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Plurisubharmonic Subextensions As Envelopes of Disc Functionals

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1. Introduction

The theory of disc functionals was founded just over twenty years ago. Its main goal is to provide disc formulas for important extremal plurisubharmonic functions in pluripotential theory, that is, to describe these functions as envelopes of disc functionals. This brings the geometry of analytic discs into play in pluripotential theory. Disc formulas have been proved for largest plurisubharmonic minorants (some of the main references, in historical order, are [14; 2; 15; 9; 19; 13]) and for various Green functions (see for example [16; 5; 10; 18; 12]). For recent generalizations to singular spaces, see [3; 4].

We continue this project by proving a disc formula for largest plurisubharmonic subextensions. Consider domains $W \subset X$ in complex affine space $\mathbb{C}^n$ or, more generally, in a Stein manifold. Let $\phi\colon W \to [-\infty, \infty)$ be upper semicontinuous, for example plurisubharmonic, and let $S\phi$ be the supremum of all plurisubharmonic functions u on X with $u|W \le \phi$. If X is covered by analytic discs with boundaries in W, then $S\phi$ is a plurisubharmonic function on X, the largest plurisubharmonic subextension of ϕ to X. Under suitable conditions on W and X, we prove that for every $x \in X$, $S\phi(x)$ is the infimum of the averages of ϕ over the boundaries of all analytic discs in X with boundary in W and center x (Theorems 3 and 4). In general, however, the disc formula can fail (Example 3).

A recent Stein neighborhood theorem of Forstnerič [6, Thm. 1.2] allows us to work with analytic discs that are merely continuous on the closed unit disc $\overline{\mathbb{D}}$. This is technically easier than the traditional approach that uses germs of holomorphic maps from open neighborhoods of $\overline{\mathbb{D}}$.

A new equivalence relation on the space $\mathcal{A}_X^W$ of analytic discs in X with boundary in W naturally appears in the proof of our disc formula. We call analytic discs in $\mathcal{A}_X^W$ *center-homotopic* if they have the same center and can be joined by a path in $\mathcal{A}_X^W$ of discs with that same center. The quotient of $\mathcal{A}_X^W$ by this equivalence relation, if it is Hausdorff, is a complex manifold with a local biholomorphism to X (Theorem 6). The sufficient conditions in Theorems 3 and 4 are naturally

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expressed in terms of the quotient (Theorem 7). Finally, we use our disc formula to generalize Kiselman's minimum principle [8] and to give a new proof of a special case of it; this proof is based on the observation that Kiselman's infimum function may be viewed as a plurisubharmonic subextension to a suitable larger domain (Theorem 8).

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2. A Disc Formula for Plurisubharmonic Subextensions

We start by establishing some notation. For $r > 0$, let $D_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} = D_1$. Let λ denote the normalized arc-length measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. For a complex manifold X, let $\mathcal{A}_X$ denote the set of analytic discs in X, here taken to be continuous maps $f : \overline{\mathbb{D}} \to X$ that are holomorphic on $\mathbb{D}$. We call $f(0)$ the center of f. We endow $\mathcal{A}_X$ with the compact-open topology, that is, the topology of uniform convergence on $\overline{\mathbb{D}}$. This endowment makes $\mathcal{A}_X$ a complete metrizable space. For any topological space Y, a continuous map $Y \to \mathcal{A}_X$ is nothing but a continuous map $Y \times \overline{\mathbb{D}} \to X$ that is holomorphic when restricted to $\{y\} \times \mathbb{D}$ for every $y \in Y$. If $W \subset X$, write $\mathcal{A}_X^W$ for the set of analytic discs f in X with $f(\mathbb{T}) \subset W$. If W is open, then $\mathcal{A}_X^W$ is open in $\mathcal{A}_X$.

Let $f \in \mathcal{A}_X$. By a theorem of Forstnerič [6, Thm. 1.2], the graph $\Gamma_f = \{(z, f(z)) : z \in \overline{\mathbb{D}}\}$ of f has a basis of nice Stein open neighborhoods in $\mathbb{C} \times X$. More precisely, there is a basis of Stein open neighborhoods V of Γ_f in $\mathbb{C} \times X$, each with a biholomorphism onto an open subset of $\mathbb{C} \times \mathbb{C}^{\dim X}$, mapping $(\{z\} \times X) \cap V$ onto an open convex subset of $\{z\} \times \mathbb{C}^{\dim X}$ for each $z \in \mathbb{C}$. The sets $V^* = \{g \in \mathcal{A}_X : \Gamma_g \subset V\}$, as V ranges over such a basis of open neighborhoods of Γ_f, form a basis of open neighborhoods of f in $\mathcal{A}_X$. It follows that there is a neighborhood W of $f(0)$ in X and a continuous map $F : W \to \mathcal{A}_X$ such that $F(f(0)) = f$ and $F(x)(0) = x$ for each $x \in W$. Hence the center map $c : \mathcal{A}_X \to X$, $f \mapsto f(0)$, is not only continuous but also open. Each neighborhood V^* is contractible, so $\mathcal{A}_X$ is locally contractible. In particular, the connected components and the path components of $\mathcal{A}_X$ are the same, and they are open in $\mathcal{A}_X$.

For an upper semicontinuous function $\phi : X \to [-\infty, \infty)$, let $H_\phi : \mathcal{A}_X \to [-\infty, \infty)$ be the *Poisson functional* associated to ϕ, defined by the formula

$$H_\phi(f) = \int_{\mathbb{T}} \phi \circ f \, d\lambda.$$

For $\mathcal{B} \subset \mathcal{A}_X$, the *Poisson envelope* $E_{\mathcal{B}}\phi : X \to [-\infty, \infty]$ of ϕ with respect to $\mathcal{B}$ is defined by the formula

$$E_{\mathcal{B}}\phi(x) = \inf_{\substack{f \in \mathcal{B} \\ f(0)=x}} H_\phi(f).$$

It is well known that $P\phi = E_{\mathcal{A}_X}\phi$ is the largest plurisubharmonic minorant of ϕ on X (see the references given in the Introduction). If $W \subset X$ is open, $\mathcal{B} \subset \mathcal{A}_X^W$, and $\phi\colon W \to [-\infty, \infty)$ is upper semicontinuous, then the envelope $E_{\mathcal{B}}\phi\colon X \to [-\infty, \infty]$ is defined as before.

Let W be a domain (a connected, nonempty, open subset) in a complex manifold X. Let $\phi\colon W \to [-\infty, \infty)$ be upper semicontinuous, for example plurisubharmonic (we take the constant function $-\infty$ to be plurisubharmonic). A plurisubharmonic function u on X is a *subextension* of ϕ if $u|W \le \phi$. Let

$$S\phi = \sup\{u \in \mathrm{PSH}(X) : u|W \le \phi\}.$$

Now $S\phi$ is upper semicontinuous (in particular nowhere ∞) and hence plurisubharmonic if and only if the class $\{u \in \mathrm{PSH}(X) : u|W \le \phi\}$ has local upper bounds on X, which holds for example if X is covered by analytic discs with boundaries in W. Then $S\phi$ is the largest plurisubharmonic subextension of ϕ to X. It is easily seen that $S(S\phi) = S\phi$ and that $S\phi$ is maximal on $X \setminus \overline{W}$. Also, $S\phi \le E_{\mathcal{A}_X^W}\phi$, with equality if and only if $E_{\mathcal{A}_X^W}\phi$ is plurisubharmonic on X. We will prove the disc formula $S\phi = E_{\mathcal{A}_X^W}\phi$ under suitable conditions on X and W.

