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The Pompeiu Formula for Slice Hyperholomorphic Functions

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1. Introduction

The fundamental result that makes complex analysis into a new discipline, independent from the theory of real variables, is the Cauchy formula, which allows the representation of any holomorphic function through a reproducing holomorphic kernel. This result is in fact an almost immediate application of the Stokes formula, which, in the more general case, offers an integral representation formula for $\mathcal{C}^1$ functions. This general representation formula is often known as the *Pompeiu formula* and can be stated as follows.

Theorem 1.1 (Pompeiu formula). *Let $U \subset \mathbb{C}$ be a bounded open set such that its boundary consists of a union of a finite number of $\mathcal{C}^1$ Jordan curves. If $f \in \mathcal{C}^1(\bar{U})$, we have*

$$f(\zeta) = \frac{1}{2\pi i}\left(\int_{\partial U} \frac{f(z)}{z-\zeta}\,dz + \int_U \frac{\partial f/\partial \bar{z}}{z-\zeta}\,dz \wedge d\bar{z}\right), \quad \zeta \in U.$$

It is clear that, when f is actually holomorphic on U, the Pompeiu formula becomes the usual Cauchy integral formula.

A classical direction for contemporary research has been the search for the analogue of the theory of complex variables when one considers instead functions from the algebra $\mathbb{H}$ of quaternions into itself, or even functions from the Euclidean space $\mathbb{R}^{n+1}$ into the real Clifford algebra $\mathbb{R}_n$. Although several possible theories have been proposed, it is fair to say that the most successful theory in the case of quaternionic functions is the theory of regular functions in the sense of Cauchy–Fueter, which was originally introduced by Fueter and whose main results are summarized in [8; 15; 16]. In the case of Clifford algebra–valued functions, the most successful theory has been the theory of monogenic functions, for which we refer the readers to [1]. In both cases it is possible to reconstruct analogues of the Cauchy kernel and to prove versions of the Cauchy formula. An analogue of the Pompeiu formula is available as well; see [1, Thm. 9.5].

In the last several years, two new notions of hyperholomorphic functions have been suggested by Gentili and Struppa (in the case of quaternionic-valued functions

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defined on the space of quaternions; see [13; 14] for the theory of slice regular functions) and by the authors (in the case of Clifford algebra–valued functions defined on a suitable Euclidean space; see [5; 10; 11; 12] for the theory of slice monogenic functions). The two settings are quite different, but the two theories have several similarities and together offer a rather far-reaching analogue of the classical theory of holomorphic functions. Among the most important results are that in both cases it is possible to define the analogue of the Cauchy kernel (despite the noncommutative nature of the settings) and that it is possible also to prove an analogue of the Cauchy formula. What makes these new theories fairly attractive is that polynomials in the quaternionic (resp. paravector) variables are in fact slice regular (resp. slice monogenic) functions. This allows, in particular, the possibility of considering power series expansions in a way that was not possible in the classical theory of regular or monogenic functions. It is also worth noticing that these new definitions have allowed the authors and their collaborators to develop a new functional calculus for quaternionic operators [3; 4; 6] as well as for n-tuples of noncommuting operators [5; 9]. Therefore, it is of intrinsic interest to see what can be proved in this new context.

In this paper we will show that a recent structure theorem for slice regular and slice monogenic functions (see [2] and [5], respectively) can be used to prove a new quaternionic and Clifford algebra version of the Pompeiu formula. The result that we prove is particularly interesting because it offers more than the simple representation of a $\mathcal{C}^1$ function and, in fact, portrays a new phenomenon.

In order to make our paper self-contained, we have recalled the basic properties of slice regular functions in Section 2, where we also restate their fundamental representation theorem. Section 3 gives the proof of the Pompeiu formula in the quaternionic setting. Finally, Section 4 shows that these same ideas can be used to prove a Pompeiu formula in the setting of Clifford algebras.

