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# On Degree Growth and Stabilization of Three-Dimensional Monomial Maps

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## 1. Introduction

Given a rational self-map $f: X \dashrightarrow X$ on an $n$-dimensional Kähler manifold $X$, one can define a pullback map $f^*: H^{p,p}(X) \to H^{p,p}(X)$ for $0 \le p \le n$. In general, the pullback does not commute with iteration; that is, $(f^*)^k \neq (f^k)^*$. Following Sibony ([Si]; see also [FoSi]), we call the map $f$ (algebraically) *stable* if the action on the cohomology of $X$ is compatible with iterations. More precisely, $f$ is called *p-stable* if the pullback on $H^{p,p}(X)$ satisfies $(f^*)^k = (f^k)^*$ for all $k \in \mathbb{N}$.

If $f$ is not $p$-stable on $X$, one might try to find a birational change of coordinate $h: X' \dashrightarrow X$ such that $\tilde{f} = h^{-1} \circ f \circ h$ is $p$-stable. This is not always possible even for $p = 1$, as shown by Favre [Fa]. However, for $p = 1$ and $n = 2$, one can find such a stable model (with at worst quotient singularities) for quite a few classes of surface maps [DFa; FaJ]. Also for $p = 1$, such a model can be obtained for certain monomial maps [Fa; JW; L].

For an $n \times n$ integer matrix $A = (a_{i,j})$, the associated monomial map $f_A: (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is defined by

$$f_A(x_1, \dots, x_n) = \left( \prod_j x_j^{a_{1,j}}, \dots, \prod_j x_j^{a_{n,j}} \right).$$

The morphism $f_A$ extends to a rational map, which is also denoted $f_A$, on any $n$-dimensional toric variety. The question of finding a stable model for $f_A$ (or showing that there is no stable model for certain $f_A$) has been studied in [Fa; JW; L]. In particular, the stabilization problem is fully classified for dimension two in [Fa] and [JW].

In this paper, we focus on the case when $n = 3$ and $A$ is diagonalizable. We deal with both the 1-stable and the 2-stable problems. There are more cases than dimension two. A main case where a model that is both 1-stable and 2-stable can be obtained by performing proper modification is summarized in the following theorem.

Theorem 1.1. *Let $\Delta$ be a fan in $N \cong \mathbb{Z}^3$, and let $f_A: X(\Delta) \dashrightarrow X(\Delta)$ be the monomial map associated to $A$. Suppose that $A$ is diagonalizable and that, for each eigenvalue $\mu$ of $A$, $\mu/\bar{\mu}$ is a root of unity. Then there exists a complete*

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*simplicial refinement $\Delta'$ of $\Delta$ and $k_0 \in \mathbb{N}$ such that the map $f_A^k : X(\Delta') \dashrightarrow X(\Delta')$ is both 1-stable and 2-stable for all $k \geq k_0$.*

We note that even better results will hold in certain subcases. For example, if the eigenvalues have different modulus then we can make $X(\Delta')$ smooth and projective. Another particularly nice subcase is when the set $\{A^k \mid k \in \mathbb{N}\}$ is a finite set; then we can find a smooth projective $X(\Delta')$ such that $f_A$ is an automorphism on $X(\Delta')$. This is related to the resolution of indeterminacy of pairs for birational maps (see Sections 3–5 for more details).

The case when there are two complex eigenvalues $\mu, \bar{\mu}$ of $A$ with $\mu/\bar{\mu}$ *not* a root of unity is more complicated (see Section 6). This case contains several interesting subcases. For instance, the following phenomena are possible.

- The map $f_A$ can be 1-stable on a smooth projective variety while not having a 2-stable model (and vice versa).
- Given a toric variety $X$, there can be a birational model for $f_A$ that is stable even when $f_A$ cannot be made stable by performing the blowup process on $X$.
- The map $f_A$ may have no 1-stable model and also no 2-stable model.

The rest of this paper is organized as follows. In Section 2 we review known facts about toric varieties and monomial maps, after which we develop some technical tools about stabilization and the degree sequence in Section 3. A case that is related to the resolution of indeterminacy of pairs is studied in Section 4, and Theorem 1.1 is proved in Section 5. Finally, in Section 6 we discuss the more complicated case of two complex eigenvalues $\mu, \bar{\mu}$ of $A$ with $\mu/\bar{\mu}$ not a root of unity.

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## 2. Preliminaries on Toric Varieties

A toric variety is a (partial) compactification of the torus $T \cong (\mathbb{C}^*)^m$, which contains $T$ as a dense subset and which admits an action of $T$ that extends the natural action of $T$ on itself. We briefly recall some of the basic definitions. All results stated in this section are known, so the proofs are omitted. We refer the reader to [F] and [O] for details about toric varieties.

### *2.1. Fans and Toric Varieties*

Let $N$ be a lattice isomorphic to $\mathbb{Z}^n$ and let $M = \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice. The algebraic torus $T = T_N \cong (\mathbb{C}^*)^n$ is canonically identified with the group $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. Set $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and define $M_{\mathbb{Q}}$ and $M_{\mathbb{R}}$ analogously. Let $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the sets of nonnegative and nonpositive numbers, respectively.

A *convex rational polyhedral cone* $\sigma$ is of the form $\sigma = \sum \mathbb{R}_+ v_i$ for some $v_i \in N$. We will simply say that $\sigma$ is a cone *generated* by the vectors $v_i$. If $\sigma$ is convex and does not contain any line in $N_{\mathbb{R}}$, then it is said to be *strictly convex.* A *face* of $\sigma$ is the intersection of $\sigma$ and a supporting hyperplane. The *dimension* of $\sigma$ is the dimension of the linear space spanned by $\sigma$. One-dimensional cones are called *rays.* Given a ray $\gamma$, the associated *ray generator* is the first nonzero lattice point on $\gamma$. A $k$-dimensional cone is *simplicial* if it can be generated by $k$ vectors. A cone is *regular* if it is generated by part of a basis for $N$.