As a start, let us derive a preliminary disc formula for largest plurisubharmonic subextensions directly from the disc formula for largest plurisubharmonic minorants.

Proposition 1. *Let W be a domain in a complex manifold X such that X is covered by analytic discs with boundaries in W. Let $\phi\colon W \to [-\infty, \infty)$ be upper semicontinuous and bounded above. Then, for every $x \in X$,*

$$S\phi(x) = \lim_{\varepsilon\to 0+} \inf_f \int_{(f|\mathbb{T})^{-1}(W)} \phi \circ f \, d\lambda,$$

where, for each $\varepsilon > 0$, the infimum is taken over all $f \in \mathcal{A}_X$ such that $f(0) = x$ and $\lambda((f|\mathbb{T})^{-1}(W)) > 1 - \varepsilon$.

Proof. Suppose ϕ is bounded above on W by $m \in \mathbb{N}$. For each $n \ge m$, define an upper semicontinuous function ϕ_n on X as ϕ on W and as n on $X \setminus W$. Now $P\phi_n$ is plurisubharmonic on X and $P\phi_n \le \phi$ on W, so $P\phi_n \le S\phi$. Also, $S\phi \le m$ on X, so $S\phi \le P\phi_n$. Hence $S\phi = P\phi_n$ for all $n \ge m$.

Let $x \in X$. If $f \in \mathcal{A}_X$ has $\lambda((f|\mathbb{T})^{-1}(W)) > 1 - \varepsilon$ and $f(0) = x$, then

$$S\phi(x) \le \int_{\mathbb{T}} \phi_m \circ f \, d\lambda < \int_{(f|\mathbb{T})^{-1}(W)} \phi \circ f \, d\lambda + m\varepsilon.$$

Thus $S\phi(x) \le \lim_{\varepsilon\to 0+} \inf_f \int_{(f|\mathbb{T})^{-1}(W)} \phi \circ f \, d\lambda$. On the other hand, for each $n \ge m$, there is $f_n \in \mathcal{A}_X$ with $f_n(0) = x$ and $H_{\phi_n}(f_n) \le P\phi_n(x) + 1/n = S\phi(x) + 1/n$. Then $\lambda((f_n|\mathbb{T})^{-1}(W)) \to 1$ and

$$\int_{(f_n|\mathbb{T})^{-1}(W)} \phi \circ f_n \, d\lambda \le \int_{\mathbb{T}} \phi_n \circ f_n \, d\lambda \le S\phi(x) + \frac{1}{n}.$$

Thus $S\phi(x) \ge \lim_{\varepsilon\to 0+} \inf_f \int_{(f|\mathbb{T})^{-1}(W)} \phi \circ f \, d\lambda$. □

The disc formula in Proposition 1 is rather clumsy. It is natural to ask whether we can "pass to the limit" and use only analytic discs whose entire boundary lies in W. This turns out to be a subtle question involving a mix of complex analysis and topology. The answer is affirmative only if suitable restrictions are imposed on the pair $W \subset X$.

We say that $f_0, f_1 \in \mathcal{A}_X^W$ with $f_0(0) = f_1(0)$ are *center-homotopic* if there is a continuous map $f : \overline{\mathbb{D}} \times [0,1] \to X$ with $f(\cdot, t) \in \mathcal{A}_X^W$ for all $t \in [0,1]$ and $f(\cdot, t) = f_t$ for $t = 0, 1$ (that is, a continuous path in $\mathcal{A}_X^W$ joining f_0 and f_1) such that $f(0,t) = f_0(0)$ for all $t \in [0,1]$.

A *W-disc structure* on X is a family $\beta = (\beta_\nu)$ of continuous maps $\beta_\nu : U_\nu \to \mathcal{A}_X^W$, where (U_ν) is an open cover of X, satisfying the following two conditions.

For all $x \in U_\nu$, $\beta_\nu(x)(0) = x$.

(S) If $x \in U_\nu \cap U_\mu$, then $\beta_\nu(x)$ and $\beta_\mu(x)$ are center-homotopic.

We will be interested in the following condition that β may or may not satisfy.

(N) There is μ such that $U_\mu = W$ and $\beta_\mu(w)$ is the constant disc at w for all $w \in W$.

The class $\mathcal{B}_\beta \subset \mathcal{A}_X^W$ associated to a W-disc structure β on X is the union $\bigcup_\nu \beta_\nu(U_\nu)$. It is easily seen that if ϕ is upper semicontinuous on W, then the envelope $E_\beta\phi = E_{\mathcal{B}_\beta}\phi$ is upper semicontinuous on X. We say that X is a *schlicht disc extension* of W if X carries a W-disc structure satisfying (N). These definitions will be viewed in a more abstract light in Section 3.

EXAMPLE 1. Here is a simple example to illustrate the definitions. For $n \geq 2$ and $r > 0$, let $B_r^n = \{x \in \mathbb{C}^n : \|x\| < r\}$. Set $X = B_4^n$ and $W = B_4^n \setminus \overline{B_1^n}$. Let $U_0 = W$ and $U_1 = B_2^n$. Then $\{U_0, U_1\}$ is an open cover of X, and a W-disc structure on X satisfying (N) is given by setting $\beta_0(x)(\zeta) = x$ for all $x \in U_0$ and setting

$$\beta_1(x)(\zeta) = \left(\rho(x)\frac{\rho(x)\zeta + x_1}{\rho(x) + \bar{x}_1\zeta}, x_2, \ldots, x_n\right),$$

where

$$\rho(x) = \sqrt{9 - |x_2|^2 - \cdots - |x_n|^2},$$

for all $x = (x_1, \ldots, x_n) \in U_1$. Condition (S) is evident.

The following lemma is proved along the lines of the original proof in [14] of the plurisubharmonicity of the Poisson envelope. We follow the exposition in [9]. Note that we are now restricting our discussion to domains in affine space.

LEMMA 2. *Let $W \subset X$ be domains in $\mathbb{C}^n$, and let β be a W-disc structure on X. If $\phi : W \to [-\infty, \infty)$ is upper semicontinuous, then*

$$E_{\mathcal{A}_X^W}\phi \leq PE_\beta\phi.$$

Before proving the lemma, we state and prove our first theorem.

THEOREM 3. *Let $W \subset X$ be domains in $\mathbb{C}^n$ such that X is a schlicht disc extension of W. If $\phi : W \to [-\infty, \infty)$ is upper semicontinuous, then*

$$S\phi = E_{\mathcal{A}_X^W}\phi.$$

Proof. Let β be a W-disc structure on X satisfying (N). We have

$$S\phi \le E_{\mathcal{A}_X^W}\phi \le PE_\beta\phi \le S\phi.$$

The first inequality is obvious. The second is Lemma 2. The third holds because $E_\beta\phi \le \phi$ on W by (N). □

This approach to proving a disc formula using an auxiliary class such as $\mathcal{B}_\beta$ first appeared in [11].