2. Preliminaries on Slice Regular Functions

Let $\mathbb{H}$ be the real associative algebra of quaternions endowed with the standard basis $\{1, i, j, k\}$ satisfying the relations

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k,$$
$$jk = -kj = i, \qquad ki = -ik = j.$$

We will write a quaternion q as $q = x_0 + ix_1 + jx_2 + kx_3$ ($x_i \in \mathbb{R}$). As is customary, we will denote its conjugate $x_0 - ix_1 - jx_2 - kx_3$ by $\bar{q}$, its real part x_0 by $\mathrm{Re}(q)$, and its imaginary part $ix_1 + jx_2 + kx_3$ by $\mathrm{Im}(q)$. The square of the module $|q|$ of a quaternion q can be obtained as $|q|^2 = q\bar{q}$.

In the sequel, we will use the symbol $\mathbb{S}$ to denote the 2-sphere of purely imaginary, unit quaternions:

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We will often use the fact that any nonreal quaternion q can be written in a unique way as $q = x + I_q y$ for $x, y \in \mathbb{R}$, $y > 0$, and $I_q \in \mathbb{S}$. The imaginary unit I_q is defined as

$$I_q = \begin{cases} \dfrac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} & \text{if } \mathrm{Im}(q) \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

As pointed out in the Introduction, a major weakness of the classical Fueter theory of regular functions is that not even polynomials are regular in the sense of Fueter. This has several unpleasant consequences, one of which is the difficulty of developing a functional calculus on the basis on this notion of regularity. In [14] the authors introduced a new notion of regularity that addresses this issue and allows polynomials to satisfy a new notion of regularity, the so-called slice regularity. We recall here the definition of slice regular functions, and we include some of their basic properties that will be useful for our purposes.

DEFINITION 2.1. Let $U \subseteq \mathbb{H}$ be an open set and let $f\colon U \to \mathbb{H}$ be a function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $L_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I. We say that f is a *left slice regular* (or *regular*) function if, for every $I \in \mathbb{S}$,

$$\bar{\partial}_I f_I := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0;$$

and we say that f is *right slice regular* (or *right regular*) if, for every $I \in \mathbb{S}$,

$$f_I\bar{\partial}_I := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x+Iy) + \frac{\partial}{\partial y} f_I(x+Iy)I\right) = 0.$$

DEFINITION 2.2. The function defined by

$$S^{-1}(s,q) = -(q^2 - 2q\,\mathrm{Re}(s) + |s|^2)^{-1}(q - \bar{s}) \tag{1}$$

will be called the *Cauchy kernel function*.

These definitions place no stipulations on the nature of the open set U, but several results in the theory of slice regular functions can be obtained only on a special class of open sets. Specifically, we give the following definition.

DEFINITION 2.3. Let $U \subseteq \mathbb{H}$ be a domain in $\mathbb{H}$. We say that U is a *slice domain* (*s-domain* for short) if $U \cap \mathbb{R}$ is nonempty and if $L_I \cap U$ is a domain in L_I for all $I \in \mathbb{S}$. We say that U is *axially symmetric* if, for all $x + Iy \in U$, the whole 2-sphere $x + \mathbb{S}y$ is contained in U.

The following lemma is an immediate consequence of the Stokes formula.

LEMMA 2.4. *Let f, g be quaternionic-valued, continuously (real) differentiable functions on a bounded open set U_I of the plane L_I. Then, for every open $W \subset U_I$ whose boundary consists of a finite number of piecewise smooth closed curves, we have*

$$\int_{\partial W} g\, ds_I f = 2\int_W ((g\bar{\partial}_I)f + g(\bar{\partial}_I f))\, d\sigma,$$

where $s = x + Iy$ is the variable on L_I, $ds_I = -I\, ds$, and $d\sigma = dx \wedge dy$.

One of the most striking properties of s-regular functions is the following representation formula, which allows one to write the value of an s-regular function in terms of its values on a given complex plane.

