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Proper Holomorphic Maps between Reinhardt Domains in $\mathbb{C}^2$

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0. Introduction

Let D_1, D_2 be bounded Reinhardt domains in $\mathbb{C}^2$ and let $f : D_1 \to D_2$ be a proper holomorphic map. Such maps are often *elementary algebraic*; that is, have the "monomial" form

$$z \mapsto \operatorname{const} z^a w^b,$$

$$w \mapsto \operatorname{const} z^c w^d,$$

where z, w denote variables in $\mathbb{C}^2$ and where a, b, c, d are integers such that $ad - bc \neq 0$. For brevity we shall call such maps *elementary* maps. All elementary maps are well-defined outside I, the union of the coordinate complex lines, but not necessarily at points in I. The question of the existence of an elementary proper holomorphic map between two given domains is resolved by passing to the logarithmic diagrams of the domains. Several classes of domains between which only elementary proper holomorphic maps are possible have been described in [S].

The aim of this paper is to identify situations in which f is not elementary and to explicitly describe all forms that the map f and the domains D_1, D_2 may have in such cases. If f is biholomorphic, then it can be represented as the composition of an elementary biholomorphism between D_1 and D_2 and automorphisms of these domains (see [Kr; Sh]). Therefore, nonelementary biholomorphisms can occur only between domains that are equivalent by means of an elementary map and having nonelementary automorphisms (and that are hence straightforward to determine).

Proper maps that are not biholomorphic are harder to deal with. Nonelementary maps may occur, for example, if each of D_1, D_2 is a bidisc, in which case at least one component of f contains a Blaschke product with a zero away from the origin. In [BeP] and [LS], the problem of describing nonelementary proper holomorphic maps was studied for complete Reinhardt domains; it turns out that, apart from the example of bidiscs, such maps can arise only if D_1 and D_2 are certain pseudo-ellipsoids. On the other hand, all proper holomorphic maps between pseudo-ellipsoids in $\mathbb{C}^n$ for $n \geq 2$ can be found using arguments from [D-SP]. All

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proper holomorphic maps for another special class of domains (a generalization to higher dimensions of domains of the form (0.2), to follow) were determined in [L]. Similarly to the case of bidiscs, the only nonelementary maps for this class are expressed in terms of Blaschke products with at least one zero away from the origin.

In this paper we describe all nonelementary proper holomorphic maps between Reinhardt domains in $\mathbb{C}^2$, as well as the corresponding pairs of domains. First of all, the map f can be extended to a proper holomorphic map between the envelopes of holomorphy $\hat{D}_1$ and $\hat{D}_2$ of D_1 and D_2, respectively. The passage to the pseudoconvex domains $\hat{D}_1, \hat{D}_2$ is essential for our arguments because we shall often use the connectedness of the boundaries of $\hat{D}_1, \hat{D}_2$ and the convexity of their logarithmic diagrams. Furthermore, f extends holomorphically to a neighborhood of $\partial\hat{D}_1 \setminus I$. Next, one can show that f can be nonelementary only if $\partial\hat{D}_1 \setminus I$ either consists of two or three Levi-flat pieces or is a connected spherical hypersurface (see Section 1).

The Levi-flat and spherical cases are considered in Sections 2 and 3, respectively, and the results are summarized in Theorem 0.1. In the spherical case (see (iv)–(vi) of Theorem 0.1), the map f can be represented as the composition of three maps of special forms: two elementary maps and an automorphism of an intermediate Reinhardt domain. We note that a factorization result of a different kind for proper maps into the unit ball was obtained in [KLS]. In the case where D_1 is a strongly pseudoconvex smoothly bounded Reinhardt domain in $\mathbb{C}^n$ for $n \geq 2$ that does not intersect the coordinate hyperplanes (while D_2 is not necessarily Reinhardt), another factorization theorem was proved in [BDa]. We also observe that the nonelementary proper holomorphic map between pseudo-ellipsoids in the example given in [D-SP] factors as in Theorem 0.1(vi).

Our results immediately imply that, if there exists a proper holomorphic map between two bounded Reinhardt domains, then there also exists an elementary proper map between the domains (Corollary 0.2). Another consequence of our classification is that the domains described in (i)–(iii) of Theorem 0.1 are the only domains for which there exist nonelementary nonbiholomorphic proper self-maps (Corollary 0.3).

THEOREM 0.1. *Let D_1, D_2 be bounded Reinhardt domains in $\mathbb{C}^2$ and let $f\colon D_1 \to D_2$ be a proper holomorphic map. Assume that f is not elementary. Then one of the following six scenarios obtains.*

(i) *Up to permutation of the components of f and the variables, the map f has the form*

$$\begin{aligned} z &\mapsto \operatorname{const} z^a w^b B(A_1 z^{p_1} w^{q_1}), \\ w &\mapsto \operatorname{const} w^c, \end{aligned} \tag{0.1}$$

where $a, b, c, p_1, q_1 \in \mathbb{Z}$, $a > 0$, $c > 0$, $p_1 > 0$, $q_1 \leq 0$, p_1 and q_1 are relatively prime, $aq_1 - bp_1 \leq 0$, $A_1 > 0$, and B is a nonconstant Blaschke product in the unit disc that is nonvanishing at 0. *In this case, D_1 either has the form*

$$\{(z,w)\in\mathbb{C}^2 : A_1|z|^{p_1}|w|^{q_1} < 1,\ 0<|w|<C_1\} \tag{0.2}$$

for some $C_1 > 0$ *or is a bidisc* (*in the latter case we have* $b=0$, $p_1=1$, *and* $q_1 = 0$ *in* (0.1)). *The domain* D_2 *is, respectively, either a domain of the form*

$$\{(z,w)\in\mathbb{C}^2 : A_2|z|^{p_2}|w|^{q_2} < 1,\ 0<|w|<C_2\}$$

(*where* $p_2, q_2\in\mathbb{Z}$ *are relatively prime,* $p_2>0$, $q_2\le 0$, $q_2/p_2=(aq_1-bp_1)/(cp_1)$, $A_2>0$, *and* $C_2>0$) *or a bidisc.*

(ii) *Up to permutation of the components of* f *and the variables, the map* f *has the form* (0.1), *where* $a,b,c,p_1,q_1\in\mathbb{Z}$, $a>0$, $c\neq 0$, $p_1>0$, p_1 *and* q_1 *are relatively prime,* $A_1>0$, *and* B *is a nonconstant Blaschke product in the unit disc that is nonvanishing at* 0. *In this case the domains have the forms*

$$\begin{aligned} D_1 &= \{(z,w)\in\mathbb{C}^2 : A_1|z|^{p_1}|w|^{q_1}<1,\ E_1<|w|<C_1\},\\ D_2 &= \{(z,w)\in\mathbb{C}^2 : A_2|z|^{p_2}|w|^{q_2}<1,\ E_2<|w|<C_2\}, \end{aligned} \tag{0.3}$$

where $p_2,q_2\in\mathbb{Z}$ *are relatively prime,* $p_2>0$, $q_2/p_2=(aq_1-bp_1)/(cp_1)$, *and* $C_1>0$, $E_1>0$, $A_2>0$, $C_2>0$, *and* $E_2>0$.

(iii) *Up to permutation of the components of* f, *the map* f *has the form*

$$\begin{aligned} z &\mapsto \operatorname{const} z^a B_1(Az),\\ w &\mapsto \operatorname{const} w^b B_2(Cw), \end{aligned}$$

where $a,b\in\mathbb{Z}$, $a\ge 0$, $b\ge 0$, $A>0$, $C>0$, *and* B_1, B_2 *are nonconstant Blaschke products in the unit disc that are nonvanishing at* 0. *In this case,* D_1 *and* D_2 *are bidiscs.*

(iv) *The map* f *is a composition* $f=\mathbf{g}\circ\mathbf{f}\circ\mathbf{h}$, *where* $\mathbf{h}$ *is an elementary map from* D_1 *into the domain* $D := \{(z,w)\in\mathbb{C}^2 : |w| > \exp[|z|^2]\}$, $\mathbf{f}$ *is an automorphism of* D, *and* $\mathbf{g}$ *is an elementary map from a subdomain of* D *onto* D_2. *Up to permutation of the variables, the map* $\mathbf{h}$ *has the form*

$$\begin{aligned} z &\mapsto \operatorname{const} z^{a_1} w^{-b_1},\\ w &\mapsto \operatorname{const} w^{-c_1}, \end{aligned}$$

where $a_1,b_1,c_1\in\mathbb{N}$; *the map* $\mathbf{f}$ *has the form*

$$\begin{aligned} z &\mapsto e^{it_1}z+s,\\ w &\mapsto e^{it_2}\exp[2\bar{s}e^{it_1}z+|s|^2]w, \end{aligned}$$

where $t_1,t_2\in\mathbb{R}$ *and* $s\in\mathbb{C}^*$; *and, up to permutation of its components, the map* $\mathbf{g}$ *has the form*

$$\begin{aligned} z &\mapsto \operatorname{const} z^{a_2} w^{-b_2},\\ w &\mapsto \operatorname{const} w^{-c_2}, \end{aligned}$$

where $a_2,b_2,c_2\in\mathbb{N}$. *In this case the domains have the forms*

$$D_1 = \{(z, w) \in \mathbb{C}^2 : C_1' \exp[-E_1|z|^{2a_1}|w|^{-2b_1}] < |w| < C_1 \exp[-E_1|z|^{2a_1}|w|^{-2b_1}]\},$$

$$D_2 = \{(z, w) \in \mathbb{C}^2 : C_2' \exp[-E_2|z|^{2/a_2}|w|^{-2b_2/a_2c_2}] < |w| < C_2 \exp[-E_2|z|^{2/a_2}|w|^{-2b_2/a_2c_2}]\},$$

where $0 \le C_1' < C_1$, $0 \le C_2' < C_2$, $E_1 > 0$, *and* $E_2 > 0$.