Proof of Lemma 2. Let β be a W-disc structure on X. To prove the desired inequality, we show that for every $h \in \mathcal{A}_X$, $\varepsilon > 0$, and a continuous function $v\colon X \to \mathbb{R}$ with $v \ge E_\beta\phi$, there exists $g \in \mathcal{A}_X^W$ such that $g(0) = h(0)$ and

$$H_\phi(g) \le H_v(h) + \varepsilon. \tag{1}$$

The proof is carried out in three steps. First we show that there exists a continuous map $F\colon \overline{\mathbb{D}} \times \mathbb{T} \to X$, such that $F(\cdot, w) \in \mathcal{A}_X^W$ and $F(0, w) = h(w)$ for all $w \in \mathbb{T}$, and

$$\int_{\mathbb{T}} H_\phi(F(\cdot, w))\, d\lambda(w) \le H_v(h) + \frac{\varepsilon}{2}. \tag{2}$$

Next we show that there exists a continuous map $G\colon \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to X$, holomorphic on $\mathbb{D} \times \mathbb{D}$, such that $G(0, w) = h(w)$ for all $w \in \overline{\mathbb{D}}$, $G(\mathbb{T} \times \mathbb{T}) \subset W$, and

$$\int_{\mathbb{T}} H_\phi(G(\cdot, w))\, d\lambda(w) \le \int_{\mathbb{T}} H_\phi(F(\cdot, w))\, d\lambda(w) + \frac{\varepsilon}{2}. \tag{3}$$

Finally we show that there is $\theta_0 \in [0, 2\pi]$ such that if $g \in \mathcal{A}_X^W$ is defined by the formula $g(z) = G(e^{i\theta_0}z, z)$, then

$$H_\phi(g) \le \int_{\mathbb{T}} H_\phi(G(\cdot, w))\, d\lambda(w). \tag{4}$$

By combining (2), (3), and (4), we get (1).

Step 1. Let $h \in \mathcal{A}_X$, $\varepsilon > 0$, and $v\colon X \to \mathbb{R}$ be continuous with $v \ge E_\beta\phi$. Let $w_0 \in \mathbb{T}$ and set $x_0 = h(w_0)$. Find ν such that $x_0 \in U_\nu$ and $H_\phi(\beta_\nu(x_0)) < v(x_0) + \varepsilon/(8\pi)$. For all x in a small enough neighborhood of x_0, we have $H_\phi(\beta_\nu(x)) < v(x) + \varepsilon/(8\pi)$. By compactness, there is a cover of $\mathbb{T}$ by closed arcs $I_1, \dots, I_m$ that meet only in endpoints, such that, for each $j = 1, \dots, m$, there is ν_j with $h(I_j) \subset U_{\nu_j}$ and $H_\phi(\beta_{\nu_j}(h(w))) < v(h(w)) + \varepsilon/(8\pi)$ for all $w \in I_j$.

Let w_0 be a common endpoint of I_1 and I_2, say. By (S), $\beta_{\nu_1}(h(w_0))$ and $\beta_{\nu_2}(h(w_0))$ are center-homotopic. Choose a center-homotopy between them, use small translations to spread the centers of the analytic discs in the homotopy over a small arc centered at w_0, and reparameterize $\beta_{\nu_1} \circ h$ and $\beta_{\nu_2} \circ h$ by small translations over subarcs slightly smaller than I_1 and I_2, respectively. In this way we obtain a continuous map $F\colon \overline{\mathbb{D}} \times \mathbb{T} \to X$, such that $F(\cdot, w) \in \mathcal{A}_X^W$ and $F(0, w) = h(w)$ for each $w \in \mathbb{T}$, and

$$\int_{\mathbb{T}} H_\phi(F(\cdot,w))\,d\lambda(w) \le \sum_{j=1}^{m} \int_{I_j} H_\phi(\beta_{v_j}\circ h)\,d\lambda + \frac{\varepsilon}{4}$$
$$\le \sum_{j=1}^{m} \int_{I_j} v\circ h\,d\lambda + \frac{\varepsilon}{2} = H_v(h) + \frac{\varepsilon}{2};$$

thus we have proved (2).

Step 2. For each $j \ge 1$ we define the Cesàro mean $F_j\colon \overline{\mathbb{D}} \times \overline{\mathbb{D}}^* \to \mathbb{C}^n$, where $\overline{\mathbb{D}}^*$ denotes $\overline{\mathbb{D}} \setminus \{0\}$, by

$$F_j(z,w) = h(w) + \frac{1}{j+1} \sum_{m=0}^{j} \sum_{k=-m}^{m} \biggl(\int_{\mathbb{T}} (F(z,\zeta) - h(\zeta))\zeta^{-k}\,d\lambda(\zeta) \biggr) w^k.$$

The well-known theorem on the uniform convergence of the Cesàro means of a continuous function on $\mathbb{T}$ holds for maps into a Banach space, such as the space of continuous maps $\overline{\mathbb{D}} \to \mathbb{C}^n$ that are holomorphic on $\mathbb{D}$. We conclude that $F_j \to F$ uniformly on $\overline{\mathbb{D}} \times \mathbb{T}$ as $j \to \infty$. Hence $F_j(\overline{\mathbb{D}} \times \mathbb{T}) \subset X$ and $F_j(\mathbb{T} \times \mathbb{T}) \subset W$ for j large enough. For simplicity, assume that this holds for all $j \ge 1$.

For every $z \in \overline{\mathbb{D}}$, the map $w \mapsto F_j(z,w) - h(w)$ has a pole of order at most j at the origin, and for every $w \in \overline{\mathbb{D}}^*$, the map $z \mapsto F_j(z,w) - h(w)$ has a zero at the origin. Thus $(z,w) \mapsto F_j(zw^k, w)$ extends to a continuous map $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}^n$, holomorphic on $\mathbb{D} \times \mathbb{D}$, for every $k \ge j$.

Since $F_j(0,w) = h(w) \in X$ for all $w \in \overline{\mathbb{D}}^*$, there is $\delta_j > 0$ such that $F_j(zw^k, w) \in X$ for all integers $k \ge j$ and $(z,w) \in D_{\delta_j} \times \overline{\mathbb{D}}$. Since $F_j(\overline{\mathbb{D}} \times \mathbb{T}) \subset X$, there is $r_j \in (0,1)$ such that $F_j(\overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus D_{r_j})) \subset X$, so $F_j(zw^k, w) \in X$ for all $(z,w) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus D_{r_j})$ and all $k \ge j$.

Take $k_j \ge j$ so large that $|zw^{k_j}| < \delta_j$ for all $(z,w) \in \overline{\mathbb{D}} \times D_{r_j}$. Then we have $F_j(zw^{k_j}, w) \in X$ for all $(z,w) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Define a continuous map $G_j\colon \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to X$, holomorphic on $\mathbb{D} \times \mathbb{D}$, by $G_j(z,w) = F_j(zw^{k_j}, w)$. Then $G_j(\mathbb{T} \times \mathbb{T}) \subset W$ and $G_j(0,w) = h(w)$ for all $w \in \overline{\mathbb{D}}$.