A *fan* $\Delta$ in $N$ is a finite collection of rational strongly convex cones in $N_{\mathbb{R}}$ such that (a) each face of a cone in $\Delta$ is also a cone in $\Delta$ and (b) the intersection of two cones in $\Delta$ is a face of each of them. Let $\Delta(k)$ denote the set of cones in $\Delta$ of dimension $k$. A fan $\Delta$ determines a toric variety $X(\Delta)$ by patching together affine toric varieties $U_\sigma$ corresponding to the cones $\sigma \in \Delta$. The *support* $|\Delta|$ of the fan $\Delta$ is the union of all cones of $\Delta$. In fact, given any collection of cones $\Sigma$, we will use $|\Sigma|$ to denote the union of the cones in $\Sigma$. If $|\Delta| = N_{\mathbb{R}}$, then the fan $\Delta$ is said to be *complete*. If all cones in $\Delta$ are simplicial then $\Delta$ is said to be *simplicial,* and if all cones are regular then $\Delta$ is said to be *regular.* A fan $\Delta'$ is a *refinement* of $\Delta$ if, for each cone $\sigma'$ in $\Delta'$, there is a cone $\sigma \in \Delta$ such that $\sigma' \subset \sigma$.

A toric variety $X(\Delta)$ is compact if and only if $\Delta$ is complete. If $\Delta$ is simplicial, then $X(\Delta)$ has at worst quotient singularities; that is, $X(\Delta)$ is an orbifold. Also, $X(\Delta)$ is smooth (nonsingular) if and only if $\Delta$ is regular. For any fans $\Delta_1$ and $\Delta_2$ in $N$, there is a common refinement such that there exist resolutions of singularities $X(\Delta') \to X(\Delta_j)$. In particular, $X(\Delta_1)$ and $X(\Delta_2)$ are birationally equivalent.

### 2.2. Monomial Maps and the Condition for Being Stable

Suppose $A\colon N \to N'$ is a homomorphism of lattices, $\Delta$ is a fan in $N$, and $\Delta'$ is a fan in $N'$. A homomorphism of lattices $A\colon N \to N'$ induces a group homomorphism $f_A\colon T_N \to T_{N'}$ that is given by monomials on each coordinate. One can extend $f_A$ to an *equivariant rational map* $f_A\colon X(\Delta) \dashrightarrow X(\Delta')$. On a complete toric variety, $f_A$ is *dominant* if and only if $A_{\mathbb{R}} = (A \otimes \mathbb{R})\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$ is surjective. The map $f_A$ is called *semisimple* if $A$ is diagonalizable. Given a cone $\sigma \in \Delta$, we say that $\sigma$ *maps regularly* to $\Delta'$ by $A$ if there is a cone $\sigma' \in \Delta'$ such that $A(\sigma) \subseteq \sigma'$. In this case, we call the smallest such cone in $\Delta'$ the *cone closure* of the image of $\sigma$ and denote it by $\overline{A(\sigma)}$.

For a complete toric variety $X(\Delta)$ associated to a complete fan $\Delta$, the group of torus-invariant Cartier divisors on $X(\Delta)$ is denoted by $\mathrm{CDiv}_T(X(\Delta))$ and the *Picard group* is denoted by $\mathrm{Pic}(X(\Delta))$. Given a monomial map $f_A\colon X(\Delta) \dashrightarrow X(\Delta')$, we can define a *pullback map* $f_A^*\colon \mathrm{Pic}(X(\Delta)) \to \mathrm{Pic}(X(\Delta'))$ as well as the pullback map $f_A^*\colon \mathrm{CDiv}_T(X(\Delta)) \to \mathrm{CDiv}_T(X(\Delta'))$. A toric rational map $f_A\colon X(\Delta) \dashrightarrow X(\Delta)$ is *strongly 1-stable* if $(f_A^k)^* = (f_A^*)^k$ as maps of $\mathrm{CDiv}_T(X(\Delta))_{\mathbb{Q}}$ for all $k \in \mathbb{N}$; it is simply *1-stable* if $(f_A^k)^* = (f_A^*)^k$ as maps of $\mathrm{Pic}(X(\Delta))_{\mathbb{Q}}$ for all $k$. On a projective toric variety, $f_A$ is strongly 1-stable if and only if it is 1-stable. The following theorem gives a geometric condition for being strongly 1-stable.

THEOREM 2.1. *A toric rational map $f_A\colon X(\Delta) \dashrightarrow X(\Delta)$ is strongly 1-stable if and only if, for all rays $\tau \in \Delta(1)$ and all $n \in \mathbb{N}$, $\overline{A^n(\tau)}$ maps regularly to $\Delta$ by $A$.*

For the proof of Theorem 2.1 and for additional details on the 1-stability of monomial maps, see [JW; L]. Notice that what we call “1-stable” here is called “(algebraically) stable” in those papers.

## 3. Degrees and Stabilization

In this section we give various results about the degrees and stabilization of monomial maps. Although they serve as tools to prove Theorem 1.1, these results hold not only in dimension three but also, in many cases, in arbitrary dimensions.