(v) *The map* f *is a composition* $f = \mathbf{g} \circ \mathbf{f} \circ \mathbf{h}$, *where* $\mathbf{h}$ *is an elementary map from* D_1 *into the domain* $\Omega^\alpha := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^\alpha < 1\}$ *for some* $\alpha > 0$, $\mathbf{g}$ *is an elementary map from a subdomain of* Ω^α *onto* D_2, *and* $\mathbf{f}$ *is an automorphism of* Ω^α. *Up to permutation of the variables, the map* $\mathbf{h}$ *has the form*

$$\begin{aligned} z &\mapsto \operatorname{const} z^{a_1} w^{-b_1}, \\ w &\mapsto \operatorname{const} w^{c_1}, \end{aligned} \tag{0.4}$$

where $a_1, b_1, c_1 \in \mathbb{Z}$, $a_1 > 0$, $b_1 \ge 0$, *and* $c_1 > 0$; *the map* $\mathbf{f}$ *has the form*

$$\begin{aligned} z &\mapsto e^{it_1} \frac{z - a}{1 - \bar{a}z}, \\ w &\mapsto e^{it_2} \frac{(1 - |a|^2)^{1/\alpha}}{(1 - \bar{a}z)^{2/\alpha}} w, \end{aligned}$$

where $|a| < 1$, $a \ne 0$, *and* $t_1, t_2 \in \mathbb{R}$; *and, up to permutation of its components, the map* $\mathbf{g}$ *has the form*

$$\begin{aligned} z &\mapsto \operatorname{const} z^{a_2} w^{b_2}, \\ w &\mapsto \operatorname{const} w^{c_2}, \end{aligned} \tag{0.5}$$

where $a_2, b_2, c_2 \in \mathbb{Z}$, $a_2 > 0$, $b_2 \ge 0$, *and* $c_2 > 0$. *In this case the domains have either the forms*

$$D_1 = \{(z, w) \in \mathbb{C}^2 : C_1|z|^{2a_1} + E_1|w|^{\alpha c_1} < 1\},$$
$$D_2 = \{(z, w) \in \mathbb{C}^2 : C_2|z|^{2/a_2} + E_2|w|^{\alpha/c_2} < 1\}$$

or the forms

$$D_1 = \{(z, w) \in \mathbb{C}^2 : C_1|z|^{2a_1}|w|^{-2b_1} < 1, E_1'(1 - C_1|z|^{2a_1}|w|^{-2b_1})^{1/\alpha c_1} < |w| < E_1(1 - C_1|z|^{2a_1}|w|^{-2b_1})^{1/\alpha c_1}\},$$

$$D_2 = \{(z, w) \in \mathbb{C}^2 : C_2|z|^{2/a_2}|w|^{-2b_2/a_2c_2} < 1, E_2'(1 - C_2|z|^{2/a_2}|w|^{-2b_2/a_2c_2})^{c_2/\alpha} < |w| < E_2(1 - C_2|z|^{2/a_2}|w|^{-2b_2/a_2c_2})^{c_2/\alpha}\}$$

for some $C_1 > 0$, $C_2 > 0$, $0 \le E_1' < E_1$, *and* $0 \le E_2' < E_2$. (*In the first case of* (0.4) *and* (0.5) *we have* $b_1 = 0$ *and* $b_2 = 0$.)

(vi) *The map* f *is a composition* $f = \mathbf{g} \circ \mathbf{f} \circ \mathbf{h}$, *where* $\mathbf{h}$ *is an elementary map from* D_1 *onto the unit ball* $B^2 := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$, $\mathbf{f}$ *is an automorphism of* B^2, *and* $\mathbf{g}$ *is an elementary map from* B^2 *onto* D_2. *Up to permutation of the variables, the map* $\mathbf{h}$ *has the form*

$$z \mapsto \operatorname{const} z^{a_1},$$
$$w \mapsto \operatorname{const} w^{b_1},$$

where $a_1, b_1 \in \mathbb{N}$; *the map* $\mathbf{f}$ *is such that* $\mathbf{f}(B^2 \cap \mathcal{L}_z) \not\subset B^2 \cap I$ *and* $\mathbf{f}(B^2 \cap \mathcal{L}_w) \not\subset B^2 \cap I$, *where* $\mathcal{L}_z := \{z = 0\}$, $\mathcal{L}_w := \{w = 0\}$, *and* $I := \mathcal{L}_z \cup \mathcal{L}_w$; *and, up to permutation of the variables, the map* $\mathbf{g}$ *has the form*

$$z \mapsto \operatorname{const} z^{a_2},$$
$$w \mapsto \operatorname{const} w^{b_2},$$

where $a_2, b_2 \in \mathbb{N}$. *In this case the domains have the forms*

$$D_1 = \{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1} + E_1|w|^{2b_1} < 1\},$$
$$D_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{2/a_2} + E_2|w|^{2/b_2} < 1\},$$

where $C_1 > 0$, $E_1 > 0$, $C_2 > 0$, *and* $E_2 > 0$.

We will now state two corollaries of Theorem 0.1, as mentioned previously.

Corollary 0.2. *Let* D_1 *and* D_1 *be bounded Reinhardt domains in* $\mathbb{C}^2$. *If there exists a proper holomorphic map from* D_1 *onto* D_2, *then there also exists an elementary proper map from* D_1 *onto* D_2.

Corollary 0.3. *Let* D *be a bounded Reinhardt domain in* $\mathbb{C}^2$ *that admits a nonelementary, nonbiholomorphic, proper holomorphic self-map. Then either* D (*up to permutation of the variables*) *has one of the forms* (0.2) *or* (0.3) *or* D *is a bidisc.*

On the other hand, if a bounded pseudoconvex Reinhardt domain D admits an elementary, nonbiholomorphic, proper holomorphic self-map, then either D is a bidisc or D (up to permutation of the variables) has one of the forms (0.2) or (0.3) or the form

$$\{(z,w) \in \mathbb{C}^2 : A|z|^p|w|^q < 1,\ E < |z|^{p'}|w|^{q'} < C\},$$

where $A > 0$, $C > 0$, $E \geq 0$, and p, q, p', q' are integers satisfying conditions similar to those in (0.2) and (0.3). This is easy to see from the following observation: if the logarithmic diagram of D is unbounded, then the two asymptotes of its convex boundary define the eigendirections of either the linear part of the affine transformation of the logarithmic diagram corresponding to the elementary map or the square of this operator.

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1. Preliminaries

As we pointed out in the Introduction, f can be extended to a proper holomorphic map (that we also denote by f) between the envelopes of holomorphy $\hat{D}_1, \hat{D}_2$ of D_1, D_2, respectively (see [Ke]). Moreover, it follows from [Ba] (see also [L]) that f extends holomorphically to a neighborhood of $\partial\hat{D}_1 \setminus I$, where $I := \mathcal{L}_z \cup \mathcal{L}_w$ with $\mathcal{L}_z := \{z = 0\}$ and $\mathcal{L}_w := \{w = 0\}$. Since f is proper, $f(\partial\hat{D}_1 \setminus I) \subset \partial\hat{D}_2$. Denote by J_f the zero set of the Jacobian of f in $\partial\hat{D}_1 \setminus I$, and let $C_f := f^{-1}(f(\partial\hat{D}_1 \setminus I) \cap I)$.

For every $p = (z, w) \in \mathbb{C}^2$, let $\mathbb{T}(p)$ be the torus $\{(e^{i\alpha}z, e^{i\beta}w) \in \mathbb{C}^2 : \alpha, \beta \in \mathbb{R}\}$ and let $\mathbb{T}$ be the standard torus $\mathbb{T}((1,1))$. We shall think of $\mathbb{T}$ as a group acting on $\mathbb{C}^2$. Set

$$\mathcal{S}_f := \{p \in \partial\hat{D}_1 \setminus I : f(\mathbb{T}(p)) \subset \mathbb{T}(f(p))\}.$$

We assume that $\mathcal{S}_f$ does not contain three distinct tori that do not lie in a Levi-flat Reinhardt hypersurface, since otherwise f is elementary by Lemma 4.4 of [S]. Hereafter we refer to this assumption as Condition $(*)$.

For $j = 1, 2$ we denote by H_j the union of all locally holomorphically homogeneous, connected, real-analytic hypersurfaces lying in $\partial\hat{D}_j \setminus I$. Hypersurfaces making up H_j are either strongly pseudoconvex or Levi-flat, and we denote by H_j^{spher}, H_j^{nonspher}, and H_j^{flat} the unions of all spherical (i.e., locally biholomorphically equivalent to the unit sphere in $\mathbb{C}^2$), strongly pseudoconvex nonspherical, and Levi-flat hypersurfaces from H_j, respectively, for $j = 1, 2$.

Note that locally holomorphically homogeneous, nonspherical, Reinhardt hypersurfaces do exist, so Lemma 3.3 of [S] stating otherwise is incorrect. Consider, for example, the nonspherical tube hypersurface

$$\mathbf{T} := \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w = (\operatorname{Re} z)^3,\ \operatorname{Re} z > 0\}.$$

The base of $\mathbf{T}$ is an affinely homogeneous curve (a complete list of affinely homogeneous curves in $\mathbb{R}^2$ can be found in [NSa]) and hence $\mathbf{T}$ is holomorphically homogeneous. The map $\Pi\colon (z, w) \mapsto (e^z, e^w)$ takes suitable portions of $\mathbf{T}$ to locally holomorphically homogeneous, nonspherical, Reinhardt hypersurfaces.

