phi\,\bar{\partial}\phi < C\, ds^2$$

on $U \cap D$, where a, b, and C are positive numbers with $1 \le a \le b < a+3$. Then, for any positive integer p and q with $p+q=n$, we have

$$\dim H_2^{p,q}(D) = \infty;$$

here $H_2^{p,q}(D)$ denotes the L^2 $\bar{\partial}$-cohomology group relative to ds^2.

REMARK 8.3. If in Theorem 8.2 we assume also that ds^2 is complete and Kähler, then for any positive integer p and q with $p+q=n$ we have

$$\dim \mathcal{H}_2^{p,q}(D) = \infty,$$

where $\mathcal{H}_2^{p,q}(D)$ is the space of square-integrable harmonic (p,q)-forms on D relative to ds^2.

To apply these results to the Λ-metric, let D be a C^∞-smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ and let ds^2 be the Λ-metric on D. Then the Kähler form ω of ds^2 is given by

$$\omega = i \sum_{\alpha=1}^{n} \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta}\, dz_\alpha \wedge d\bar{z}_\beta = d\eta,$$

where

$$\eta = -i \sum_{\alpha=1}^{n} \frac{\partial \log(-\Lambda)}{\partial z_\alpha}\, dz_\alpha.$$

Now let ψ be a C^∞-smooth defining function for D. Then, using (7.1), for $v \in \mathbf{C}^n$ we have

$$\eta(v) = -i \sum_{\alpha=1}^{n} \frac{\partial \log(-\Lambda)}{\partial z_\alpha} v^\alpha = -i(\lambda^{-1}\langle v, \bar{\partial}\lambda\rangle - 2(n-1)\psi^{-1}\langle v, \bar{\partial}\psi\rangle)$$

and

$$|\eta(v)|^2 = \lambda^{-2}|\langle v, \bar\partial\lambda\rangle|^2 - 4(n-1)\lambda^{-1}\psi^{-1}\Re\big(\langle v, \bar\partial\lambda\rangle\overline{\langle v, \bar\partial\psi\rangle}\big) + 4(n-1)^2\psi^{-2}|\langle v, \bar\partial\psi\rangle|^2. \tag{8.1}$$

LEMMA 8.4. *Let D be a C^∞-smoothly bounded pseudoconvex domain in $\mathbf{C}^n$, and let ψ be a C^∞-smooth defining function for D. Let $z_0 \in \partial D$ and $v \in \mathbf{C}^n$ with $|v| = 1$. Then*

$$\lim_{z\to z_0}(-\psi(z))^2|\eta_z(v)|^2 = 4(n-1)^2|\langle v, \bar\partial\psi(z_0)\rangle|^2.$$

Also, if $\langle v, \bar\partial\psi(z_0)\rangle = 0$ then

$$\lim_{z\to z_0}(-\psi(z))|\eta_z(v)|^2 = 0.$$

Finally, the limits are uniform in z_0 and v.

Proof. Since λ is C^2-smooth up to $\bar D$ and since ψ is C^∞-smooth, the terms

$$\langle v, \partial\bar\lambda(z)\rangle \quad \text{and} \quad \langle v, \partial\bar\psi(z)\rangle$$

are uniformly bounded for all $z \in \bar D$ and all $v \in \mathbf{C}^n$ with $|v| = 1$. Also, since $\lambda = -|\partial\psi|^{2n-2}$ on ∂D, it follows that λ^{-1} is bounded near ∂D.

By the preceding observation it is evident from (8.1) that

$$\lim_{z\to z_0}(-\psi(z))^2|\eta_z(v)|^2 = 4(n-1)^2|\langle v, \bar\partial\psi(z_0)\rangle|^2$$

uniformly for $z_0 \in \partial D$ and unit vectors v. This proves the first part of the lemma.

To prove the second part we observe that, as in the proof of Lemma 7.2,

$$|\langle v, \bar\partial\psi(z)\rangle| \lesssim (-\psi(z))$$

uniformly for z near z_0 and for unit vectors v satisfying $\langle v, \bar\partial\psi(z_0)\rangle = 0$. Combining this with our previous observation, it now follows from (8.1) that

$$\lim_{z\to z_0}(-\psi(z))|\eta_z(v)|^2 = 0$$

uniformly for $z_0 \in \partial D$ and for unit vectors v satisfying $\langle v, \bar\partial\psi(z_0)\rangle = 0$. Thus the lemma is proved. □

PROPOSITION 8.5. *Let D be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$. Then the ratio*

$$\frac{|\eta_z(v)|^2}{ds_z^2(v,v)} \tag{8.2}$$

is uniformly bounded for $z \in D$ and for vectors $v \in \mathbf{C}^n$ with $v \neq 0$.