### *3.1. Duality between $p$ and $(n-p)$*

For $A \in M_n(\mathbb{Z})$, let

$$A' = |\det(A)| \cdot A^{-1} = \operatorname{sgn}(\det(A)) \cdot \operatorname{ad}(A),$$

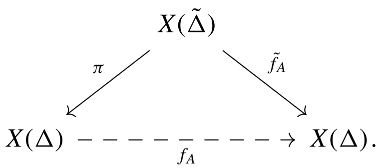
where $\operatorname{sgn}(\cdot)$ is the sign function and $\operatorname{ad}(A)$ is the classical adjoint matrix of $A$. Notice that $A' \in M_n(\mathbb{Z})$ and that, if $\det(A) \neq 0$, then $\det(A') = \det(A)^{n-1}$ is also nonzero.

PROPOSITION 3.1. *For a monomial map $f_A$, its pullback map $f_A^*$ on $H^{p,p}$ is adjoint* (*up to a scalar*) *to the pullback map $f_{A'}^*$ on $H^{n-p,n-p}$ under the intersection pairing.*

*More precisely, let $\langle \cdot, \cdot \rangle$ be the intersection pairing, and let $\alpha \in H^{p,p}(X(\Delta))$ and $\beta \in H^{n-p,n-p}(X(\Delta))$. Then*

$$\langle f_A^* \alpha, \beta \rangle = |\det(A)|^{p-n+1} \cdot \langle \alpha, f_{A'}^* \beta \rangle.$$

*Proof.* First assume that $X(\Delta)$ is simplicial and projective. Under this assumption, $H^{*,*}(X(\Delta)) \cong A^*(X(\Delta))$ is generated by $\operatorname{Pic}(X(\Delta))_{\mathbb{R}}$. Moreover, every divisor in a projective variety is a difference of two ample divisors. Therefore, it suffices to prove the equation for products of ample divisors of $X(\Delta)$. Recall the following diagram for toric rational map $f_A$:




Let $\alpha = [D_1]. \cdots .[D_p]$ and $\beta = [D_{p+1}]. \cdots .[D_n]$, where $D_i$ is an ample divisor with associated polytope $P_i$, $i = 1, \ldots, n$. Then, as a consequence of the Riemann–Roch theorem for toric varieties (see [F, pp. 116–117]), we have

$$\begin{aligned}
\langle f_A^* \alpha, \beta \rangle &= \pi_* \tilde{f}_A^*(D_1. \cdots .D_p).(D_{p+1}. \cdots .D_n) \\
&= \tilde{f}_A^*(D_1. \cdots .D_p).\pi^*(D_{p+1}. \cdots .D_n) \\
&= n! \cdot \operatorname{MV}({}^{\mathrm{t}}AP_1, \ldots, {}^{\mathrm{t}}AP_p, P_{p+1}, \ldots, P_n) \\
&= n! \cdot |\det({}^{\mathrm{t}}A)| \cdot \operatorname{MV}(P_1, \ldots, P_p, {}^{\mathrm{t}}A^{-1}P_{p+1}, \ldots, {}^{\mathrm{t}}A^{-1}P_n)
\end{aligned}$$

$$= |\det({}^{\mathrm{t}}A)|^{p-n+1} \cdot n! \cdot \mathrm{MV}(P_1, \dots, P_p, {}^{\mathrm{t}}A' P_{p+1}, \dots, {}^{\mathrm{t}}A' P_n)$$
$$= |\det({}^{\mathrm{t}}A)|^{p-n+1} \cdot \langle \alpha, f_{A'}^* \beta \rangle.$$

Here $\mathrm{MV}(\cdot)$ denotes the mixed volume of polytopes.

Finally, for a general toric variety $X(\Delta)$, we can subdivide $\Delta$ to obtain a refinement $\Delta'$ such that $X(\Delta')$ is simplicial and projective. The induced map $A^*(X(\Delta)) \to A^*(X(\Delta'))$ will be injective, and the formula for $X(\Delta)$ follows from the result for $X(\Delta')$. □

Two consequences of Proposition 3.1 are described by the next two corollaries.

Corollary 3.2. *The map $f_A$ is $p$-stable if and only if $f_{A'}$ is $(n-p)$-stable.*

*Proof.* We have

$$(f_A^*)^k = (f_A^k)^* \iff \langle (f_A^*)^k \alpha, \beta \rangle = \langle (f_A^k)^* \alpha, \beta \rangle \quad \forall \alpha \in H^{p,p},\ \forall \beta \in H^{n-p,n-p}$$
$$\iff \langle \alpha, (f_{A'}^*)^k \beta \rangle = \langle \alpha, f_{(A^k)'}^* \beta \rangle$$
$$\iff (f_{A'}^*)^k = (f_{A'}^k)^*.$$

The last implication follows because we have $(A^k)' = (A')^k$ for all $k$. Thus the corollary is proved. □

Given an ample divisor $D$ on a projective toric variety $X(\Delta)$, we define the $p$th degree of $f_A$ with respect to $D$, denoted $\deg_{D,p}(f_A)$, as

$$\deg_{D,p}(f_A) = \langle f_A^* D^p, D^{n-p} \rangle.$$

Because the intersection pairing is symmetric, we have the following relation between the $p$th degree and the $(n-p)$th degree of a monomial map with respect to $D$.

Corollary 3.3. *For any ample divisor $D$,*

$$\deg_{D,p}(f_A) = |\det(A)|^{p-n+1} \deg_{D,n-p}(f_{A'}).$$

In particular, for $p = n-1$ we have $\deg_{D,n-1}(f_A) = \deg_{D,1}(f_{A'})$.

### *3.2. Stability and Linear Recurrence of the Degree Sequence*

The following proposition describes the relation between being $p$-stable and the $p$th degree satisfying a linear recurrence relation. This result is probably well known to experts; we include it here for completeness.