We will now show that, for nonelementary f, the set H_1^{nonspher} must be empty.

Proposition 1.1. *Condition $(*)$ implies that $H_1^{\text{nonspher}} = \emptyset$.*

Proof. A Reinhardt hypersurface $N \subset \mathbb{C}^2 \setminus I$ is locally biholomorphically equivalent to the tube hypersurface $T_N := \log(N) + i\mathbb{R}^2$, whose base is the logarithmic diagram $\log(N) \subset \mathbb{R}^2$ of N (cf. the foregoing example). If N is real analytic, strongly pseudoconvex, nonspherical, and locally holomorphically homogeneous, then infinitesimal CR-transformations of T_N form a three-dimensional Lie algebra $\mathfrak{g}_{T_N}$ (see e.g. [C]). Further, it follows from [Lo] that the curve $\log(N)$ is locally affinely homogeneous. Taking into account that translations in the imaginary directions form a two-dimensional subalgebra $\mathfrak{h}_{T_N}$ in $\mathfrak{g}_{T_N}$, we see that $\mathfrak{g}_{T_N}$ is generated by $\mathfrak{h}_{T_N}$ and the one-dimensional algebra of local affine transformations of $\log(N)$. Hence $\mathfrak{h}_{T_N}$ is an ideal in $\mathfrak{g}_{T_N}$, and it is straightforward to observe that there are no other ideals in $\mathfrak{g}_{T_N}$. Let $\mathfrak{h}_N$ be the ideal corresponding to $\mathfrak{h}_{T_N}$ in the Lie algebra $\mathfrak{g}_N$ of all infinitesimal CR-transformations of N. The ideal $\mathfrak{h}_N$ consists of infinitesimal transformations corresponding to the action of $\mathbb{T}$ on N.

Suppose that $H_1^{\text{nonspher}} \neq \emptyset$ and let M be a hypersurface contained in H_1^{nonspher}. Fix $p \in M \setminus (C_f \cup J_f)$. Clearly, f maps a neighborhood of p in M biholomorphically onto a hypersurface $M' \subset H_2^{\text{nonspher}}$. The homomorphism between $\mathfrak{g}_M$ and $\mathfrak{g}_{M'}$ induced by f maps $\mathfrak{h}_M$ into $\mathfrak{h}_{M'}$. Hence we have $f(\mathbb{T}(p)) \subset \mathbb{T}(f(p))$ and thus $M \setminus (C_f \cup J_f) \subset \mathcal{S}_f$, which contradicts Condition $(*)$. Therefore, $H_1^{\text{nonspher}} = \emptyset$ as required. □

Assume now that $H_1^{\text{spher}} \neq \emptyset$. Define $S_1 := \partial\hat{D}_1 \setminus (H_1 \cup I)$. We will need the following general lemma (see also [LS] and [S]).

LEMMA 1.2. *We have $S_1 \subset \mathcal{S}_f$.*

Proof. Assume that $f(\mathbb{T}(p)) \not\subset \mathbb{T}(f(p))$ and let $p' \in \mathbb{T}(p)$ be a point close to p such that $p' \notin C_f \cup J_f$ and $f(\mathbb{T}(p')) = f(\mathbb{T}(p))$ is not tangent to $\mathbb{T}(f(p'))$. Choose a neighborhood U of p' in which f is biholomorphic and let $V := f(U)$. We may assume that $f(\mathbb{T}(q)) \not\subset \mathbb{T}(f(p'))$ for all $q \in U$. Let $T := V \cap \mathbb{T}(f(p'))$ and let $\gamma \subset f(\mathbb{T}(p'))$ be the image of the orbit of p' on $\mathbb{T}(p')$ under the action of a 1-parameter subgroup of $\mathbb{T}$ such that γ is not tangent to T. Consider now the set $\Gamma := \bigcup_{s\in\gamma} \mathbb{T}(s)$. This is clearly a real-analytic hypersurface in $\partial\hat{D}_2$, which is, moreover, locally holomorphically homogeneous because we have on it actions of a two-dimensional torus and a 1-parameter group and the orbits of one action are transversal to those of the other. Thus, $f(p') \in H_2$ and therefore $p' \in H_1$. This means that $p \in H_1$, which contradicts the assumptions of the lemma. Hence $f(\mathbb{T}(p)) \subset \mathbb{T}(f(p))$, as required. □

Let M be a connected component of H_1^{spher}. Arguing as in a similar situation in the proof of Proposition 3.2 of [S] and using Lemma 1.2, one sees that if $\bar{M}$ intersects S_1 then $M \subset \mathcal{S}_f$, which contradicts Condition $(*)$. Thus, $\bar{M} \cap S_1 = \emptyset$. Hence, if there exists another hypersurface N lying in H_1^{spher} or in H_1^{flat} such that $M \cap N = \emptyset$ and $\bar{M} \cap \bar{N} \neq \emptyset$, then $\bar{M} \cap \bar{N} \subset I$. However, passing to the logarithmic diagram of $\hat{D}_1$, we immediately see from its convexity that this is impossible. As a result, if $H_1^{\text{spher}} \neq \emptyset$ then $\partial\hat{D}_1 \setminus I$ is a connected spherical hypersurface. Finally, if $H_1 = H_1^{\text{flat}}$ then Lemma 1.2 implies that S_1 contains at most two distinct tori.

To summarize, nonelementary proper holomorphic maps can exist only in the following two cases: either $H_1 = H_1^{\text{flat}}$ and S_1 consists of one or two distinct tori; or $\partial\hat{D}_1 \setminus I$ is a connected spherical hypersurface. These cases are considered in Sections 2 and 3, respectively. We remark that a related decomposition result in the context of biholomorphic maps of not necessarily bounded Reinhardt domains in $\mathbb{C}^2$ was obtained in Theorem 8 of [So].

2. Levi-Flat Case

In this section we assume that $H_1 = H_1^{\text{flat}}$. Note that in this case the logarithmic diagram of $\hat{D}_1$ is an unbounded polygon with either one or two vertices, depending on the number (one or two, respectively) of tori in S_1.

The Case of a Single Torus. Let f_1, f_2 be the components of f and assume first that S_1 is a single torus $\mathbb{T}_1$. Let $\log(\hat{D}_1)$ be the logarithmic diagram of $\hat{D}_1$. The set H_1 can be represented as the union of two distinct Levi-flat hypersurfaces L_1^1, L_1^2 whose boundaries in $\mathbb{C}^2 \setminus I$ coincide with $\mathbb{T}_1$. Furthermore, since $\hat{D}_1$ is bounded, $\log(\hat{D}_1)$ is a sector lying in the interior of a right angle of the form $\{(x, y) \in \mathbb{R}^2 : x < x_0, y < y_0\}$ for some $(x_0, y_0) \in \mathbb{R}^2$. We can describe this sector as follows:

$$\log(\hat{D}_1) = \{(x, y) \in \mathbb{R}^2 : a_1 x - y > -\ln C,\ x + d_1 y < -\ln A\},$$

where $a_1 \geq 0$, $d_1 \leq 0$, $a_1 d_1 > -1$, and $A, C > 0$. Note that $\hat{D}_1$ can contain the origin only if $a_1 = d_1 = 0$.

Each of L_1^1 and L_1^2 is foliated by complex curves, and every such curve intersects $\mathbb{T}_1$ along a real-analytic curve. Hence, we obtain two distinct families of curves $\mathcal{C}_1^j$ $(j = 1, 2)$ on $\mathbb{T}_1$. If $\psi_1 \colon \mathbb{R}^2 \to \mathbb{T}_1$ is the covering map, then the inverse images of $\mathcal{C}_1^j$ under ψ_1 are two distinct families of parallel lines $\mathcal{L}_1^j$ in $\mathbb{R}^2$, $j = 1, 2$.

For $p \in \mathbb{T}_1 \setminus J_f$ consider the torus $\mathbb{T}_2 := \mathbb{T}(f(p))$. By Lemma 1.2 we obtain $f(\mathbb{T}_1) \subset \mathbb{T}_2$ and $\mathbb{T}_2 \not\subset I$. Clearly, if U is a small neighborhood of p, then in a neighborhood of $f(p)$ the torus $\mathbb{T}_2$ lies in the boundaries of two distinct Levi-flat hypersurfaces $f(L_1^1 \cap U)$ and $f(L_1^2 \cap U)$. Hence $\mathbb{T}_2$ entirely lies in the boundaries of two distinct Levi-flat hypersurfaces L_2^j, $j = 1, 2$. The hypersurfaces L_2^j produce two distinct families of curves $\mathcal{C}_2^j$ on $\mathbb{T}_2$, and $f(\mathcal{C}_1^j) \subset \mathcal{C}_2^j$, $j = 1, 2$. Each $\mathcal{C}_2^j$ is invariant under the action of $\mathbb{T}$ on $\mathbb{T}_2$ and thus, if $\psi_2 \colon \mathbb{R}^2 \to \mathbb{T}_2$ is the covering map, then the inverse images of $\mathcal{C}_2^j$ under ψ_2 are two distinct families of parallel lines $\mathcal{L}_2^j$ in $\mathbb{R}^2$, $j = 1, 2$. Further, if $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a real-analytic covering map for $f|_{\mathbb{T}_1} \colon \mathbb{T}_1 \to \mathbb{T}_2$, then $\tilde{f}(\mathcal{L}_1^j) \subset \mathcal{L}_2^j$ for $j = 1, 2$.