LEMMA 2.5. *Let $U \subseteq \mathbb{H}$ be an axially symmetric s-domain. Let $f: U \to \mathbb{H}$ be an s-regular function. Then, for all $q = x + I_q y \in U$ and $I \in \mathbb{S}$, the following formula holds*:

$$f(q) = \frac{1}{2}[1 - I_q I] f(x + Iy) + \frac{1}{2}[1 + I_q I] f(x - Iy). \tag{2}$$

This lemma is a key ingredient in proving that, on axially symmetric s-domains, it is possible to obtain a Cauchy formula in which the integrals (computed on a given complex plane L_I) do not depend on the imaginary unit $I \in \mathbb{S}$. A version of it was originally proved in [14], but this version was actually first given in [2].

THEOREM 2.6 (Cauchy formula). *Let $U \subseteq \mathbb{H}$ be an axially symmetric bounded open set such that $\partial(U \cap L_I)$ is a union of a finite number of rectifiable Jordan arcs. Let f be a regular function on U and, for any $I \in \mathbb{S}$, set $ds_I = -I\, ds$. Then, for every $q \in U$, we have*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} -(q^2 - 2\,\mathrm{Re}(s) q + |s|^2)^{-1} (q - \bar{s})\, ds_I f(s). \tag{3}$$

Moreover, the value of the integral depends neither on U nor on the imaginary unit $I \in \mathbb{S}$.

3. The Pompeiu Formula in the Quaternionic Setting

In this section we prove the Pompeiu formula for a quaternionic-valued function f defined on a suitable subset of $\mathbb{H}$.

THEOREM 3.1 (Pompeiu formula). *Let $U \subset \mathbb{H}$ be an axially symmetric open bounded set such that $\partial(U \cap L_I)$ is a union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Let $f: \bar{U} \to \mathbb{H}$ be a function of class $\mathcal{C}^1$ and set $ds_I = -I\, ds$. For every $q \in U$, $q = x + I_q y$, and $I \in \mathbb{S}$, we have*

$$\begin{aligned}
&\frac{1}{2}[1 - I_q I] f(x + Iy) + \frac{1}{2}[1 + I_q I] f(x - Iy) \\
&\quad = \frac{1}{2\pi} \left(\int_{\partial(U \cap L_I)} S^{-1}(s, q)\, ds_I f(s) + \int_{U \cap L_I} S^{-1}(s, q) \bar{\partial}_I f(s)\, ds_I \wedge d\bar{s} \right).
\end{aligned} \tag{4}$$

In particular, when $I = I_q$ we have

$$f(q) = \frac{1}{2\pi} \left(\int_{\partial(U \cap L_{I_q})} S^{-1}(s, q)\, ds_{I_q} f(s) + \int_{U \cap L_{I_q}} S^{-1}(s, q) \bar{\partial}_{I_q} f(s)\, ds_{I_q} \wedge d\bar{s} \right). \tag{5}$$

Proof. Let us set $q = x + I_q y$ and define

$$U_\varepsilon = \{ s = x' + I_s y' \in U \mid |(x + Iy) - (x' + Iy')| > \varepsilon \ \forall I \in \mathbb{S} \},$$

where ε is a positive number less than the distance from the 2-sphere $x+\mathbb{S}y$ defined by q to the complement of U. The zeros of the function $q^2 - 2\,\mathrm{Re}(s)q + |s|^2 = 0$ consist of either a real point or a 2-sphere $x + Iy$. On L_{I_q} we find only the point q as a singularity, and the result then follows from the Pompeiu formula on the complex plane L_{I_q}. When the singularity is a real number, S^{-1} is the standard Cauchy kernel and again the statement follows from the Pompeiu formula on the complex plane L_I for every $I \in \mathbb{S}$. If the zero is not real, then on any complex plane L_I we find two zeros $s_{1,2} = x \pm Iy$. Thus $\partial U_\varepsilon = \partial U - \partial B_1 - \partial B_2$, where ∂B_i is the boundary of ball B_i with center s_i and radius ε.