Take j large enough that

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F_j(e^{it}, e^{i\theta}))\,dt\,d\theta$$
$$\le \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F(e^{it}, e^{i\theta}))\,dt\,d\theta + \frac{\varepsilon}{2}$$
$$= \int_0^{2\pi} H_\phi(F(\cdot,w))\,d\lambda(w) + \frac{\varepsilon}{2},$$

and let $G = G_j$. Then

$$\int_{\mathbb{T}} H_\phi(G(\cdot,w))\,d\lambda(w) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F_j(e^{i(t+k_j\theta)}, e^{i\theta}))\,dt\,d\theta$$
$$= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F_j(e^{it}, e^{i\theta}))\,dt\,d\theta,$$

and (3) is proved.

Step 3. The right-hand side of (4) is

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\phi(G(e^{it},e^{i\theta}))\,dt\,d\theta = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\phi(G(e^{i\theta}e^{it},e^{it}))\,dt\,d\theta.$$

There is $\theta_0 \in [0,2\pi]$ such that

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\phi(G(e^{it},e^{i\theta}))\,dt\,d\theta \geq \frac{1}{2\pi}\int_0^{2\pi}\phi(G(e^{i\theta_0}e^{it},e^{it}))\,dt.$$

If we set $g(z) = G(e^{i\theta_0}z, z)$, then $g(0) = G(0,0)$ and (4) holds. □

We now provide another sufficient condition for our disc formula to hold.

THEOREM 4. *Let $W \subset X$ be domains in $\mathbb{C}^n$. Suppose $\mathcal{A}_X^W$ has a connected component, call it $\mathcal{B}$, with the following properties.*

(i) *$\mathcal{B}$ covers X.*

(ii) *If two analytic discs in $\mathcal{B}$ have the same center, then they are center-homotopic.*

Then, for every upper semicontinuous function $\phi\colon W \to [-\infty,\infty)$,

$$S\phi = E_{\mathcal{A}_X^W}\phi.$$

Proof. The proof is the same as the proof of Theorem 3, except for the inequality $PE_{\mathcal{B}}\phi \leq S\phi$, which now requires more work. We can show that $E_{\mathcal{A}_X^W}\phi \leq PE_{\mathcal{B}}\phi$ exactly as we proved Lemma 2, using (i) and (ii) and the fact that $\mathcal{B}$ is open in $\mathcal{A}_X^W$. We need to show that $E_{\mathcal{B}}\phi \leq \phi$ on W, which previously was an immediate consequence of (N).

Let $p \in W$ and $\varepsilon > 0$. We need an analytic disc $f \in \mathcal{B}$ with $f(0) = p$ and $H_\phi(f) < \phi(p) + \varepsilon$. Take $g \in \mathcal{B}$ with $g(0) = p$. Extend g to a continuous map $g\colon \overline{\mathbb{D}} \cup [1,2] \to X$ such that $g|[1,2]$ is a path in W with $g(2) = p$. By Mergelyan's theorem, g can be approximated uniformly on $\overline{\mathbb{D}} \cup [1,2]$ by polynomial maps $\mathbb{C} \to \mathbb{C}^n$. Since W and $\mathcal{B}$ are open, we may assume that g is the restriction to $\overline{\mathbb{D}} \cup [1,2]$ of a polynomial map that we will still call g.

Let $\Omega \subset \mathbb{C}$ be the simply connected domain of all points within distance $\delta > 0$ of $\overline{\mathbb{D}} \cup [1,2]$. Let μ be the harmonic measure of Ω with respect to the point 2. Choose δ so small that $g(\overline{\Omega}\setminus\mathbb{D}) \subset W$ and $\int_{\partial\Omega}\phi\circ g\,d\mu < \phi(g(2)) + \varepsilon = \phi(p) + \varepsilon$.

A theorem of Radó (see [17] or [7, II.5, Thm. 2]) states that as a simply connected bounded domain in $\mathbb{C}$ is continuously varied in a suitable sense, its normalized Riemann map also varies continuously. More precisely, for each $n \geq 0$, let U_n be a simply connected domain in $\mathbb{C}$ containing 0 and bounded by Jordan curves, and let $\psi_n\colon \mathbb{D} \to U_n$ be the biholomorphism with $\psi_n(0) = 0$ and $\psi_n'(0) > 0$. Then $\psi_n \to \psi_0$ uniformly on $\overline{\mathbb{D}}$ if and only if, for every $\varepsilon > 0$, there is $m \geq 1$ such that, for every $n \geq m$, there is a homeomorphism $\partial U_n \to \partial U_0$ such that the distance between corresponding points is at most ε.

Radó's theorem provides a continuous map $\Psi\colon \overline{\mathbb{D}} \times [0,1] \to \overline{\Omega}$ such that $\Psi_t = \Psi(\cdot,t)$ is a homeomorphism onto its image and holomorphic on $\mathbb{D}$ with $\Psi_t(0) = 0$ for every $t \in [0,1]$, $\Psi(\mathbb{T} \times [0,1]) \subset \overline{\Omega} \setminus \mathbb{D}$, Ψ_0 is the inclusion $\overline{\mathbb{D}} \hookrightarrow \overline{\Omega}$, and $\Psi_1(\overline{\mathbb{D}}) = \overline{\Omega}$. Then $g \circ \Psi_t \in \mathcal{A}_X^W$ for all $t \in [0,1]$, so $h = g \circ \Psi_1 \in \mathcal{B}$ and $h(0) = p$.

Let $a = \Psi_1^{-1}(2)$. Then $h(a) = p$. By precomposing h by a path in $\operatorname{Aut}\mathbb{D}$ joining the identity to an automorphism α of $\mathbb{D}$ taking 0 to a, we obtain $f \in \mathcal{B}$ with $f(0) = p$. Finally,

$$H_\phi(f) = \int_{\mathbb{T}} \phi \circ g \circ \Psi_1 \circ \alpha \, d\lambda = \int_{\partial\Omega} \phi \circ g \, d\mu < \phi(p) + \varepsilon. \qquad \square$$

In Theorems 3 and 4, $\mathbb{C}^n$ can be replaced by any Stein manifold S. Only minor modifications of Step 2 in the proof of Lemma 2 and of the proof of Theorem 4 are needed, using a tubular neighborhood of S viewed as a submanifold of $\mathbb{C}^m$ for some m. Whether $\mathbb{C}^n$ can be replaced by an arbitrary complex manifold is an open question.

EXAMPLE 2. Take $X = D_2$ and $W = D_2 \setminus \overline{\mathbb{D}}$. Then we have one connected component of $\mathcal{A}_X^W$ for each nonnegative winding number. The hypotheses of Theorem 4 hold for each positive winding number. Namely, (i) is obvious and (ii) follows from writing an analytic disc in $\mathcal{A}_X^W$ as the product of a holomorphic function without zeros and a Blaschke product whose degree equals the winding number. Thus our disc formula holds for the pair $W \subset X$. Theorem 3 does not apply because X is not a schlicht disc extension of W.