Let g be a linear transformation of $\mathbb{R}^2$ mapping $\mathcal{L}_1^1$ and $\mathcal{L}_1^2$ into the families of horizontal and vertical lines, respectively, and let h be a similar transformation for the families $\mathcal{L}_2^j$, $j = 1, 2$. Consider $\hat{f} = h \circ \tilde{f} \circ g^{-1}$. Clearly, $\hat{f} = (\hat{f}_1, \hat{f}_2)$ is a real-analytic map such that $\hat{f}_1$ is constant on every vertical line and $\hat{f}_2$ is constant on every horizontal line in $\mathbb{R}^2$. Hence $\hat{f}_1$ is a function of x and $\hat{f}_2$ is a function of y alone. We choose g to be the linear transformation with the matrix

$$\begin{pmatrix} a_1 & -1 \\ 1 & d_1 \end{pmatrix},$$

and we let the matrix of h be

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Since $h \circ \tilde{f} = \hat{f} \circ g$, it follows that

$$a_2 \tilde{f}_1(x, y) + b_2 \tilde{f}_2(x, y) = \hat{f}_1(a_1 x - y),$$
$$c_2 \tilde{f}_1(x, y) + d_2 \tilde{f}_2(x, y) = \hat{f}_2(x + d_1 y).$$

This implies that there exist holomorphic functions of one variable F and G such that, in a neighborhood U of $p \in \mathbb{T}_1$, we have

$$\begin{aligned} f_1^{a_2} f_2^{b_2} &= F(z^{a_1} w^{-1}), \\ f_1^{c_2} f_2^{d_2} &= G(z w^{d_1}). \end{aligned} \tag{2.1}$$

We shall consider the case $a_1, d_1 \in \mathbb{Q}$ first. Let $a_1 = a_1'/a_1''$, where $a_1' \geq 0$ and $a_1'' > 0$ are relatively prime integers. For a fixed $\alpha_1 \neq 0$ consider the curve $P_1^{\alpha_1}$ with the equation $z^{a_1'} w^{-a_1''} = \alpha_1$. The logarithmic diagram $\log(P_1^{\alpha_1})$ of $P_1^{\alpha_1}$ is a straight line parallel to one side of $\log(\hat{D}_1)$. We choose α_1 so that $P_1^{\alpha_1} \cap \hat{D}_1 \cap U \neq \emptyset$. The intersection $P_1^{\alpha_1} \cap \hat{D}_1$ is biholomorphically equivalent to either a disc or a punctured disc, and the equivalence is given by $\zeta_1 \mapsto (\mu_1 \zeta_1^{a_1''}, \nu_1 \zeta_1^{a_1'})$, where ζ_1 is the variable in a disc of a suitable radius and $\mu_1^{a_1'} \nu_1^{-a_1''} = \alpha_1$. We note that $P_1^{\alpha_1} \cap \hat{D}_1$ can be equivalent to a disc only if $a_1' = 0$.

We have $f_1^{a_2} f_2^{b_2} = \alpha_2 := F(\alpha_1^{1/a_1''})$ on an open subset of $U \cap P_1^{\alpha_1}$ in which $f_1^{a_2}$ and $f_2^{b_2}$ are defined as single-valued holomorphic functions. Hence $f(P_1^{\alpha_1} \cap \hat{D}_1)$ is contained in $P_2^{\alpha_2} \cap \hat{D}_2$, where $P_2^{\alpha_2}$ is obtained by the analytic continuation of a connected component of the analytic set defined by the equation $z^{a_2} w^{b_2} = \alpha_2$ near $f(p)$. Since $P_1^{\alpha_1} \cap \hat{D}_1$ is closed, so is $f(P_1^{\alpha_1} \cap \hat{D}_1)$; therefore, $f(P_1^{\alpha_1} \cap \hat{D}_1) = P_2^{\alpha_2} \cap \hat{D}_2$ and a_2, b_2 are rationally dependent. Changing the function F if necessary, we can assume either that $a_2 \in \mathbb{Q}$, $a_2 \geq 0$, and $b_2 = -1$ or that $a_2 = 1$ and $b_2 = 0$. Clearly, the restriction of f to $P_1^{\alpha_1} \cap \hat{D}_1$ is proper. Furthermore, $P_2^{\alpha_2} \cap \hat{D}_2$ is equivalent to either a disc or a punctured disc. If $b_2 = -1$ and $a_2 = a_2'/a_2''$ for some relatively prime integers $a_2' \geq 0$ and $a_2'' > 0$, then this equivalence has the form $\zeta_2 \mapsto (\mu_2 \zeta_2^{a_2''}, \nu_2 \zeta_2^{a_2'})$, where ζ_2 is the variable in a disc of a suitable radius and where $\mu_2^{a_2'} \nu_2^{-a_2''} = \alpha_2^{a_2''}$. If $a_2 = 1$ and $b_2 = 0$, the equivalence has the form $\zeta_2 \mapsto (\alpha_2, \zeta_2)$.

Assume first that, for some α_1 such that $P_1^{\alpha_1} \cap \hat{D}_1 \cap U \neq \emptyset$, the intersections $P_1^{\alpha_1} \cap \hat{D}_1$ and $P_2^{\alpha_2} \cap \hat{D}_2$ are equivalent to punctured discs $r_1 \mathring{\Delta}$ and $r_2 \mathring{\Delta}$, respectively (we denote by Δ and $\mathring{\Delta}$ the unit disc and the punctured unit disc, respectively). A proper holomorphic map between $r_1 \mathring{\Delta}$ and $r_2 \mathring{\Delta}$ has the form $\zeta_2 = \text{const}\, \zeta_1^k$, where k is a positive integer. Hence, from the second equation in (2.1), we obtain

$$\text{const}\, \zeta_1^{\sigma} = G(\text{const}\, \zeta_1^{\mu})$$

for all ζ_1 in an open subset of $r_1 \mathring{\Delta}$ and some nonzero $\sigma, \mu \in \mathbb{R}$. This means that $G(t) = \text{const}\, t^{\tau}$ for some $\tau \in \mathbb{R}$.

Assume next that, for some α_1 such that $P_1^{\alpha_1} \cap \hat{D}_1 \cap U \neq \emptyset$, the intersections $P_1^{\alpha_1} \cap \hat{D}_1$ and $P_2^{\alpha_2} \cap \hat{D}_2$ are equivalent to discs $r_1\Delta$ and $r_2\Delta$, respectively (in this case, $a_1' = 0$ and $a_1'' = 1$). A proper holomorphic map between $r_1\Delta$ and $r_2\Delta$ has the form $\zeta_2 = r_2 B(\zeta_1/r_1)$, where B is a Blaschke product in the unit disc. Hence, from the second equation in (2.1), we obtain

$$\text{const}\, B(\zeta)^\tau = G(\text{const}\, \zeta^\mu)$$

for all ζ in an open subset of the unit disc Δ and some nonzero $\tau, \mu \in \mathbb{R}$. This means that $G(t) = \text{const}\, B(\text{const}\, t^\sigma)^\tau$ for some $\sigma, \tau \in \mathbb{R}$.

In a similar way, considering the curves $Q_1^{\beta_1}$ and $Q_2^{\beta_2}$ with the equations $z^{d_1''} w^{d_1'} = \beta_1$ (where $d_1 = d_1'/d_1''$ for some relatively prime integers $d_1' \leq 0$ and $d_1'' > 0$) and $z^{c_2} w^{d_2} = \beta_2 := G(\beta_1^{1/d_1''})$, respectively, we obtain that $F(t) = \text{const}\, t^\rho$ if $Q_1^{\beta_1} \cap \hat{D}_1$ and $Q_2^{\beta_2} \cap \hat{D}_2$ are equivalent to punctured discs and $F(t) = \text{const}\, B(\text{const}\, t^\eta)^\rho$ for some Blaschke product B in the unit disc if $Q_1^{\beta_1} \cap \hat{D}_1$ and $Q_2^{\beta_2} \cap \hat{D}_2$ are equivalent to discs (in the second case $d_1' = 0$ and $d_1'' = 1$), $\eta, \rho \in \mathbb{R}$.

If $F(t) = \text{const}\, t^\rho$ and $G(t) = \text{const}\, t^\tau$, then it follows from (2.1) that f is elementary.

Let $F(t) = \text{const}\, t^\rho$ and $G(t) = \text{const}\, B(\text{const}\, t^\sigma)^\tau$, where B is a Blaschke product in the unit disc with a zero away from 0. In this case, $a_1' = 0$ and $a_1'' = 1$. It now follows from (2.1) that f has either the form

$$\begin{aligned} f_1(z, w) &= \text{const}\, z^a w^b \tilde{B}(A^{d_1''} z^{d_1''} w^{d_1'}), \\ f_2(z, w) &= \text{const}\, w^d, \end{aligned} \tag{2.2}$$

where $a, b, d \in \mathbb{Z}$ and $\tilde{B}$ is a nonconstant Blaschke product in the unit disc that is nonvanishing at 0, or the form

$$\begin{aligned} f_1(z, w) &= \text{const}\, z^a w^b \hat{B}(A^{d_1''} z^{d_1''} w^{d_1'}), \\ f_2(z, w) &= \text{const}\, z^c w^d \tilde{B}(A^{d_1''} z^{d_1''} w^{d_1'}), \end{aligned} \tag{2.3}$$

where $a, b, c, d \in \mathbb{Z}$, $\tilde{B}$ is a nonconstant Blaschke product in the unit disc that is nonvanishing at 0, $\hat{B}$ is either a Blaschke product in the unit disc with the same zeroes as $\tilde{B}$ or a constant, and a may be nonzero only if $\hat{B}$ is nonconstant. Forms (2.2) and (2.3) correspond to the cases $a_2 = 0$ and $a_2 \neq 0$, respectively.