Proposition 3.4. *Given a projective simplicial toric variety $X = X(\Delta)$, suppose that the monomial map $f_A$ is $p$-stable on $X$. Then, for any $T_N$-invariant ample divisor $D$ of $X$, the degree sequence $\{\deg_{D,p}(f_A^k)\}_{k=1}^{\infty}$ satisfies a linear recurrence relation.*

*Proof.* Since $D^n = n!\,\mathrm{Vol}(P_D) > 0$ for an ample divisor $D$, we know that the cohomology class $[D^p]$ is nonzero in $H^{p,p}(X)$. Hence we can extend $[D^p]$ to form a basis $\mathcal{B} = \{[D^p], \theta_1, \ldots, \theta_r\}$ for $H^{p,p}(X)$ such that $D^{n-p}.\theta_i = 0$ for all $i = 1, \ldots, r$. Then $\deg_{D,p}(f_A)$ is the $(1,1)$-entry of the matrix $\mathcal{A}$ for $f_A^*$ with respect to $\mathcal{B}$. Since $f_A$ is $p$-stable (i.e., $(f_A^k)^* = (f_A^*)^k$ for all $k \in \mathbb{N}$), we know that $\deg_{D,p}(f_A^k)$ is the $(1,1)$-entry of the matrix $\mathcal{A}^k$. Therefore, by the Cayley–Hamilton theorem, the sequence $\{\deg_{D,p}(f_A^k)\}_{k=1}^{\infty}$ satisfies the linear recurrence relation induced by the characteristic polynomial of $\mathcal{A}$. □

In practice, however, it is usually difficult to make a birational change of coordinate to make the map stable—even for $p = 1$. Instead, we impose the following slightly weaker condition. This is a "stable version of $p$-stable"; thus we write $p$-SS (stably stable) for the condition.

($p$-SS) There exists an integer $k_0 \geq 1$ such that, for all $k, l \geq k_0$, we have $(f^{k+l})^* = (f^k)^* \circ (f^l)^*$.

REMARK. We have the following consequences of $p$-SS.

(1) For all $k \geq k_0$, the map $f^k$ is $p$-stable.
(2) The degree sequences $\{\deg_{D,p}(f_A^{kj+l})\}_{j=1}^{\infty}$ satisfy a (same) linear recurrence relation for all $k, l \geq k_0$. The proof is similar to the proof of Proposition 3.4. Thus, the sequence $\{\deg_{D,p}(f_A^k)\}_{k=1}^{\infty}$ also satisfies a linear recurrence relation.

### *3.3. Simultaneous Stabilization*

Given an $n \times n$ matrix $A \in M_n(\mathbb{R})$ with $\det(A) \neq 0$, let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $A$, counting multiplicities. We say that $A$ has *isolated spectrum* if $\mu_i \neq \mu_j$ for $i \neq j$ and that $A$ has *absolutely isolated spectrum* if $|\mu_i| \neq |\mu_j|$ for $i \neq j$.

THEOREM 3.5. *Let $\Delta$ be a fan in $N \cong \mathbb{Z}^n$, and let $A_1, \ldots, A_m \in M_n(\mathbb{Z})$ be matrices with absolutely isolated spectrum. Then there exists a complete refinement $\Delta'$ of $\Delta$ such that $X(\Delta')$ is smooth and projective and the induced maps $f_{A_i}: X(\Delta') \dashrightarrow X(\Delta')$ are 1-SS for all $i = 1, \ldots, m$.*

The $m = 1$ case of the theorem is proved in [JW, Thm. A′]. The proof here basically follows theirs, with a slight modification. So we begin by briefly recalling their proof.

In [JW], the authors introduced the notion of *adapted systems of cones,* which is a collection of simplicial cones satisfying certain conditions. We list some properties of adapted systems of cones in the following lemmas. For details, see [JW, Sec. 4].

LEMMA 3.6 [JW, Lemma 4.5]. *Let $\mathcal{S} = \{\sigma(V, \eta)\}$ be an adapted system of cones, and let $v \in N$. Then there exists a $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, $A^k(v) \in \sigma(V, \eta)$ for some $\sigma(V, \eta) \in \mathcal{S}$.*

LEMMA 3.7 [JW, Lemma 4.6]. *Let $\mathcal{S} = \{\sigma(V, \eta)\}$ be an adapted system of cones, and let $v \in N$. Then there exists a $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, $\mathcal{S}$ is invariant under $A^k$.*

Lemma 3.8 [JW, Lemma 4.11]. *Let $\Delta$ be a fan in $N$ that contains an adapted system of cones, and let $\Delta'$ be a refinement of $\Delta$. For every invariant rational subspace $V$ of $N_{\mathbb{R}}$, suppose there is a subfan of $\Delta'$ with support $V$. Then $\Delta'$ contains a unique adapted system of cones.*

Lemma 3.9 [JW, Lemma 4.12]. *There exists a rational adapted system of cones for every $A \in M_n(\mathbb{Z})$ with absolutely isolated spectrum.*

*Proof of Theorem 3.5.* First, we refine the fan $\Delta$ so that, for each rational invariant subspace $V$ of each $A_i$, there is a subfan of $\Delta$ whose support is $V$.

Next, we refine $\Delta$ so that it contains an adapted system of cones for each $A_i$. We achieve this goal as follows. Let $\mathcal{S}_i$ be a rational adapted system of cones for $A_i$; its existence is guaranteed by Lemma 3.9. Let $\Delta_i$ be a complete fan generated by cones in $\mathcal{S}_i$. Now take a common refinement of $\Delta$ and all $\Delta_i$, and then take a further refinement to make the fan both regular and projective. Call this refined fan $\Delta'$. By Lemma 3.8, $\Delta'$ contains a unique adapted system of cones $\mathcal{S}_i'$ for each $A_i$.