The following example shows that, without the hypotheses of Theorems 3 and 4, our disc formula can fail.

EXAMPLE 3. Fix $\delta \in \left(0, \frac{1}{2}\right)$ and let

$$W_1 = \mathbb{D} \times \{z_2 \in \mathbb{C} : |z_2| < \delta\},$$
$$W_2 = \mathbb{D} \times \{z_2 \in \mathbb{C} : 1 - \delta < |z_2| < 1\}.$$

Join W_1 and W_2 by the curve $[0,1] \to \mathbb{C}^2$, $t \mapsto \left(1 + e^{2\pi i(2t-1)/3}, \left(1 - \frac{\delta}{2}\right)t\right)$. Let W_3 be a thin open tubular neighborhood of the image of the curve, and let W be the domain $W_1 \cup W_2 \cup W_3$. We may assume that the intersection with $\mathbb{R}$ of the projection of W onto the z_1-plane is $(-1,1) \cup (a,b)$ with $1 < a < 2 < b$ and also that b is the supremum of $\operatorname{Re} z_1$ on W. Let $I = (a,b)$, and let

$$W^- = W_1 \cup \{z \in W_3 : \operatorname{Im} z_1 < 0\},$$
$$W^+ = W_2 \cup \{z \in W_3 : \operatorname{Im} z_1 > 0\}.$$

Let U_1 and U_2 be thin open neighborhoods of the semicircles

$$\gamma_1 = \{e^{it}/2 : t \in [0,\pi]\} \quad \text{and} \quad \gamma_2 = \{e^{it}/2 : t \in [\pi, 2\pi]\}$$

in $\mathbb{D}$, respectively, and let

$$V_1 = U_1 \times \{z_2 \in \mathbb{C} : |z_2| < \delta\} \subset W_1,$$
$$V_2 = U_2 \times \{z_2 \in \mathbb{C} : 1 - \delta < |z_2| < 1\} \subset W_2.$$

Define an upper semicontinuous function ϕ on W as -1 on $V_1 \cup V_2$ and as 0 elsewhere.

Let X be any domain in $\mathbb{C}^2$ containing W such that X is covered by images $f(\overline{\mathbb{D}})$ of analytic discs $f \in \mathcal{A}_{\mathbb{C}^2}^W$ and such that X contains the image of the analytic disc $\zeta \mapsto \big(z_1, \big(1 - \frac{\delta}{2}\big)\zeta\big)$ for every $z_1 \in \gamma_2$. For example, X could be $\mathbb{D}^2 \cup W_3$.

By making W_3, U_1, and U_2 thin enough, we can ensure that there is an $s < 1$ such that, for every $g \in \mathcal{A}_{\mathbb{C}^2}$ with $g(0) = (0,0)$, each of the sets $g^{-1}(I \cup V_1) \cap \mathbb{T}$ and $g^{-1}(I \cup V_2) \cap \mathbb{T}$ has harmonic measure (with respect to $0 \in \mathbb{D}$, that is, normalized arc-length measure) less than s.

We will show that $S\phi(0,0) = -1$, but $E_{\mathcal{A}_X^W}\phi(0,0) \geq -s > -1$ and so our disc formula fails for the pair $W \subset X$.

First, if $z_1 \in \gamma_1$, then $\phi(z_1, 0) = -1$, so $S\phi(z_1, 0) = -1$. If $z_1 \in \gamma_2$, then the analytic disc $\zeta \mapsto \big(z_1, \big(1 - \frac{\delta}{2}\big)\zeta\big)$ has its boundary in V_2, so $\phi = -1$ on the boundary and $S\phi(z_1, 0) = -1$. It follows that $S\phi(0,0) = -1$.

Second, let $f = (f_1, f_2) \in \mathcal{A}_X^W$ with $f(0) = (0,0)$. We may assume that f extends holomorphically to an open neighborhood of $\overline{\mathbb{D}}$, that f_1 has no critical values in $\bar{I}$, and that $f_1|\mathbb{T}$ is transverse to I, so in particular $f_1^{-1}(I) \cap \mathbb{T}$ is finite. We claim that the harmonic measure of $f^{-1}(V_1 \cup V_2) \cap \mathbb{T}$ is less than s, so $H_\phi(f) > -s$.

If $f_1^{-1}(I) = \emptyset$, then $f^{-1}(V_1 \cup V_2) \cap \mathbb{T}$ is either $f^{-1}(V_1) \cap \mathbb{T}$ or $f^{-1}(V_2) \cap \mathbb{T}$, so the claim is clear. Assume that $f_1^{-1}(I) \neq \emptyset$. Then $f_1^{-1}(I)$ is the disjoint union of finitely many embedded 1-dimensional submanifolds. None of them are loops, for otherwise f_1 would be constant, so they are all arcs. None of them are relatively compact in $\mathbb{D}$ since f_1 cannot take a point in $\mathbb{D}$ to b, so each arc has one endpoint on $\mathbb{T}$ and the other on $\mathbb{T}$ or in $\mathbb{D}$.

Say an arc in $f_1^{-1}(I)$ is *good* if both its endpoints lie on $\mathbb{T}$. Call the other arcs *bad*. Let Ω be the set of all points in $\mathbb{D}$ that lie on the same side of each good arc as 0. Then Ω is a simply connected domain containing 0, bounded by some of the good arcs and some arcs in $\mathbb{T}$ that we will call *circular arcs,* unless there are no good arcs, in which case $\Omega = \mathbb{D}$ and $\mathbb{T}$ is the one circular "arc".

The finitely many points on each circular arc that lie in $f_1^{-1}(I)$ divide the circular arc into open *subarcs,* where each subarc has its f-image in W^- or W^+. Suppose there is a subarc with its f-image in W^+. Moving counterclockwise along the subarc, we come to an endpoint p.

Suppose that on the other side of p there is a subarc of the same circular arc. Then p is an endpoint of a bad arc, and $\operatorname{Im} f_1|\mathbb{T}$ changes sign at p from positive to negative. Since f_1 is conformal at p, $\operatorname{Re} f_1$ is decreasing as we approach p along the bad arc. Now $\operatorname{Re} f_1$ has no critical points on the bad arc; hence $\operatorname{Re} f_1$ is decreasing all along the bad arc in the direction of p, and the maximum principle is violated at the interior endpoint of the bad arc. A slight elaboration of this argument shows that $\Omega \neq \mathbb{D}$.

Therefore, p is an endpoint of a good arc. Since f_1 is conformal at p, $\operatorname{Re} f_1$ is increasing as we leave p along the good arc. Since $\operatorname{Re} f_1$ has no critical points on the good arc, $\operatorname{Re} f_1$ is still increasing as we approach the other endpoint q of the good arc. Since f_1 is conformal at q, it follows that $\operatorname{Im} f_1$ is increasing as we leave q along the next subarc, which therefore has its f-image in W^+.