We shall now show that form (2.3) can be simplified. Assume first that $\hat{B}$ is nonconstant. Then $\hat{D}_2$ contains the origin and $f^{-1}(0)$ contains the intersection with $\hat{D}_1$ of a curve of the form $z^{d_1''} w^{d_1'} = \text{const}$. Hence $f^{-1}(0)$ is not compact, which contradicts the assumption that f is proper. Thus, $\hat{B} \equiv \text{const}$ and therefore $a = 0$. So (2.3), in fact, differs from (2.2) only by permutation of the components of the maps.

We shall now study form (2.2) of proper maps and the domains $\hat{D}_1, \hat{D}_2$ in more detail. For every α_1 such that $|\alpha_1| > 1/C$ we have $f(P_1^{\alpha_1} \cap \hat{D}_1) = P_2^{\alpha_2} \cap \hat{D}_2$, where $\alpha_2 := F(\alpha_1)$ and where each of the curves $P_1^{\alpha_1} \cap \hat{D}_1$ and $P_2^{\alpha_2} \cap \hat{D}_2$ is equivalent to either a disc or a punctured disc. However, $\tilde{B}$ has a zero away from 0 and therefore $P_2^{\alpha_2} \cap \hat{D}_2$ is, in fact, equivalent to a disc; hence so is $P_1^{\alpha_1} \cap \hat{D}_1$. This shows that either

$$\hat{D}_1 = \{(z, w) \in \mathbb{C}^2 : A^{d_1''}|z|^{d_1''}|w|^{d_1'} < 1,\ 0 < |w| < C\} \tag{2.4}$$

or

$$\hat{D}_1 = \{(z, w) \in \mathbb{C}^2 : |z| < 1/A,\ |w| < C\}$$

(in the second case, we have $d_1' = 0$ and $d_1'' = 1$). Moreover, f of the form (2.2) is a proper map from $\hat{D}_1$ onto a bounded Reinhardt domain only if $d > 0$ and $ad_1' - bd_1'' \le 0$.

It is straightforward to observe that there exists no proper subdomain of $\hat{D}_1$ mapped properly by f onto a bounded Reinhardt domain and whose envelope of holomorphy coincides with $\hat{D}_1$. Thus, $D_1 = \hat{D}_1$ and hence $D_2 = \hat{D}_2$. If (2.4) holds then

$$D_2 = \{(z, w) \in \mathbb{C}^2 : \tilde{A}|z|^{\tilde{d}_1''}|w|^{\tilde{d}_1'} < 1,\ 0 < |w| < \tilde{C}\}$$

for some relatively prime integers $\tilde{d}_1', \tilde{d}_1''$, with $\tilde{d}_1' \le 0$ and $\tilde{d}_1'' > 0$, such that $\tilde{d}_1'/\tilde{d}_1'' = (ad_1' - bd_1'')/(dd_1'')$ and $\tilde{A}, \tilde{C} > 0$. If D_1 is the bidisc then f can be proper only if $b = 0$, and in this case D_2 is also a bidisc. We have thus obtained (i) of Theorem 0.1.

The case $F(t) = \text{const}\, B(\text{const}\, t^\eta)^\rho$ and $G(t) = \text{const}\, t^\tau$, where B is a Blaschke product in the unit disc with a zero away from 0, leads to the same description of f and D_1, D_2 up to permutation of the components of f and the variables.

Let $F(t) = \text{const}\, B_1(\text{const}\, t^\eta)^\rho$ and $G(t) = \text{const}\, B_2(\text{const}\, t^\sigma)^\tau$, where B_1, B_2 are Blaschke products in the unit disc with zeroes away from 0. In this case, $a_1' = d_1' = 0$ and $a_1'' = d_1'' = 1$. From (2.1) we see that either: f has the form

$$\begin{aligned} f_1(z, w) &= \text{const}\, z^a w^b \tilde{B}_1(Az)\hat{B}_1(w/C), \\ f_2(z, w) &= \text{const}\, w^d \hat{B}_2(w/C), \end{aligned} \tag{2.5}$$

where $a, b, d \in \mathbb{Z}$, $\tilde{B}_1, \hat{B}_2$ are nonconstant Blaschke products in the unit disc that are nonvanishing at 0, $\hat{B}_1$ is either a Blaschke product in the unit disc with the same zeroes as $\hat{B}_2$ or a constant, and b can be nonzero only if $\hat{B}_1$ is nonconstant; or f has the form

$$\begin{aligned} f_1(z, w) &= \text{const}\, z^a w^b \tilde{B}_1(Az)\hat{B}_1(w/C), \\ f_2(z, w) &= \text{const}\, z^c w^d \tilde{B}_2(Az)\hat{B}_2(w/C), \end{aligned} \tag{2.6}$$

where $a, b, c, d \in \mathbb{Z}$, $\hat{B}_1, \tilde{B}_2$ are nonconstant Blaschke products in the unit disc that are nonvanishing at 0, $\tilde{B}_1$ is either a Blaschke product in the unit disc with the same zeroes as $\tilde{B}_2$ or a constant, $\hat{B}_2$ is either a Blaschke product in the unit disc with the same zeroes as $\hat{B}_1$ or a constant, a can be nonzero only if $\tilde{B}_1$ is nonconstant, and d can be nonzero only if $\hat{B}_2$ is nonconstant. Forms (2.5) and (2.6) correspond to the cases $a_2 = 0$ and $a_2 \neq 0$, respectively.

We shall now show that forms (2.5) and (2.6) can be simplified. Assume that in (2.5) $\hat{B}_1$ is nonconstant. Then $\hat{D}_2$ contains the origin and $f^{-1}(0)$ contains the intersection of a complex line of the form $w = \text{const}$ with $\hat{D}_1$. Hence $f^{-1}(0)$ is not compact, which contradicts the assumption that f is proper. Therefore $\hat{B}_1 \equiv$ const and hence $b = 0$. A similar argument shows that in (2.6) we have $\tilde{B}_1 \equiv$ const and $\hat{B}_2 \equiv$ const, so $a = d = 0$. Thus (2.5) reduces to

$$\begin{aligned} f_1(z,w) &= \text{const}\, z^a \tilde{B}_1(Az), \\ f_2(z,w) &= \text{const}\, w^d \hat{B}_2(w/C), \end{aligned} \tag{2.7}$$

where $a,d \in \mathbb{Z}$, $a \geq 0$, $d \geq 0$, $\tilde{B}_1, \hat{B}_2$ are nonconstant Blaschke products in the unit disc that are nonvanishing at 0, and (2.6) reduces to (2.7) up to permutation of the components of the maps.

Furthermore, repeating the argument preceding formula (2.4) shows that, for every α_1, β_1 with $|\alpha_1| > 1/C$ and $|\beta_1| < 1/A$, the intersections $P_1^{\alpha_1} \cap \hat{D}_1$ and $Q_1^{\beta_1} \cap \hat{D}_1$ are equivalent to discs and therefore

$$\hat{D}_1 = \{(z,w) \in \mathbb{C}^2 : |z| < 1/A,\ |w| < C\}.$$

Again, there exists no proper subdomain of $\hat{D}_1$ mapped properly by f onto a bounded Reinhardt domain and whose envelope of holomorphy coincides with $\hat{D}_1$. Thus, $D_1 = \hat{D}_1$ and hence $D_2 = \hat{D}_2$. Therefore, D_1 and D_2 are bidiscs and we have obtained (iii) of Theorem 0.1.

Assume now that $a_1, d_1 \notin \mathbb{Q}$. For a suitable $\alpha_1 \neq 0$, consider the curve $P_1^{\alpha_1}$ obtained by the analytic continuation of the curve defined by the equation $z^{a_1} w^{-1} = \alpha_1$ in U. As before, we choose α_1 to ensure that $P_1^{\alpha_1} \cap \hat{D}_1 \cap U \neq \emptyset$. The intersection $P_1^{\alpha_1} \cap \hat{D}_1$ is not closed in $\hat{D}_1$ and is biholomorphically equivalent to a half-plane; the equivalence is given by $\sigma_1 \colon \zeta_1 \mapsto (\exp[\zeta_1 + \mu_1], \exp[a_1\zeta_1 + \nu_1])$, where ζ_1 is the variable in a suitable half-plane $R_1 := \{\zeta_1 \in \mathbb{C} : \operatorname{Re}\zeta_1 < s_1\}$ and $\exp[\mu_1 a_1 - \nu_1] = \alpha_1$.