Finally, by Lemma 3.6, there is a number $k_0$ such that, for all $k \geq k_0$ and $\gamma \in \Delta'(1)$, we have $A_i^k(\gamma) \subset \sigma$ for some $\sigma \in \mathcal{S}_i'$. Furthermore, by Lemma 3.7, we can choose an even larger $k_0$ such that, for all $k \geq k_0$, each $\mathcal{S}_i'$ is invariant under $A_i^k$. These conditions will imply that each $A_i$ is 1-SS, which concludes our proof. □

Corollary 3.10. *Let $\Delta$ be a fan in $N \cong \mathbb{Z}^n$, and let $A \in M_n(\mathbb{Z})$ be a matrix with absolutely isolated spectrum. Then there exists a complete refinement $\Delta'$ of $\Delta$ such that $X(\Delta')$ is smooth and projective and such that the induced map $f_A \colon X(\Delta') \dashrightarrow X(\Delta')$ is both 1-SS and $(n-1)$-SS.*

*Proof.* By Corollary 3.2, the claim follows from applying $A$ and $A'$ to Theorem 3.5. □

### *3.4. Stabilization in a Special Case*

Our next proposition addresses a case that arises when we prove Theorem 1.1. We prove a version of it that is valid in all dimensions.

Proposition 3.11. *Let $\Delta$ be a fan in $N \cong \mathbb{Z}^n$, and let $A \in M_n(\mathbb{Z})$ be a diagonalizable matrix whose eigenvalues are two distinct real numbers $\mu_1, \mu_2 \in \mathbb{R}$ (each with some multiplicities) such that $|\mu_1| \neq |\mu_2|$. Then there exists a complete refinement $\Delta'$ of $\Delta$ such that $X(\Delta')$ is simplicial and projective and such that the induced map $f_A \colon X(\Delta') \dashrightarrow X(\Delta')$ is both 1-stable and $(n-1)$-stable.*

*Proof.* For each ray $\gamma \in \Delta(1)$, consider its ray generator $v$. Write $v = v_1 + v_2$, where $v_i$ is an eigenvector of $\mu_i$ for $i = 1, 2$. If one of the $v_i$ is zero, then $\gamma$ is actually invariant under $A$. Now suppose that both $v_1$ and $v_2$ are nonzero, and let $\Gamma_\gamma := \operatorname{span}\{v_1, v_2\}$ be the two-dimensional vector space spanned by $v_1$ and $v_2$. Then $\Gamma_\gamma$ is invariant under $A$ (i.e., $A(\Gamma_\gamma) = A^{-1}(\Gamma_\gamma) = \Gamma_\gamma$) and $A^k(\gamma) \subset \Gamma_\gamma$ for all $k \in \mathbb{Z}$.

Observe that $\Gamma_\gamma \cap N$ has rank 2, since both $v$ and $A(v)$ are in $\Gamma_\gamma \cap N$ and are independent. Thus, $\Gamma_\gamma \cap \Delta := \{\Gamma_\gamma \cap \sigma \mid \sigma \in \Delta\}$ is a fan of rational polyhedral cones with support $\Gamma_\gamma$. We can refine $\Gamma_\gamma \cap \Delta$ to obtain a refinement $\Delta_\gamma$ such that, under the map $A|_{\Gamma_\gamma}$, each ray in $\Delta_\gamma$ either maps to another ray or maps into a two-dimensional cone $\sigma$ of $\Delta_\gamma$ with $A(\sigma) \subset \sigma$. Moreover, we can find a refinement $\Delta_\gamma$ such that the same requirement holds for $A'$ as well.

Now take a common simplicial refinement of $\Delta$ and $\Delta_\gamma$, $\gamma \in \Delta(1)$, without adding rays, to obtain a new fan $\Delta'$. We then have $\Delta'(1) = \bigcup_{\gamma \in \Delta(1)} \Delta_\gamma(1)$ and so, under the map $A$, each ray in $\Delta'$ is either invariant or eventually maps to some two-dimensional cone that is invariant. Thus $f_A$ is 1-stable. Similarly, we know $f_{A'}$ is 1-stable; hence $f_A$ is $(n-1)$-stable, too. □

More generally, we now assume that $A$ is diagonalizable and that there exist $r_1 > r_2 > 0$ such that, for each eigenvalue $\mu$ of $A$, $\mu/\bar{\mu}$ is a root of unity and either $|\mu| = r_1$ or $|\mu| = r_2$. Then there exists some $\ell$ such that $A^\ell$ satisfies the conditions of Proposition 3.11. For any ray $\gamma$ with generator $v$, let $\Gamma_\gamma$ be the plane spanned by $v$ and $A^\ell(v)$; then the orbit $\{A^k(v) \mid k \in \mathbb{Z}\}$ is contained in finitely many planes $A^k(\Gamma_\gamma), k = 0, \ldots, \ell-1$. We can then simultaneously refine the fans $A^k(\Gamma_\gamma) \cap \Delta$ so that, under $A$, each ray in the subdivided fan either maps into another ray or maps into a two-dimensional cone $\sigma_0$; in this case there exist two-dimensional cones $\sigma_1, \ldots, \sigma_\ell = \sigma_0$ in the refined cones such that $A(\sigma_k) \subset \sigma_{k+1}$ for $k = 0, \ldots, \ell - 1$. This implies that $A^\ell(\sigma_k) \subset \sigma_k$. We can also achieve the same requirement for $A'$ simultaneously, as in the proof of Proposition 3.11. Now take a common refinement of $\Delta$ with all these refined fans, without adding more rays, to obtain a new fan $\Delta'$ such that $f_A$ is both 1-stable and $(n-1)$-stable on $X(\Delta')$.