This shows that $f(\partial\Omega)$ cannot intersect both W^- and W^+. Thus either $f^{-1}(V_1) \cap \mathbb{T}$ or $f^{-1}(V_2) \cap \mathbb{T}$ lies behind $\partial\Omega \cap \mathbb{D}$ as seen from 0, so the harmonic measure of $f^{-1}(V_1 \cup V_2) \cap \mathbb{T}$ is at most the harmonic measure with respect to $0 \in \Omega$ of either $f^{-1}(I \cup V_1) \cap \partial\Omega$ or $f^{-1}(I \cup V_2) \cap \partial\Omega$ and is therefore less than s, proving our claim.

By Theorem 3, X is not a schlicht disc extension of W. By Theorem 4, one or both of the following statements hold (we only know that they are not both false).

- No disc in $\mathcal{A}_X^W$ can be deformed in $\mathcal{A}_X^W$ to a disc with an arbitrary center in X.
- There are discs in $\mathcal{A}_X^W$ with the same center that are homotopic but not center-homotopic.

3. The Center-Homotopy Relation on Spaces of Analytic Discs

Let X be a complex manifold and, as before, let $\mathcal{A}_X$ be the space of analytic discs in X, that is, continuous maps $\overline{\mathbb{D}} \to X$ that are holomorphic on $\mathbb{D}$, with the topology of uniform convergence. Let $f \in \mathcal{A}_X$. Recall that by [6, Thm. 1.2] there is a basis of Stein open neighborhoods V of the graph Γ_f of f in $\mathbb{C} \times X$, each with a biholomorphism onto an open subset of $\mathbb{C} \times \mathbb{C}^{\dim X}$, mapping $(\{z\} \times X) \cap V$ onto an open convex subset of $\{z\} \times \mathbb{C}^{\dim X}$ for each $z \in \mathbb{C}$. The sets $V^* = \{g \in \mathcal{A}_X : \Gamma_g \subset V\}$, as V ranges over such a basis of open neighborhoods of Γ_f, form a basis of open neighborhoods of f in $\mathcal{A}_X$. We already noted that V^* is contractible; moreover, V^* intersects each fibre of the center map $\mathcal{A}_X \to X$, $f \mapsto f(0)$, in a contractible set. Let W be a domain in X. If $f \in \mathcal{A}_X^W$ and if V as above is small enough, then V^* lies in $\mathcal{A}_X^W$ and intersects each fibre of the restricted center map $c\colon \mathcal{A}_X^W \to X$ in a contractible set.

Define an equivalence relation $\sim_c$ on $\mathcal{A}_X^W$ by taking $f \sim_c g$ if f and g are center-homotopic; that is, the equivalence classes of $\sim_c$ are the connected components—or, equivalently, the path components—of the fibres of c. Let $q\colon \mathcal{A}_X^W \to X_W = \mathcal{A}_X^W/{\sim_c}$ be the quotient map, and endow X_W with the quotient topology. Let X_W^0 be the connected component of X_W containing the equivalence classes of the constant discs in W. Since c is continuous, it factors through q by a continuous map $\pi\colon X_W \to X$.

PROPOSITION 5. *Let W be a domain in a complex manifold X. The equivalence relation $\sim_c$ on $\mathcal{A}_X^W$ is open; that is, the quotient map $q\colon \mathcal{A}_X^W \to X_W$ is open.*

Proof. Let $f \sim_c g$ in $\mathcal{A}_X^W$ and let V be a neighborhood of f. Take a path $\gamma\colon [0,1] \to c^{-1}(f(0))$ with $\gamma(0) = f$ and $\gamma(1) = g$. Cover the image of γ by a chain of open sets $V_1^*, \dots, V_k^*$ of the kind described previously, whose intersection with each fibre of c is contractible, such that $f \in V_1^* \subset V$, $g \in V_k^*$, and $V_j^* \cap V_{j+1}^* \cap \gamma([0,1]) \neq \emptyset$ for $j = 1, \dots, k-1$. Recall that c is open and let U be the neighborhood $\bigcap_{j=1}^{k-1} c(V_j^* \cap V_{j+1}^*)$ of $f(0)$ in X. Then each analytic disc in the neighborhood $V_k^* \cap c^{-1}(U)$ of g is $\sim_c$-related to an analytic disc in V. Thus $\sim_c$ is open. $\square$

THEOREM 6. *Let W be a domain in a complex manifold X. Then the map $\pi\colon X_W \to X$ is a local homeomorphism.*

Proof. First, π is open since c is. We need to show that π is locally injective. Let $f \in \mathcal{A}_X^W$. Let U be a neighborhood of $q(f)$ in X_W. Then $q^{-1}(U)$ is a neighborhood of f in $\mathcal{A}_X^W$. Find a neighborhood V^* of f in $\mathcal{A}_X^W$ as described before with $V^* \subset q^{-1}(U)$. By Proposition 5, $q(V^*) \subset U$ is a neighborhood of $q(f)$ and π is injective on $q(V^*)$. □

The Poincaré–Volterra theorem now implies that each connected component of X_W is second countable [1, I.11.7, Cor. 2]. Thus we could turn X_W into a complex manifold with the unique complex structure that makes π holomorphic, except that we do not know whether X_W is Hausdorff. It is evident, though, that X_W has closed points, that is, is T_1. By Proposition 5 and [1, I.8.3, Prop. 8], X_W is Hausdorff if and only if the graph of $\sim_c$ is closed in $\mathcal{A}_X^W \times \mathcal{A}_X^W$, that is, if whenever $f_n \to f$ and $g_n \to g$ in $\mathcal{A}_X^W$ and $f_n \sim_c g_n$ for all n, we have $f \sim_c g$.

Note that q has local continuous sections since c does, and that a W-disc structure on X is therefore nothing but a continuous section $\sigma\colon X \to X_W$ of π (so, in particular, π is surjective). Condition (N) says that σ extends the tautological section $W \to X_W^0$ that takes a point in W to the class of the constant disc at that point. Thus we can restate Theorems 3 and 4 as follows.

THEOREM 7. *Let $W \subset X$ be domains in $\mathbb{C}^n$. Suppose that one of the following two conditions holds.*

- *There is a continuous section $X \to X_W^0$ of $\pi\colon X_W \to X$ extending the tautological section on W.*
- *The restriction of π to some connected component of X_W is a bijection onto X.*

Then, for every upper semicontinuous function $\phi\colon W \to [-\infty, \infty)$,

$$S\phi = E_{\mathcal{A}_X^W}\phi.$$

4. Hartogs Domains and Kiselman's Minimum Principle

Let Y be a domain in $\mathbb{C}^{n-1}$, $n \geq 2$. Let $r\colon Y \to [0,\infty)$ be an upper semicontinuous function and let $R\colon Y \to (0,\infty]$ be lower semicontinuous with $r < R$. Then

$$W = \{(z', z_n) \in Y \times \mathbb{C} : r(z') < |z_n| < R(z')\}$$

is a Hartogs domain in $\mathbb{C}^n$. It is well known that if W is pseudoconvex, then $\log r$ is plurisubharmonic and $\log R$ is plurisuperharmonic. Let

$$X = \{(z', z_n) \in Y \times \mathbb{C} : |z_n| < R(z')\}$$

be the completion of W.