As before, we observe that $f(P_1^{\alpha_1} \cap \hat{D}_1)$ lies in $P_2^{\alpha_2} \cap \hat{D}_2$, where $P_2^{\alpha_2}$ for $\alpha_2 := F(\alpha_1)$ is obtained by the analytic continuation of a connected component of the set given by $z^{a_2} w^{b_2} = \alpha_2$ near $f(p)$. If a_2 and b_2 were rationally dependent then the intersection $P_2^{\alpha_2} \cap \hat{D}_2$ would be closed in $\hat{D}_2$. Hence $f^{-1}(P_2^{\alpha_2} \cap \hat{D}_2)$ would contain the closure of $P_1^{\alpha_1} \cap \hat{D}_1$ in $\hat{D}_1$, which is $|P_1^{\alpha_1}| \cap \hat{D}_1$, where $|P_1^{\alpha_1}|$ is the Levi-flat hypersurface with the equation $|z|^{a_1}|w|^{-1} = |\alpha_1|$. Therefore, a_2 and b_2 are in fact rationally independent, and $P_2^{\alpha_2} \cap \hat{D}_2$ is biholomorphically equivalent to either a half-plane or a strip with the equivalence map $\sigma_2 \colon \zeta_2 \mapsto (\exp[-b_2\zeta_2 + \mu_2], \exp[a_2\zeta_2 + \nu_2])$, where ζ_2 is the variable in either a suitable half-plane $R_2 := \{\zeta_2 \in \mathbb{C} : \operatorname{Re}\zeta_2 < s_2\}$ or a suitable strip $R_2' := \{\zeta_2 \in \mathbb{C} : s_2' < \operatorname{Re}\zeta_2 < s_2\}$ and where $\exp[\mu_2 a_2 + \nu_2 b_2] = \alpha_2$. Changing the function F if necessary, we can assume that in the first case we have $a_2 > 0$ and $b_2 < 0$.

It is now straightforward to show that $f(P_1^{\alpha_1} \cap \hat{D}_1) = P_2^{\alpha_2} \cap \hat{D}_2$ and that the restriction of f to $P_1^{\alpha_1} \cap \hat{D}_1$ is proper. This restriction gives rise to a proper holomorphic map $\varphi := \sigma_2^{-1} \circ f \circ \sigma_1$ either between R_1 and R_2 or between R_1 and R_2'.

We shall now show that φ is one-to-one. Assume the contrary and let l_1 be the line given by the equation $\operatorname{Re}\zeta_1 = s_1$. Since φ is not one-to-one, $\varphi^{-1}(\infty)$ contains a point $\xi \in l_1$. Observe that $\sigma_1^{-1}(P_1^{\alpha_1} \cap (\partial\hat{D}_1 \setminus I)) = l_1$ and therefore $\sigma_1(\xi) \in \partial\hat{D}_1 \setminus I$. In particular, f is defined near $\sigma_1(\xi)$ and $f(\sigma_1(\xi)) \in \partial\hat{D}_2$. On the other hand, consider in either R_2 or R_2' a sequence $\{\xi_n\}$ converging to ∞ such that the sequence $\{\sigma_2(\xi_n)\}$ converges to a point in $\hat{D}_2$. Let $\{\xi_n'\}$ be a sequence in R_1 converging to ξ such that $\varphi(\xi_n') = \xi_n$ for all n. Then $\{f(\sigma_1(\xi_n'))\}$ converges to a point in $\hat{D}_2$, which is impossible. Hence φ is one-to-one. This argument also shows that either $\varphi(l_1) = l_2$ or $\varphi(l_1) = l_2'$, where l_2 and l_2' are the lines given by the equations $\operatorname{Re}\zeta_2 = s_2$ and $\operatorname{Re}\zeta_2 = s_2'$, respectively. It follows that $P_2^{\alpha_2} \cap \hat{D}_2$ is in fact equivalent to R_2 and that $\varphi(l_1) = l_2$ and $\varphi(\zeta_1) = r\zeta_1 + q$ for $r > 0$ and $q \in i\mathbb{R}$.

Then, from the second equation in (2.1), we obtain

$$\operatorname{const}\exp[\sigma\zeta_1] = G(\operatorname{const}\exp[\mu\zeta_1])$$

for all ζ_1 in an open subset of R_1 and some nonzero $\sigma, \mu \in \mathbb{R}$. Hence $G(t) = \operatorname{const} t^\tau$ for some $\tau \in \mathbb{R}$. Similarly, $F(t) = \operatorname{const} t^\rho$ for some $\rho \in \mathbb{R}$. It thus follows from (2.1) that f is elementary.

We shall now assume that $a_1 \notin \mathbb{Q}$ and $d_1 \in \mathbb{Q}$. Repeating the preceding arguments, we obtain that $G(t) = \operatorname{const} t^\tau$ for some $\tau \in \mathbb{R}$ and that either $F(t) = \operatorname{const} t^\rho$ or $F(t) = \operatorname{const} B(\operatorname{const} t^\eta)^\rho$ for some $\eta, \rho \in \mathbb{R}$, where B is a Blaschke product in the unit disc with a zero away from 0. In the first case we can show (similarly to our previous demonstration) that f is elementary. In the second case it is easy to see using (2.1) that f is necessarily a multivalued map. This shows that the formula for F does not actually contain a Blaschke product with a zero away from 0 and hence f is elementary. Likewise, if $a_1 \in \mathbb{Q}$ and $d_1 \notin \mathbb{Q}$ then f is elementary.

The Case of Two Tori. Assume now that S_1 is a union of two tori. In this case $\log(\hat{D}_1)$ has the form

$$\begin{aligned}\log(\hat{D}_1) = \{(x, y) \in \mathbb{R}^2 : a_1x + b_1y > -\ln C,\ c_1x + d_1y < -\ln A,\\ u_1x + v_1y < -\ln E\}\end{aligned}$$

for some u_1, v_1, where $a_1 \geq 0, b_1 \leq 0, c_1 \geq 0, d_1 \leq 0, b_1c_1 \leq a_1d_1$, and $A, C, E > 0$. Note that u_1 and v_1 are not arbitrary: the line $u_1x + v_1y = 0$ must intersect the other two "to the left" of their intersection point.

The logarithmic diagram $\log(\hat{D}_1)$ has two vertices, and we shall first concentrate on the one made by the lines $a_1x + b_1y = -\ln C$ and $u_1x + v_1y = -\ln E$. Let $\mathbb{T}_1$ be the torus in S_1 corresponding to this vertex. As before, we can show that there exist holomorphic functions of one variable F and G such that, in a neighborhood U of $p \in \mathbb{T}_1$,

$$\begin{aligned} f_1^{a_2} f_2^{b_2} &= F(z^{a_1} w^{b_1}),\\ f_1^{u_2} f_2^{v_2} &= G(z^{u_1} w^{v_1}) \end{aligned} \tag{2.8}$$

for some $a_2, b_2, u_2, v_2 \in \mathbb{R}$.

Assume first that both pairs a_1, b_1 and u_1, v_1 are rationally dependent. As before, we obtain that either $G(t) = \text{const}\, t^\tau$ or $G(t) = \text{const}\, B(\text{const}\, t^\sigma)^\tau$, where B is a Blaschke product in the unit disc with a zero away from 0 and where $\sigma, \tau \in \mathbb{R}$ (in the second case, either $a_1 = 0$ or $b_1 = 0$). Similarly, considering the intersections $Q_1^{\beta_1} \cap \hat{D}_1$, where $Q_1^{\beta_1}$ is the curve with the equation $z^{u_1} w^{v_1} = \beta_1$, we see that $F(t) = \text{const}\, t^\rho$ for some $\rho \in \mathbb{R}$. For the proof one must note that every connected component of $Q_1^{\beta_1} \cap \hat{D}_1$ is biholomorphically equivalent to an annulus with nonzero inner radius and that every proper map between two such annuli has the form $\zeta \mapsto \text{const}\, \zeta^k$, where $k \in \mathbb{Z} \setminus \{0\}$.

For $G(t) = \text{const}\, t^\tau$ it follows from (2.8) that f is elementary, and we shall therefore assume that $G(t) = \text{const}\, B(\text{const}\, t^\sigma)^\tau$, where B is a Blaschke product in the unit disc with a zero away from 0 (in this case, either $a_1 = 0$ or $b_1 = 0$). Now (2.8) implies that, up to permutation of its components, f has either the form

$$\begin{aligned} f_1(z, w) &= \text{const}\, z^a w^b \tilde{B}(E^{u_1'/u_1} z^{u_1'} w^{v_1'}), \\ f_2(z, w) &= \text{const}\, w^d, \end{aligned} \tag{2.9}$$

where $a, b, d \in \mathbb{Z}$ and $\tilde{B}$ is a nonconstant Blaschke product in the unit disc that is nonvanishing at 0, or the form

$$\begin{aligned} f_1(z, w) &= \text{const}\, z^a w^b \tilde{B}(E^{v_1'/v_1} z^{u_1'} w^{v_1'}), \\ f_2(z, w) &= \text{const}\, z^c, \end{aligned}$$

where $a, b, c \in \mathbb{Z}$ and $\tilde{B}$ is a nonconstant Blaschke product in the unit disc that is nonvanishing at 0. These forms correspond to the cases $a_1 = 0$ and $b_1 = 0$, respectively. In the first case $u_1 > 0$, and $u_1' > 0$ and v_1' are relatively prime integers such that $v_1/u_1 = v_1'/u_1'$. In the second case $v_1 > 0$, and $u_1', v_1' > 0$ are relatively prime integers such that $u_1/v_1 = u_1'/v_1'$. The preceding forms are obtained from one another by permutation of the variables, and we shall assume that (2.9) holds.

For $a_1 = 0$, the image of $\hat{D}_1$ under a map of the form (2.9) is a Reinhardt domain only if $c_1 = 0$, and we obtain

$$\hat{D}_1 = \{(z, w) \in \mathbb{C}^2 : E^{u_1'/u_1} |z|^{u_1'} |w|^{v_1'} < 1,\ A^{-1/d_1} < |w| < C^{-1/b_1}\}.$$

A map of the form (2.9) is a proper map from $\hat{D}_1$ onto a Reinhardt domain only if $d \neq 0$ and $a \geq 0$. As before, there exists no proper subdomain of $\hat{D}_1$ mapped properly by f onto a bounded Reinhardt domain and whose envelope of holomorphy coincides with $\hat{D}_1$. Thus, $D_1 = \hat{D}_1$ and hence $D_2 = \hat{D}_2$. Then we have

$$D_2 = \{(z, w) \in \mathbb{C}^2 : \tilde{E} |z|^{\tilde{u}_1'} |w|^{\tilde{v}_1'} < 1,\ \tilde{A} < |w| < \tilde{C}\}$$

for some relatively prime integers $\tilde{u}_1'$ and $\tilde{v}_1'$, where $\tilde{u}_1' > 0$ and such that $\tilde{v}_1'/\tilde{u}_1' = (a v_1' - b u_1')/(d u_1')$ and $\tilde{A}, \tilde{C}, \tilde{E} > 0$. We have thus obtained (ii) of Theorem 0.1.