From Lemma 2.4 applied to the function $S^{-1}(s,q)f(s)$ and since $S^{-1}(s,q)$ is right s-regular in the variable s, we obtain

$$\frac{1}{2\pi}\int_{U_\varepsilon\cap L_I} S^{-1}(s,q)\bar{\partial}_I f(s)\, ds_I \wedge d\bar{s} + \frac{1}{2\pi}\int_{\partial(U_\varepsilon\cap L_I)} S^{-1}(s,q)\, ds_I f(s) = \mathfrak{I}_1^\varepsilon(q) + \mathfrak{I}_2^\varepsilon(q),$$

where

$$\mathfrak{I}_1^\varepsilon(q) := \frac{1}{2\pi}\int_{\partial(B_1\cap L_I)} S^{-1}(s,q)\, ds_I f(s),$$

$$\mathfrak{I}_2^\varepsilon(q) := \frac{1}{2\pi}\int_{\partial(B_2\cap L_I)} S^{-1}(s,q)\, ds_I f(s).$$

Let us start with $\mathfrak{I}_1^\varepsilon(q)$ by setting the positions

$$s = s_1 + \varepsilon e^{I\theta} = x + Iy + \varepsilon e^{I\theta},$$

$$\mathrm{Re}(s) = x + \varepsilon\cos\theta,$$

$$\bar{s} = x - Iy + \varepsilon e^{-I\theta},$$

$$ds_I = -[\varepsilon I e^{I\theta}]I\, d\theta = \varepsilon e^{I\theta}\, d\theta,$$

$$|s|^2 = x^2 + 2x\varepsilon\cos\theta + \varepsilon^2 + y^2 + 2\varepsilon\sin\theta y,$$

from which we obtain

$$\mathfrak{I}_1^\varepsilon(q) := \frac{1}{2\pi}\int_0^{2\pi} S^{-1}(s_1 + \varepsilon e^{I\theta}, q)\, d\theta f(s_1 + \varepsilon e^{I\theta}).$$

With computations similar to those in the proof of [5, Thm. 2.15], we have

$$\mathfrak{I}_1^\varepsilon(q) = \frac{1}{2\pi}\int_0^{2\pi} -(-2q\varepsilon\cos\theta + 2x\varepsilon\cos\theta + \varepsilon^2 + 2\varepsilon\sin\theta y)^{-1} (q - [x - Iy + \varepsilon e^{-I\theta}])\varepsilon e^{I\theta}\, d\theta f(x + Iy + \varepsilon e^{I\theta}).$$

Letting $\varepsilon \to 0$ and performing some computations yields

$$\mathfrak{I}_1^0(q) = \frac{1}{2}[1 - I_q I]f(x + Iy).$$

With analogous calculations, the integral related to s_2 turns out to be

$$\mathfrak{I}_2^0(q) = \frac{1}{2}[1 + I_q I]f(x - Iy).$$

So we get

$$\mathfrak{I}_1^0(q) + \mathfrak{I}_2^0(q) = \frac{1}{2}[1 - I_q I] f(x + Iy) + \frac{1}{2}[1 + I_q I] f(x - Iy),$$

and this concludes the proof. □

REMARK 3.2. Note that equation (5) is not surprising and, in fact, is the exact analogue of the Pompeiu formula in the complex case. Equation (4), on the other hand, highlights a new phenomenon: given a point q and an imaginary unit $I \in \mathbb{S}$, there are exactly two points in L_I on the same sphere of q; and equation (4) shows how to obtain an integral representation of f at those points.

An immediate consequence of Lemma 2.5 and Theorem 3.1 is the Cauchy formula (see [2]).