Proof. Let $z_0 \in \partial D$ and $v_0 \in \mathbf{C}^n$ with $|v_0| = 1$. By Lemma 7.2 and Lemma 8.4, for $\langle v_0, \bar\partial\psi(z_0)\rangle \neq 0$ we have

$$\lim_{z\to z_0}\frac{|\eta_z(v_0)|^2}{ds_z^2(v_0,v_0)} = 2(n-1);$$

for $\langle v_0, \bar\partial\psi(z_0)\rangle = 0$,

$$\lim_{z\to z_0}\frac{|\eta_z(v_0)|^2}{ds_z^2(v_0,v_0)} = 0$$

since $\mathcal{L}_\psi(z_0, v_0) > 0$ at the strongly pseudoconvex boundary point z_0. It follows that the ratio

$$\frac{|\eta_z(v)|^2}{ds_z^2(v,v)}$$

is uniformly bounded for all z near z_0 and for unit vectors v near v_0. Since ∂D and $\{v \in \mathbf{C}^n : |v| = 1\}$ are compact, this ratio is uniformly bounded for all z near ∂D and for all unit vectors v. It is clear that this ratio is uniformly bounded for all z on a compact subset of D and for all unit vectors v. Now, by the homogeneity of $\eta_z(v)$ and $ds_z^2(v,v)$ in the vector variable v, it follows that the ratio is uniformly bounded above for all $z \in D$ and for vectors $v \in \mathbf{C}^n$ with $v \neq 0$. This proves the proposition. □

We also note the following result.

PROPOSITION 8.6. *Let D be a C^∞-smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$, and let ds^2 be the Λ-metric on D. Suppose that ψ is a C^∞-smooth defining function for D. Then*

$$ds^2 \approx (-\psi)^{-1} ds_E^2 + (-\psi)^{-2} \partial\psi \bar{\partial}\psi$$

uniformly near ∂D, where ds_E^2 is the Euclidean metric on $\mathbf{C}^n$.

Proof. Let us denote the tensor on the right-hand side by h so that, for $z \in D$ and $v \in \mathbf{C}^n$,

$$h_z(v,v) = (-\psi(z))^{-1}|v|^2 + (-\psi(z))^{-2}|\langle v, \bar{\partial}\psi(z)\rangle|^2.$$

Let $z_0 \in \partial D$ and $v_0 \in \mathbf{C}^n$ with $|v_0| = 1$. Then

$$\lim_{z\to z_0} (-\psi(z))^2 h_z(v_0, v_0) = |\langle v_0, \bar{\partial}\psi(z_0)\rangle|^2.$$

Hence if $\langle v_0, \bar{\partial}\psi(z_0)\rangle \neq 0$ then, by Lemma 7.2,

$$\lim_{z\to z_0} \frac{ds_z^2(v_0,v_0)}{h_z(v_0,v_0)} = 2(n-1). \tag{8.3}$$

If $\langle v_0, \bar{\partial}\psi(z_0)\rangle = 0$ then, as in Lemma 7.2,

$$|\langle v_0, \bar{\partial}\psi(z)\rangle| \lesssim (-\psi(z))$$

and so

$$\lim_{z\to z_0} (-\psi(z)) h_z(v_0, v_0) = |v_0|^2 = 1;$$

therefore, by Lemma 7.2,

$$\lim_{z\to z_0} \frac{ds_z^2(v_0,v_0)}{h_z(v_0,v_0)} = 2(n-1)\mathcal{L}_\psi(z_0, v_0) > 0 \tag{8.4}$$

because D is strongly pseudoconvex. It follows from (8.3) and (8.4) that the ratio

$$\frac{ds_z^2(v,v)}{h_z(v,v)}$$

is uniformly bounded above and below by positive constants for all z near z_0 and for unit vectors v near v_0. Since ∂D and $\{v \in \mathbf{C}^n : |v| = 1\}$ are compact, this ratio is uniformly bounded above and below by positive constants for all z near ∂D and for unit vectors v. The proposition now follows from the homogeneity of both $ds_z^2(v, v)$ and $h_z(v, v)$ in the vector variable v. □

Proof of Theorem 1.6. Let ds^2 be the Λ-metric on D. By Proposition 7.3, ds^2 is complete. By Proposition 8.5, ds^2 satisfies the hypotheses of Theorem 8.1. Therefore,

$$\mathcal{H}_2^i(D) = 0$$

for $i \neq n$ and hence

$$\mathcal{H}_2^{p,q}(D) = 0$$

for $p + q \neq n$. Also, by Proposition 8.6, ds^2 satisfies the hypotheses of Theorem 8.2. Therefore, by Remark 8.3,

$$\dim \mathcal{H}_2^{p,q}(D) = \infty$$

for any positive integers p and q with $p + q = n$. Moreover, it is evident that $\mathcal{H}_2^{n,0}(D)$ and $\mathcal{H}_2^{0,n}(D)$ are infinite dimensional. This completes the proof. □

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