## 4. Resolution of Indeterminacy of Pairs for Toric Varieties

Let $X$ be a projective variety and let $G$ be a finite subgroup of the group of birational transformations $\mathrm{Bir}(X)$. A resolution of indeterminacy of the pair $(X, G)$ consists of a smooth variety $X'$, birationally equivalent to $X$, and a birational map $\pi : X' \to X$ such that, for every $g \in G$, the composite map $\pi^{-1} g \pi$ is an automorphism of $X$.

In a paper of de Fernex and Ein [dFE], the authors show that in characteristic 0 the resolution of indeterminacy of a pair $(X, G)$ always exists. They also obtain an explicit construction of the resolution in some two-dimensional cases. Also, in Chel′tsov's paper [Ch], an explicit construction of resolution of indeterminacy of pairs is given in dimension three using the minimal model program.

In this section, we first recall a known proposition implying the resolution of indeterminacy of pairs in the case of toric varieties and equivariant birational maps. Then we will modify the condition slightly, obtaining a theorem that can be applied to a case in our classification.

PROPOSITION 4.1 [B; CHS]. *Let $N$ be a lattice and let $\Delta \cong \mathbb{Z}^n$ be a fan of $N_\mathbb{R}$. Let $G$ be a finite group of automorphisms of $N$. Then there exists a refinement $\Delta'$ of $\Delta$ that is smooth, projective, and invariant by $G$.*

In [B; CHS], the authors use this result to show the existence of smooth projective compactification of an algebraic torus over a scheme. The proposition also gives an explicit way to construct the resolution of indeterminacy of pairs in the case of toric varieties. Moreover, this explicit resolution works for all characteristic.

THEOREM 4.2 (Resolution of Indeterminacy of Pairs). *Let $X(\Delta)$ be a toric variety, and let $G$ be a finite group of equivariant birational maps of $X(\Delta)$. Then there exist a smooth projective toric variety $X(\Delta')$ and a birational morphism $\pi: X(\Delta') \to X$ such that, for each $g \in G$, the composite map $\pi^{-1}g\pi$ is an equivariant automorphism of $X(\Delta')$. In other words, elements in $G$ are birational conjugate to automorphisms of $X(\Delta')$.*

In order to suit our general purpose, however, we shall need a different condition. Recall that the set of dominant monomial maps of $n$-dimensional toric varieties corresponds to the set of integer matrices with nonzero determinant:

$$M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q}) = \{A \in M_n(\mathbb{Z}) \mid \det(A) \neq 0\}.$$

For matrices of the form $d \cdot I_n$ with $d > 0$, every cone is invariant under the action of $d \cdot I_n$. Thus every fan is invariant, too. Denote by $\mathbb{Q}^+$ the set of positive rational numbers and identify $\mathbb{Q}^+$ with $\mathbb{Q}^+ \cdot I_n$. Let $\mathfrak{F}$ be the set of complete rational fans in $N_{\mathbb{R}}$. Then $\mathrm{GL}_n(\mathbb{Q})$ acts on $\mathfrak{F}$ and

$$\mathbb{Q}^+ = \bigcap_{\Delta \in \mathfrak{F}} \mathrm{GL}_n(\mathbb{Q})_\Delta,$$

where $\mathrm{GL}_n(\mathbb{Q})_\Delta$ is the stabilizer of $\Delta \in \mathfrak{F}$ in $\mathrm{GL}_n(\mathbb{Q})$.

Let $p: \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{PGL}_n^+(\mathbb{Q}) := \mathrm{GL}_n(\mathbb{Q})/\mathbb{Q}^+$ be the projection map.

PROPOSITION 4.3. *Let $N$ be a lattice and let $\Delta \cong \mathbb{Z}^n$ be a fan of $N_{\mathbb{R}}$. Let $G$ be a submonoid of $M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$. If $p(G)$ is finite, then there exists a refinement $\Delta'$ of $\Delta$ that is projective and invariant by $G$.*

*Proof.* First, take all the hyperplanes in $N_{\mathbb{R}}$ that contain some of the $(n-1)$-dimensional cones of $\Delta$. These hyperplanes determine a complete refinement $\Delta_1$ of $\Delta$. Moreover, the corresponding variety $X(\Delta)$ is projective (see [O, Prop. 2.17]).

Then we consider the fan

$$\Delta' = \bigcap_{g \in G} g\Delta_1,$$

where $g\Delta_1 = \{g\sigma \mid \sigma \in \Delta_1\}$. The intersection is finite because $p(G)$ is a finite set. Thus $\Delta'$ is a finite fan. Moreover, the fan $\Delta'$ is invariant under $G$, and it is also projective since it contains all the hyperplanes spanned by its $(n-1)$-dimensional cones. □

One can translate Proposition 4.3 into the following toric version.

PROPOSITION 4.4. *Let $X(\Delta)$ be a toric variety, and let $G$ be a submonoid of $M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ with $p(G)$ finite. Then there exist a projective toric variety $X(\Delta')$ and a birational morphism $\pi: X(\Delta') \to X$ such that, for each $A \in G$, the composite map $\pi^{-1} \circ f_A \circ \pi$ is an equivariant morphism of $X(\Delta')$.*

REMARK. In the proof of Proposition 4.3, one can make a simplicial refinement $\Delta''$ of $\Delta'$ such that $\Delta''(1) = \Delta'(1)$. Then we obtain a projective, simplicial toric variety on which $f_A$ is 1-stable for all $A \in G$.