COROLLARY 3.3 (Cauchy formula). *If f is s-regular on an axially symmetric bounded s-domain $U \subseteq \mathbb{H}$, then for every $q \in U$ we have*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap L_I)} S^{-1}(s, q)\, ds_I\, f(s).$$

4. The Pompeiu Formula in the Clifford Algebra Setting

The same arguments used in the previous section can be applied to the study of functions defined on a Euclidean space $\mathbb{R}^{n+1}$ and with values in the Clifford algebra $\mathbb{R}_n$. For the reader's convenience we recall that $\mathbb{R}_n$ is the real algebra generated by n imaginary units e_i such that $e_i^2 = -1$ and $e_i e_j + e_j e_i = 0$ for $i \neq j$. An element $(x_0, x_1, \dots, x_n)$ in $\mathbb{R}^{n+1}$ will be identified with an element $x = x_0 + x_1 e_1 + \cdots + x_n e_n = x_0 + \underline{x}$ in the Clifford algebra $\mathbb{R}_n$. The scalar x_0 corresponds to the real part $\mathrm{Re}(x)$ of x. In the sequel, we will denote by $\mathbb{S}$ the sphere of unit 1-vectors in $\mathbb{R}^n$; that is,

$$\mathbb{S} = \{y = e_1 x_1 + \cdots + e_n x_n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, let us set

$$I_x = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

Analogously to our treatment of the quaternionic case (see also [10]), we establish the following definitions.

DEFINITION 4.1. Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f\colon U \to \mathbb{R}_n$ be a function. Let $I \in \mathbb{S}$, let f_I be the restriction of f to the complex plane $L_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I, and denote by $u + Iv$ an element on L_I. We say that f is a *left slice monogenic* function if, for every $I \in \mathbb{S}$,

$$\bar{\partial}_I f = \frac{1}{2}\left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v}\right) f_I(u + Iv) = 0.$$

DEFINITION 4.2. The function defined by

$$S^{-1}(s,x) = -(x^2 - 2x\,\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}) \tag{6}$$

will be called *Cauchy kernel function.*

The structure formula for s-monogenic functions [5; 7] is given as follows.

LEMMA 4.3. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Let $f : U \to \mathbb{R}_n$ be an s-monogenic function. For every $x = x_0 + I_x|\underline{x}| \in U$, the following formula holds*:

$$f(x) = \frac{1}{2}[1 - I_x I]f(x_0 + I|\underline{x}|) + \frac{1}{2}[1 + I_x I]f(x_0 - I|\underline{x}|).$$

Note that, despite their formal similarities, the two theories are actually independent of each other. Nevertheless, the same arguments used in the previous section allow us to prove the following result.

THEOREM 4.4 (Pompeiu). *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain, and let $\partial(U \cap L_I)$ be a union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$. Let $f \in \mathcal{C}^1(\bar{U})$ and set $ds_I = ds/I$. For every $x \in U$, $x = x_0 + I_x|\underline{x}|$, and $I \in \mathbb{S}$, we have*

$$\begin{aligned}\frac{1}{2}&[1 - I_x I]f(x_0 + I|\underline{x}|) + \frac{1}{2}[1 + I_x I]f(x_0 - I|\underline{x}|)\\ &= \frac{1}{2\pi}\left(\int_{\partial(U\cap L_I)} S^{-1}(s,x)\,ds_I f(s) + \int_{U\cap L_I} S^{-1}(s,x)\bar{\partial}_I f(s)\,ds_I \wedge d\bar{s}\right).\end{aligned} \tag{7}$$

In particular, when $I = I_x$ we have

$$f(x) = \frac{1}{2\pi}\left(\int_{\partial(U\cap L_{I_x})} S^{-1}(s,x)\,ds_{I_x} f(s) + \int_{U\cap L_{I_x}} S^{-1}(s,x)\bar{\partial}_{I_x} f(s)\,ds_{I_x} \wedge d\bar{s}\right). \tag{8}$$

REMARK 4.5. When $n = 1$, equations (7) and (8) each reproduce the Pompeiu formula in the complex case because in that case $\mathbb{S}$ contains only the two points $\pm i$.

As a corollary to Lemma 4.3 and Theorem 4.4, we re-derive the Cauchy formula (see [5]).

COROLLARY 4.6 (Cauchy formula). *If f is s-monogenic on an axially symmetric bounded s-domain U then, for every $x \in U$, we have*

$$f(x) = \frac{1}{2\pi}\int_{\partial(U\cap L_I)} S^{-1}(s,x)\,ds_I f(s).$$

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