Assume W is pseudoconvex. We claim that every analytic disc $f = (f', f_n) \in \mathcal{A}_X^W$ is center-homotopic to a vertical disc of a special kind, where f' denotes $(f_1, \dots, f_{n-1})$. Namely, since $f(\mathbb{T}) \subset W$, f_n has no zeros on $\mathbb{T}$. Let h be a harmonic extension of $\log|f_n|$ to $\overline{\mathbb{D}}$. Now f is center-homotopic in $\mathcal{A}_X^W$ to $\zeta \mapsto f(s\zeta)$ for $s < 1$ close enough to 1, so we may assume that f is smooth on $\overline{\mathbb{D}}$. Then h has a harmonic conjugate k on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$, so $H = e^{h+ik}\colon \overline{\mathbb{D}} \to \mathbb{C}^*$ is continuous with $H|\mathbb{D}$ holomorphic and $|f_n| = |H|$ on $\mathbb{T}$.

Hence $|f_n/H| \le 1$ on $\overline{\mathbb{D}}$. Again since $f(\mathbb{T}) \subset W$, $r \circ f' < |f_n| < R \circ f'$ and so $\log r \circ f' < h < \log R \circ f'$ on $\mathbb{T}$. Since $\log r \circ f'$ is subharmonic and $\log R \circ f'$ is superharmonic, $r \circ f' < e^h = |H| < R \circ f'$ on $\overline{\mathbb{D}}$.

Consider the continuous map $[0,1] \to \mathcal{A}_X$, $t \mapsto f^t$, where

$$f^t(\zeta) = (f'(t\zeta), f_n(\zeta)H(t\zeta)/H(\zeta)), \quad \zeta \in \overline{\mathbb{D}}.$$

For $\zeta \in \mathbb{T}$, $|f_n^t(\zeta)| = |H(t\zeta)|$ and so $f^t(\mathbb{T}) \subset W$ for every $t \in [0,1]$. Also, $f^t(0) = f(0)$ for all $t \in [0,1]$ and $f^1 = f$, so f is center-homotopic to the vertical disc $g = f^0 \in \mathcal{A}_X^W$ with $g(\zeta) = (f'(0), f_n(\zeta)H(0)/H(\zeta))$. Note that $|g_n| = |H(0)|$ on $\mathbb{T}$.

There are two cases. If f_n and therefore g_n has no zeros in $\mathbb{D}$, then g is constant, so in particular $f(0) = g(0) \in W$. If f_n has $k \ge 1$ zeros in $\mathbb{D}$, then f_n/H is a Blaschke product with k factors. Hence, either $f(\overline{\mathbb{D}}) \subset W$ and f is center-homotopic to a constant disc, or f is center-homotopic to an analytic disc of the form $(f'(0), sB) \in \mathcal{A}_X^W$, where B is a Blaschke product and $r(f'(0)) < s < R(f'(0))$.

This shows that $\mathcal{A}_X^W$ has one connected component $\mathcal{A}_k$ for each winding number $k \ge 0$, and within each component, two discs with the same center are center-homotopic. Clearly, $c(\mathcal{A}_0) = W$ and $c(\mathcal{A}_k) = X$ for $k \ge 1$, where c is the center map. The quotient X_W has a component $q(\mathcal{A}_0)$ biholomorphic to W as well as components $q(\mathcal{A}_k)$, $k \ge 1$, that are each biholomorphic to X. In particular, X_W is Hausdorff. Evidently, X is not a schlicht disc extension of W. Finally, Theorem 4 implies that our disc formula holds for the pair $W \subset X$.

The disc formula provides a new proof of Kiselman's minimum principle [8, Thm. 2.2] in the present setting. Let $\phi\colon W \to [-\infty, \infty)$ be plurisubharmonic and rotation-invariant in the last variable. Define

$$\psi\colon Y \to [-\infty,\infty), \quad \psi(z') = \inf_{(z',z_n)\in W} \phi(z', z_n).$$

Let us refer to ψ as the *infimum function* of ϕ. Kiselman's minimum principle states that ψ is plurisubharmonic. We will prove it by showing that $\psi(z') = S\phi(z', z_n)$ for all $(z', z_n) \in X \setminus \overline{W}$.

It is clear that $S\phi(z', z_n) \le \psi(z')$ for $(z', z_n) \in X \setminus \overline{W}$. To prove the opposite inequality, take $z = (z', z_n) \in X$ and $\varepsilon > 0$, and then find $f = (f', f_n) \in \mathcal{A}_X^W$ with $f(0) = z$ and $H_\phi(f) < S\phi(z) + \varepsilon$. As before, there is a continuous function $H\colon \overline{\mathbb{D}} \to \mathbb{C}^*$ such that $H|\mathbb{D}$ is holomorphic, $|f_n| = |H|$ on $\mathbb{T}$, and $r \circ f' < |H| < R \circ f'$ on $\overline{\mathbb{D}}$. Let

$$F\colon \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}^n, \quad F(\zeta,\xi) = (f'(\zeta), H(\zeta)\xi).$$

Then $F(\overline{\mathbb{D}}, \mathbb{T}) \subset W$ and so, for each $\xi \in \mathbb{T}$, $F(\cdot, \xi)$ is an analytic disc in W and

$$\psi(z') \le \phi(F(0,\xi)) \le \int_{\mathbb{T}} \phi(F(\cdot,\xi))\, d\lambda.$$

If we average over ξ, change the order of integration, and use the rotation invariance of ϕ, we get

$$\begin{aligned}\psi(z') &\le \int_{\mathbb{T}}\int_{\mathbb{T}} \phi(F(\zeta,\xi))\,d\lambda(\xi)\,d\lambda(\zeta)\\ &= \int_{\mathbb{T}} \phi(F(\zeta, f_n(\zeta)/H(\zeta)))\,d\lambda(\zeta) = H_\phi(f).\end{aligned}$$

This proves that $\psi(z') \le S\phi(z', z_n)$.

The observation that Kiselman's infimum function is a plurisubharmonic subextension can be generalized. Let W be a Hartogs domain in $\mathbb{C}^n = \mathbb{C}^{n-1}\times\mathbb{C}$ such that if $(z_1,\dots,z_n)\in W$ and $\eta\in\mathbb{T}$, then $(z_1,\dots,z_{n-1},\eta z_n)\in W$. Let $p\colon W\to\mathbb{C}^{n-1}$ be the projection $(z_1,\dots,z_n)\mapsto(z_1,\dots,z_{n-1})$, and let Y be the domain $p(W)$ in $\mathbb{C}^{n-1}$. For every $y\in Y$, each vertical fibre $p^{-1}(y)$ is a disjoint union of at most one disc and some number of annuli, possibly none, all centered at the origin in $\{y\}\times\mathbb{C}$. Define an upper semicontinuous function $r\colon W\to[0,\infty)$ and a lower semicontinuous function $R\colon W\to(0,\infty]$ as follows. If $z\in W$ is contained in an annulus in $p^{-1}(p(z))$, then $r(z)$ is the inner radius and $R(z)$ the outer radius of the annulus. If z is contained in a disc in $p^{-1}(p(z))$, then $R(z)$ is the radius of the disc and $r(z)=0$.