COROLLARY 4.5. *Let $X(\Delta)$ be a toric variety and let $A \in M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ be diagonalizable. Assume that all the eigenvalues of A are of the same modules and that $\mu/\bar{\mu}$ is a root of unity for each eigenvalue $\mu$. Then*

(1) *there exist a projective toric variety $X(\Delta')$ and a birational morphism $\pi : X(\Delta') \to X$ such that $f_A$ is conjugate to a morphism on $X(\Delta')$; and*
(2) *there exist a projective, simplicial toric variety $X(\Delta'')$ and a birational morphism $\pi : X(\Delta'') \to X$ such that the conjugate $\tilde{f}_A$ is both 1-stable and 2-stable on $X(\Delta'')$.*

*Proof.* For part (1), apply Proposition 4.4 to the monoid generated by $A$. For (2), pick $\Delta''$ to be the fan in the Remark after Proposition 4.4 for $G$ to be the monoid generated by $A$. Recall that $A' = |\det(A)| \cdot A^{-1}$ and that the fan $\Delta''$ satisfies $-\Delta'' = \Delta''$. Thus the rays of $\Delta''$ still map to rays of $\Delta''$, which means that $X(\Delta'')$ is 1-stable for both $\tilde{f}_A$ and $\tilde{f}_{A'}$. Therefore, $\tilde{f}_A$ is also 2-stable on $X(\Delta'')$. This concludes the proof of part (2). □

## 5. Proof of Theorem 1.1

Under the assumptions of Theorem 1.1, there are four cases: $|\mu_1| > |\mu_2| > |\mu_3|$, $|\mu_1| = |\mu_2| > |\mu_3|$, $|\mu_1| > |\mu_2| = |\mu_3|$, and $|\mu_1| = |\mu_2| = |\mu_3|$.

The first case is proved in Corollary 3.10. Indeed, in this case the refinement $\Delta'$ can be regular and projective. The second case is shown in the discussion after Proposition 3.11. The third case is dual to the second case and is thus also accounted for.

Finally, the case $|\mu_1| = |\mu_2| = |\mu_3|$ is proved in Corollary 4.5. Thus we have proved all cases for Theorem 1.1.

## 6. Conjugate Pair of Eigenvalues Whose Argument Is an Irrational Multiple of $2\pi$

When there are two complex eigenvalues $\mu, \bar{\mu}$ with $\mu/\bar{\mu}$ not a root of unity, there are three possible cases. Assume that $\nu$ is the third (real) eigenvalue; then we can have $|\mu| > |\nu|$, $|\mu| < |\nu|$, or $|\mu| = |\nu|$. The first and the second cases are dual to each other. In fact, if $A$ is in the first case then $A'$ will be in the second, and vice versa. Hence we need only consider the 1-stability problem for each case. The results for 2-stability follow by duality.

### *6.1. Case 1: $|\mu| > |\nu|$*

For the first case, the following theorem shows that we cannot make $f_A$ 1-stable.

THEOREM [L, Thm. 4.7(2)]. *Suppose that $A \in \mathbf{M}_n(\mathbb{Z})$ is a integer matrix. If $\mu, \bar{\mu}$ are the only eigenvalues of A that have maximal modulus with algebraic multiplicity 1 and if $\mu/\bar{\mu}$ is not a root of unity, then there is no toric birational model that makes $f_A$ strongly algebraically stable.*

Next, notice that the 2-stabilization problem for the case $|\mu| > |\nu|$ is equivalent to the 1-stabilization problem for the case $|\mu| < |\nu|$.

### 6.2. *Case 2:* $|\mu| < |\nu|$

We also consider the 1-stabilization problem for this case. First, if we do not start with any given toric variety, then we can certainly find some simplicial toric variety $X(\Delta)$ such that $f_A$ is 1-SS on $X(\Delta)$ (see [L, Thm. 4.7(1)]). However, the situation is more complicated when we are given a fixed toric variety $X(\Delta)$ and want to stabilize $f_A$ by refining $\Delta$. We need to consider several subcases.

Under the prevailing assumptions, let $v$ be an eigenvector associated with $\nu$ and let $\gamma = \mathbb{R}_+ v$. Let $\Gamma$ be the two-dimensional invariant subspace associated with the eigenvalues $\mu, \bar{\mu}$. First, we need a lemma.

LEMMA 6.1. *Suppose that $\Delta$ satisfies the following conditions*:

- *$\gamma$ lies on a lower-dimensional cone* (*i.e., a cone of dimension* $\leq 2$); *and*
- *there exists a ray $\gamma_1 \in \Delta(1), \gamma_1 \neq \gamma$, such that $A^k \gamma_1 \to \gamma$ as $k \to \infty$* (*this means that the angle between $\gamma_1$ and $\gamma$ approaches zero as $k$ approaches infinity*).

*Then, for any refinement $\Delta'$ of $\Delta$, $f_A$ is not 1-stable on $\Delta'$ and so we cannot stabilize $f_A$ by subdividing $\Delta$.*

*Proof.* Observe that, for any refinement $\Delta'$ of $\Delta$, $\gamma$ still lies on a lower-dimensional cone. Moreover, for any refinement $\Delta'$, we still have $\gamma_1 \in \Delta'(1)$ and $A^k \gamma_1 \to \gamma$. Hence it suffices to show that $f_A$ is not 1-stable on $X(\Delta)$.

Under $A$, these two-dimensional cones with $\gamma$ as a face rotate around $\gamma$ with an irrational rotating angle because $\gamma$ is invariant under $A^k$ for all $k$. Thus, none of the three-dimensional cones with a face $\gamma$ can be mapped regularly under all $A^k$. However, since $A^k \gamma_1 \to \gamma$ it follows that, for large $k$, $A^k \gamma_1$ must be in the interior of some three-dimensional cone with a face $\gamma$. Therefore, the condition for 1-stability cannot be satisfied, and we conclude that $f_A$ is not 1-stable on $X(\Delta)$. □

Now we are ready to study all the subcases of Case 2.