Assume now that a_1, b_1 are rationally dependent and that u_1, v_1 are rationally independent. Then, as before, either $G(t) = \text{const}\, t^\tau$ or $G(t) = \text{const}\, B(\text{const}\, t^\sigma)^\tau$, where B is a Blaschke product in the unit disc with a zero away from 0 and where $\sigma, \tau \in \mathbb{R}$. Considering the intersections $Q_1^{\beta_1} \cap \hat{D}_1$, where $Q_1^{\beta_1}$ is the curve with the equation $z^{u_1} w^{v_1} = \beta_1$, we see that $F(t) = \text{const}\, t^\rho$ for some $\rho \in \mathbb{R}$. For the proof one must note that every connected component of $Q_1^{\beta_1} \cap \hat{D}_1$ is equivalent to a strip, with the equivalence map of the form $\zeta \mapsto (\exp[-v_1\zeta + \mu_1], \exp[u_1\zeta + \nu_1])$ where $\exp[\mu_1 u_1 + \nu_1 v_1] = \beta_1$, and every proper map between two strips has the form $\zeta \mapsto r\zeta + q$, where $r \neq 0$ and $q \in i\mathbb{R}$. If $G(t) = \text{const}\, t^\tau$ for some $\tau \in \mathbb{R}$, then it follows from (2.8) that f is elementary. If $G(t) = \text{const}\, B(\text{const}\, t^\sigma)^\tau$, where B is a Blaschke product in the unit disc with a zero away from 0, then it is easy to see from (2.8) that f is necessarily a multivalued map. This shows that the formula for G does not actually contain a Blaschke product with a zero away from 0, so f is elementary.

If a_1 and b_1 are rationally independent, we obtain $F(t) = \text{const}\, t^\rho$ and $G(t) = \text{const}\, t^\tau$ with $\rho, \tau \in \mathbb{R}$. In this case, (2.8) shows that f is elementary.

3. Spherical Case

Assume now that $U_1 := \partial\hat{D}_1 \setminus I$ is a connected, real-analytic, spherical hypersurface. Consider the logarithmic diagram $\log(\hat{D}_1)$ of $\hat{D}_1$ and the tube domain $T_1 \subset \mathbb{C}^2$ with base $\log(\hat{D}_1) \subset \mathbb{R}^2$, that is, $T_1 = \log(\hat{D}_1) + i\mathbb{R}^2$. The domain T_1 covers $\hat{D}_1 \setminus I$ by means of the map $\Pi\colon (z, w) \mapsto (e^z, e^w)$. Clearly, for every $p \in \hat{D}_1 \setminus I$, the fiber $\Pi^{-1}(p)$ is preserved by the abelian group G of translations of $\mathbb{C}^2$ of the form $(z, w) \mapsto (z + i2\pi n, w + i2\pi m)$, $n, m \in \mathbb{Z}$. The group G has two generators: $(z, w) \mapsto (z + i2\pi, w)$ and $(z, w) \mapsto (z, w + i2\pi)$. We denote these maps by Λ^z and Λ^w, respectively.

Since $L_1 := \partial T_1$ is a closed spherical tube hypersurface, it follows from [DaY] that there exists an affine transformation F_1 of the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto A\begin{pmatrix} z \\ w \end{pmatrix} + b, \tag{3.1}$$

where $A \in \mathrm{GL}_2(\mathbb{R})$ and $b \in \mathbb{R}^2$, that maps L_1 onto one of the four hypersurfaces defined by the following equations

$$\begin{aligned}
&(1) \quad \operatorname{Re} w = (\operatorname{Re} z)^2, \\
&(2) \quad \operatorname{Re} w = \exp[2 \operatorname{Re} z], \\
&(3) \quad \cos(\operatorname{Re} w) = \exp[\operatorname{Re} z], \quad -\pi/2 < \operatorname{Re} w < \pi/2, \\
&(4) \quad \exp[2 \operatorname{Re} z] + \exp[2 \operatorname{Re} w] = 1.
\end{aligned} \tag{3.2}$$

Let $\tilde{T}_1 := F_1(T_1)$. Clearly, $\tilde{T}_1$ covers $\hat{D}_1 \setminus I$ by means of the map $\Pi_1 := \Pi \circ F_1^{-1}$, and for each $p \in \hat{D}_1 \setminus I$ the group $G_1 := F_1 \circ G \circ F_1^{-1}$ preserves the fiber $\Pi_1^{-1}(p)$.

Further, for each hypersurface listed in (3.2) one can explicitly write a corresponding locally biholomorphic map onto a portion of the unit sphere S^3 as follows (see [DaY]):

$$
\begin{aligned}
&(1) \quad z \mapsto \frac{z}{\sqrt{2}}, && w \mapsto w - \frac{z^2}{2},\\
&(2) \quad z \mapsto e^z, && w \mapsto w,\\
&(3) \quad z \mapsto \exp\left[\frac{z+iw}{2}\right], && w \mapsto \exp[iw],\\
&(4) \quad z \mapsto \frac{e^z}{e^w-1}, && w \mapsto -\frac{e^w+1}{e^w-1}.
\end{aligned} \tag{3.3}
$$

In formulas (3.3), S^3 punctured at a point is realized as the hypersurface with the equation $\operatorname{Re} w = |z|^2$. The deleted point in this realization is at infinity, and we denote it by p_∞.

Let $\tilde{L}_1 := \partial \tilde{T}_1$ and let θ_1 be the map from list (3.3) corresponding to $\tilde{L}_1$. In case (1) $\tilde{L}_1$ is mapped by θ_1 onto $S^3 \setminus \{p_\infty\}$, in cases (2) and (3) onto $S^3 \setminus (\{p_\infty\} \cup \mathcal{L}_z)$, and in case (4) onto $S^3 \setminus (\{p_\infty\} \cup \mathcal{L}_z \cup \{w = 1\})$. We also point out that in case (1) the map θ_1 takes $\tilde{T}_1$ onto the unit ball B^2 realized as $\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w > |z|^2\}$, in cases (2) and (3) onto $B^2 \setminus \mathcal{L}_z$, and in case (4) onto $B^2 \setminus (\mathcal{L}_z \cup \{w = 1\})$.

For $h \in G_1$, consider now the locally defined map $\theta_1 \circ h \circ \theta_1^{-1}$ from S^3 into itself. It extends to an automorphism $\hat{h}$ of B^2, and hence G_1 gives rise to a subgroup $\hat{G}_1$ of the group $\operatorname{Aut}(B^2)$ of holomorphic automorphisms of B^2. The group $\hat{G}_1$ is clearly abelian and has at most two generators.

Formulas (3.3) yield the following descriptions of transformations in the group $\hat{G}_1$ in each of the four cases (where the vector (α_1, α_2) varies over a lattice in $\mathbb{R}^2$):

$$
\begin{aligned}
&(1) \quad z \mapsto z + i\alpha_1, && w \mapsto -2i\alpha_1 z + w + \alpha_1^2 + i\alpha_2,\\
&(2) \quad z \mapsto e^{i\alpha_1} z, && w \mapsto w + i\alpha_2,\\
&(3) \quad z \mapsto e^{i\alpha_1 + \alpha_2} z, && w \mapsto e^{2\alpha_2} w,\\
&(4) \quad z \mapsto \frac{2e^{i\alpha_1} z}{1 + e^{i\alpha_2} + (1 - e^{i\alpha_2}) w}, && w \mapsto \frac{(e^{i\alpha_2} + 1) w + 1 - e^{i\alpha_2}}{1 + e^{i\alpha_2} + (1 - e^{i\alpha_2}) w}.
\end{aligned} \tag{3.4}
$$

Next, since f is locally biholomorphic at the points of $U_1 \setminus J_f$, the set H_2 contains the real-analytic spherical hypersurface $U_2 := f(U_1 \setminus (C_f \cup J_f))$. Let T_2 be the covering tube domain for $\hat{D}_2 \setminus I$ and let $L_2 := \Pi^{-1}(U_2)$ be the portion of ∂T_2 covering U_2. By [DaY], there is an affine transformation F_2 of the form (3.1) mapping L_2 onto an open tube subset of one of hypersurfaces (3.2). Let $\tilde{L}_2 := F_2(L_2)$. Clearly, $\tilde{L}_2$ covers U_2 by means of the map $\Pi_2 := \Pi \circ F_2^{-1}$, and for every $p \in U_2$ the group $G_2 := F_2 \circ G \circ F_2^{-1}$ preserves the fiber $\Pi_2^{-1}(p)$. Let θ_2 be the map from list (3.3) corresponding to $\tilde{L}_2$. As for the group G_1, from every $h \in G_2$ we can (using the map θ_2) produce $\hat{h} \in \operatorname{Aut}(B^2)$; therefore, G_2 gives rise to an abelian subgroup $\hat{G}_2 \subset \operatorname{Aut}(B^2)$ with at most two generators. In each of the four cases, $\hat{G}_2$ is described by formulas (3.4).