Suppose W is pseudoconvex. Then $\log r$ is plurisubharmonic and $\log R$ is plurisuperharmonic. If $p^{-1}(y_0)$ contains a disc for some $y_0\in Y$, then $p^{-1}(y)$ contains a disc for all nearby $y\in Y$, so $\log r=-\infty$ on an open subset of W. Since W is connected, $\log r=-\infty$ on all of W, so every vertical fibre is a disc or a punctured disc. The latter are ruled out by the Kontinuitätssatz. Thus pseudoconvex Hartogs domains are divided into two classes: the complete ones, whose vertical fibres are discs; and the incomplete ones, whose vertical fibres are unions of annuli.

We assume that W is pseudoconvex and incomplete. Kiselman showed that identifying each annulus in each vertical fibre of W to a point yields a connected Hausdorff space A that is a pseudoconvex Riemann domain over Y [8, Prop. 2.1, Cor. 2.3]. Now r and R induce functions on A that are (respectively) plurisubharmonic and plurisuperharmonic, and W is reincarnated as the Hartogs domain

$$W = \{(a,\zeta)\in A\times\mathbb{C} : r(a) < |\zeta| < R(a)\}$$

over A with completion

$$X = \{(a,\zeta)\in A\times\mathbb{C} : |\zeta| < R(a)\},$$

in which A is embedded as $A\times\{0\}$.

Let $\phi\colon W\to[-\infty,\infty)$ be plurisubharmonic and rotation-invariant in the sense that $\phi(a,\eta\zeta)=\phi(a,\zeta)$ for all $(a,\zeta)\in W$ and $\eta\in\mathbb{T}$. Define the infimum function

$$\psi\colon A\to[-\infty,\infty),\quad \psi(a) = \inf_{(a,\zeta)\in W}\phi(a,\zeta).$$

Kiselman's minimum principle states that ψ is plurisubharmonic. Our generalization of the principle is as follows.

THEOREM 8. *Let W be an incomplete pseudoconvex Hartogs domain in $\mathbb{C}^n$ with completion X over the Kiselman quotient A of W.*

(i) *If ϕ is a plurisubharmonic and rotation-invariant function on W, then the infimum function ψ of ϕ is the restriction to A of the envelope $E_{\mathcal{A}_X^W}\phi$.*

(ii) *For every upper semicontinuous function ϕ on W, the envelope $E_{\mathcal{A}_X^W}\phi$ is plurisubharmonic on X and is therefore the largest plurisubharmonic subextension of ϕ to X.*

We remark that we are not giving a new proof of Kiselman's minimum principle because the principle is used in the proof that A is pseudoconvex, which implies that X is Stein. We need this in our proof of (ii) in order to apply Theorem 4. Statement (i), proved from scratch by the method used previously, shows that (ii) does indeed generalize Kiselman's minimum principle.

Proof of Theorem 8. (i) For $a \in A$, $\psi(a)$ is the infimum of $H_\phi(f)$ over all analytic discs $f \in \mathcal{A}_X^W$ of the form $f(\zeta) = (a, s\zeta)$ with $r(a) < s < R(a)$. Hence $\psi \geq E_{\mathcal{A}_X^W}\phi|A$. The opposite inequality can be proved as above by assuming (as we may) that f is smooth on $\overline{\mathbb{D}}$.

(ii) Let $\mathcal{B}$ be the connected component of $\mathcal{A}_X^W$ containing an analytic disc of the form $\zeta \mapsto (a, s\zeta)$ with $a \in A$ and $r(a) < s < R(a)$. Then $\mathcal{B}$ contains all analytic discs of the form $\zeta \mapsto (a, s\alpha(\zeta))$ with $a \in A$, $r(a) < s < R(a)$, and $\alpha \in \operatorname{Aut}\mathbb{D}$, so $\mathcal{B}$ covers X. As before, we can prove that every analytic disc in $\mathcal{B}$ is center-homotopic to an analytic disc of the latter form. Therefore, if two analytic discs in $\mathcal{B}$ have the same center then they are center-homotopic. Since X is Stein, Theorem 4 now implies that our disc formula holds for the pair $W \subset X$. □

References

[1] N. Bourbaki, *General topology,* Addison-Wesley, Reading, MA, 1966.

[2] S. Q. Bu and W. Schachermayer, *Approximation of Jensen measures by image measures under holomorphic functions and applications,* Trans. Amer. Math. Soc. 331 (1992), 585–608.

[3] B. Drinovec Drnovšek and F. Forstnerič, *The Poletsky–Rosay theorem on singular complex spaces,* Indiana Univ. Math. J. (to appear).

[4] ———, *Disc functionals and Siciak–Zaharyuta extremal functions on singular varieties,* Ann. Polon. Math. 106 (2012), 171–191.

[5] A. Edigarian, *On definitions of the pluricomplex Green function,* Ann. Polon. Math. 67 (1997), 233–246.

[6] F. Forstnerič, *Manifolds of holomorphic mappings from strongly pseudoconvex domains,* Asian J. Math. 11 (2007), 113–126.

[7] G. M. Goluzin, *Geometric theory of functions of a complex variable,* Transl. Math. Monogr., 26, Amer. Math. Soc., Providence, RI, 1969.

[8] C. O. Kiselman, *The partial Legendre transformation for plurisubharmonic functions,* Invent. Math. 49 (1978), 137–148.

[9] F. Lárusson and R. Sigurdsson, *Plurisubharmonic functions and analytic discs on manifolds,* J. Reine Angew. Math. 501 (1998), 1–39.

[10] ———, *Plurisubharmonicity of envelopes of disc functionals on manifolds,* J. Reine Angew. Math. 555 (2003), 27–38.

[11] ———, *The Siciak–Zahariuta extremal function as the envelope of disc functionals,* Ann. Polon. Math. 86 (2005), 177–192.

[12] ———, *Siciak–Zahariuta extremal functions, analytic discs and polynomial hulls,* Math. Ann. 345 (2009), 159–174.

[13] B. S. Magnússon, *Extremal ω-plurisubharmonic functions as envelopes of disc functionals,* Ark. Mat. 49 (2011), 383–399.

[14] E. A. Poletsky, *Plurisubharmonic functions as solutions of variational problems,* Several complex variables and complex geometry (Santa Cruz, 1989), Proc. Sympos. Pure Math., 52, pp. 163–171, Amer. Math. Soc., Providence, RI, 1991.

[15] ———, *Holomorphic currents,* Indiana Univ. Math. J. 42 (1993), 85–144.

[16] E. A. Poletsky and B. V. Shabat, *Invariant metrics,* Several complex variables III, pp. 63–111, Encyclopaedia Math. Sci., 9, Springer-Verlag, Berlin, 1989.

[17] T. Radó, *Sur la représentation conforme de domaines variables,* Acta litterarum ac scientiarum regiae universitatis hungaricae Francisco-Josephinae. Sectio scientiarum mathematicarum 1 (1922/1923), 180–186.

[18] A. Rashkovskii and R. Sigurdsson, *Green functions with singularities along complex spaces,* Internat. J. Math. 16 (2005), 333–355.

[19] J.-P. Rosay, *Poletsky theory of disks on holomorphic manifolds,* Indiana Univ. Math. J. 52 (2003), 157–169.

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