(i) Suppose that both $\gamma$ and $-\gamma$ are in the interior of some three-dimensional cones and that there is no ray of $\Delta$ in $\Gamma$. In this case, if we refine $\Delta$ to make both the three-dimensional cones containing $v$ and $-v$ (say, $\sigma_1$ and $\sigma_2$) lying on one side of $\Gamma$, then—after certain iterates of $A$—we know that one of the following statements holds for all $\ell$ large enough:

- $A^\ell(\sigma_1) \subset A^\ell(\sigma_1)$ and $A^\ell(\sigma_2) \subset A^\ell(\sigma_2)$; or
- $A^\ell(\sigma_1) \subset A^\ell(\sigma_2)$ and $A^\ell(\sigma_2) \subset A^\ell(\sigma_1)$.

Furthermore, every ray in $\Delta$ will map into either $\sigma_1$ or $\sigma_2$ after certain iterates. Thus there exist a refinement $\Delta'$ of $\Delta$ and an $\ell_0 \geq 1$ such that $f_A^\ell$ is 1-stable on $X(\Delta')$ whenever $\ell \geq \ell_0$.

(ii) If there is some ray of $\Delta$ that lies on $\Gamma$, then we can look at $A|_\Gamma$ and apply the same argument as in the proof of [L, Thm. 4.7(2)] to show that $f_A$ cannot be made stable by subdividing $\Delta$.

(iii) If either $\gamma$ or $-\gamma$ lies in the relative interior of a two-dimensional cone, we claim that $f_A$ cannot be made stable by subdividing $\Delta$.

Without loss of generality, assume that $\gamma$ lies in the relative interior of a two-dimensional cone $\sigma$. Suppose $\gamma_1$ and $\gamma_2$ are the one-dimensional faces of $\sigma$. Then, since $\sigma$ is strictly convex, one of $\gamma_1, \gamma_2$ must lie on the same side of $\Gamma$ as $\gamma$; assume it is $\gamma_1$. Hence $A^k\gamma_1 \to \gamma$ as $k \to \infty$ and so, by Lemma 6.1, $f_A$ cannot be made stable by subdividing $\Delta$.

(iv) Suppose that either $\gamma$ or $-\gamma$ is a cone in $\Delta$—say, $\gamma \in \Delta(1)$—and that there is another $\gamma_1 \in \Delta(1)$, $\gamma_1 \neq \gamma$, such that $A^k\gamma_1 \to \gamma$ as $k \to \infty$. Then, again by Lemma 6.1, one cannot make $f_A$ stable by subdividing $\Delta$.

(v) Finally, suppose $\gamma \in \Delta(1)$ but there is no $\gamma_1 \in \Delta(1)$, $\gamma_1 \neq \gamma$, with $A^k\gamma_1 \to \gamma$ (so we are not in case (i) or (iv)), and suppose $-\gamma$ is in the interior of a three-dimensional cone $\sigma$ (so we are not in case (iii)). Moreover, assume that there is no ray of $\Delta$ that lies in $\Gamma$ (so we are not in case (ii)).

Under these assumptions, $\gamma$ is invariant under $A^k$ for any $k$ and, for all $\gamma_1 \in \Delta(1)$, $\gamma_1 \neq \gamma$, we have $A^k\gamma_1 \to -\gamma$ as $k \to \infty$. Thus, $A^k\gamma_1 \in \sigma$ for large $k$. Note that $\gamma$ cannot be one of the one-dimensional faces of $\sigma$ because $\sigma$ is strictly convex. Therefore, the one-dimensional faces of $\sigma$ must also map into $\sigma$ for large $k$. This means that $A^k\sigma \subset \sigma$ for large $k$. To conclude, in this case we can find a $k_0$ such that $A^k$ is 1-stable for $k \geq k_0$.

EXAMPLE 6.1. Let $\Delta$ be the standard fan for $\mathbb{P}^3$, and let

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 4 & 1 & 2 \\ 1 & 5 & 1 \end{pmatrix}.$$

The three eigenvalues of $A$ are 7 and $-2 \pm 2\sqrt{2}i$. Note that the eigenspace associated to 7 is spanned by $v = (1,1,1)$. The ray generated by $-v$ is in $\Delta$ and is invariant under $A$. All other rays of $\Delta$, which are generated by the three standard basis elements, will tend to $\mathbb{R}_+ v$ under $A^k$. Thus we are in Case 2(v). Indeed, $f_A$ is 1-stable on $\mathbb{P}^3$. On the other hand, $A'$ is covered by Case 1 and so $f_A$ is not 2-stable for any complete toric variety.

### 6.3. *Case 3:* $|\mu| = |\nu|$

In this case we claim that, for all complete fans $\Delta$ and all $\ell$, $f_A^\ell$ is neither 1-stable nor 2-stable on $X(\Delta)$.

Observe that, after a (rational) conjugation, the action of $A$ on cones is to rotate along an axis with an angle that is irrational modulo $2\pi$. After passing to an iterate of $A$, we can make the angle as small as we like. However, for any three-dimensional polyhedral cone, if we rotate it along any axis for a small angle then it will not remain in any cone.

There is at least one ray $\gamma$ in $\Delta$ that is not in the eigenspace of $\nu$ and, for some $k$, $A^k\gamma$ will lie in the interior of some three-dimensional cone. By the argument in the previous paragraph, this three-dimensional cone does not always map into another cone. Thus $f_A$ is not 1-stable in $X(\Delta)$.

Finally, $f_{A'}$ will be in this case again and so cannot be made 1-stable. This concludes our claim for Case 3.

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