The map f induces a homomorphism from $\hat{G}_1$ into $\hat{G}_2$ as follows. We fix $p_1 \in U_1 \setminus (C_f \cup J_f)$ and let $p_2 := f(p_1)$. Clearly, $p_2 \in U_2$. For $g_1 \in G_1$ we choose $p_1', p_1'' \in \Pi_1^{-1}(p_1)$ such that $g_1(p_1') = p_1''$ (note that g_1 is fully determined by this condition). Now fix a curve $\tilde{\gamma}_1 \subset \Pi_1^{-1}(U_1 \setminus (C_f \cup J_f))$ from p_1' to p_1'' and let $\gamma_1 := \Pi_1(\tilde{\gamma}_1)$. It is evident that γ_1 is a closed curve in $U_1 \setminus (C_f \cup J_f)$ passing through p_1 and that $\gamma_2 := f(\gamma_1)$ is a closed curve in U_2 passing through p_2. For a fixed $p_2' \in \Pi_2^{-1}(p_2)$ we now consider a curve $\tilde{\gamma}_2 \subset \tilde{L}_2$ originating at p_2' such that $\Pi_2(\tilde{\gamma}_2) = \gamma_2$. Let $p_2'' \in \Pi_2^{-1}(p_2)$ be the other endpoint of $\tilde{\gamma}_2$ and let $g_2 \in G_2$ be the map such that $g_2(p_2') = p_2''$. The correspondence $g_1 \mapsto g_2$ defines a map from G_1 into G_2 that induces a map $\Phi\colon \hat{G}_1 \to \hat{G}_2$ with $\Phi(\hat{g}_1) = \hat{g}_2$ (we show in the next paragraph that Φ is indeed a well-defined map).

Denote by ψ an analytic element of θ_1^{-1} defined near $\hat{p}_1 := \theta_1(p_1')$ such that $\psi(\hat{p}_1) = p_1'$, and denote by π an analytic element of Π_2^{-1} defined near p_2 such that $\pi(p_2) = p_2'$. Consider the map $\theta_2 \circ \pi \circ f \circ \Pi_1 \circ \psi$ mapping biholomorphically a neighborhood of $\hat{p}_1$ in S^3 onto a neighborhood of $\hat{p}_2 := \theta_2(p_2')$ in S^3. By the Poincaré theorem this map extends to an automorphism φ of B^2, and one immediately observes that $\Phi(\hat{g}_1) = \varphi \circ \hat{g}_1 \circ \varphi^{-1}$ for all $\hat{g}_1 \in \hat{G}_1$. This shows, in particular, that Φ is independent of the choice of the curves $\tilde{\gamma}_1$ and is single-valued. Clearly, Φ is a homomorphism.

We now require the following result.

Lemma 3.1. *$\Phi(\hat{G}_1)$ is a finite-index subgroup of $\hat{G}_2$.*

Proof. Since f is proper, $f^{-1}(p_2)$ consists of finitely many points—say, $p_1, q^1, \dots, q^k \in U_1$, $k \geq 0$. Let $\Gamma_1^1, \dots, \Gamma_1^k$ be curves in $U_1 \setminus (C_f \cup J_f)$ joining respectively $q^1, \dots, q^k$ with p_1. Clearly, $\Gamma_2^j := f(\Gamma_1^j)$, $j = 1, \dots, k$, are closed curves in U_2 passing through p_2. As before, each curve Γ_2^j gives rise to an element g_2^j of G_2 and, consequently, to an element $\hat{g}_2^j$ of $\hat{G}_2$.

Fix $g_2 \in G_2$ and let $p_2'' := g_2(p_2')$. Let $\tilde{\Gamma}_2 \subset \tilde{L}_2$ be a curve from p_2' to p_2'' and let $\Gamma_2 := \Pi_2(\tilde{\Gamma}_2)$. Clearly, Γ_2 is a closed curve in U_2 passing through p_2. Consider the curve $\Gamma_1 \subset U_1 \setminus (C_f \cup J_f)$ originating at p_1 such that $f(\Gamma_1) = \Gamma_2$.

If Γ_1 is closed, it gives rise to an element $\hat{g}_1$ of $\hat{G}_1$ and so we obviously have $\hat{g}_2 = \Phi(\hat{g}_1)$. Hence $\hat{g}_2 \in \Phi(\hat{G}_1)$ in this case.

Assume now that Γ_1 is not closed, and let q^s $(1 \leq s \leq k)$ be its other endpoint. Let g_1 be the element of G_1 corresponding to the closed curve obtained by joining Γ_1 and Γ_1^s. Then we clearly have $\Phi(\hat{g}_1) = \hat{g}_2 + \hat{g}_2^s$ and hence $\hat{g}_2 \in -\hat{g}_2^s + \Phi(\hat{G}_1)$.

We have thus shown that, for $\hat{g}_2 \in \hat{G}_2$, we have either $\hat{g}_2 \in \Phi(\hat{G}_1)$ or $\hat{g}_2 \in -\hat{g}_2^s + \Phi(\hat{G}_1)$ for some $1 \leq s \leq k$. Therefore, $\Phi(\hat{G}_1)$ is of finite index in $\hat{G}_2$.

The proof of the lemma is complete. □

The following result imposes constraints on the possible forms of $\tilde{L}_1$ and $\tilde{L}_2$.

Proposition 3.2. *We have $\tilde{L}_2 \subset \tilde{L}_1$, and the map f is elementary unless $\tilde{L}_1$ is either hypersurface* (2) *or hypersurface* (4) *of* (3.2).

Proof. We begin by proving the first assertion. Assume that $\tilde{L}_2 \not\subset \tilde{L}_1$. Then we shall show that the group $\hat{G}_1$ either cannot be conjugate in $\mathrm{Aut}(B^2)$ to a subgroup of $\hat{G}_2$ or can only be conjugate to a subgroup of $\hat{G}_2$ of infinite index. It will then follow from Lemma 3.1 that there exists no proper holomorphic map from $\hat{D}_1$ onto $\hat{D}_2$, contradicting our assumptions. We shall consider all possibilities for $\hat{G}_1$ and $\hat{G}_2$ (see (3.4)).

Assume that $\tilde{L}_1$ is hypersurface (1) and that $\tilde{L}_2$ lies in hypersurface (2). In this case, the only fixed point of each of $\hat{G}_1$ and $\hat{G}_2$ in $\overline{B^2}$ is the point $p_\infty \in S^3$ at infinity. If $\varphi \circ \hat{G}_1 \circ \varphi^{-1} \subset \hat{G}_2$ for some $\varphi \in \operatorname{Aut}(B^2)$, then $\varphi(p_\infty) = p_\infty$ (i.e., φ is affine). The general form of affine automorphisms of B^2 is

$$\begin{aligned} z &\mapsto \lambda e^{it} z + \zeta, \\ w &\mapsto \lambda^2 w + 2\bar{\zeta}\lambda e^{it} z + |\zeta|^2 + i\mu, \end{aligned} \tag{3.5}$$

where $\lambda > 0$, $\zeta \in \mathbb{C}$, and $t, \mu \in \mathbb{R}$. It is now straightforward to show that $\hat{G}_1$ cannot be conjugate to a subgroup of $\hat{G}_2$ by means of an automorphism of the form (3.5). The same argument works for the case when $\tilde{L}_1$ is hypersurface (2) and $\tilde{L}_2$ lies in hypersurface (1).

Also, if $\tilde{L}_1$ is one of hypersurfaces (1) or (2) and if $\tilde{L}_2$ lies in hypersurface (3), then the group $\hat{G}_1$ cannot be conjugate to a subgroup of $\hat{G}_2$ because $\hat{G}_1$ has only one fixed point in S^3 (the point p_∞) whereas $\hat{G}_2$ has two (0 and p_∞).

Let $\tilde{L}_1$ be hypersurface (3) and assume that $\tilde{L}_2$ lies in one of hypersurfaces (1) or (2). Then $\hat{G}_1$ has two fixed points in S^3 (0 and p_∞) and the only fixed point of $\hat{G}_2$ is p_∞. It is clear from formula (3.4) that $\hat{G}_2$ contains nontrivial elements fixing a point in S^3 other than p_∞ only if $\tilde{L}_2$ lies in hypersurface (2); for such elements $\alpha_2 = 0$. However, a subgroup of $\hat{G}_2$ containing only elements satisfying this condition has infinite index in $\hat{G}_2$. Hence, $\hat{G}_1$ cannot be conjugate to a finite-index subgroup of $\hat{G}_2$.

Next, if $\tilde{L}_1$ is one of hypersurfaces (1), (2), or (3) and if $\tilde{L}_2$ lies in hypersurface (4), then the group $\hat{G}_1$ cannot be conjugate to a subgroup of $\hat{G}_2$ because $\hat{G}_1$ does not have any fixed points in B^2 whereas $\hat{G}_2$ fixes the point $(0,1) \in B^2$.

Finally, let $\tilde{L}_1$ be hypersurface (4) and assume that $\tilde{L}_2$ lies in one of hypersurfaces (1), (2), or (3). Then $\hat{G}_1$ has a fixed point in B^2 and $\hat{G}_2$ fixes no point in B^2. It is clear from formula (3.4) that $\hat{G}_2$ contains nontrivial elements fixing a point in B^2 only if $\tilde{L}_2$ lies in either hypersurface (2) or hypersurface (3); for such elements $\alpha_2 = 0$. However, a subgroup of $\hat{G}_2$ containing only elements satisfying this condition has infinite index in $\hat{G}_2$. Hence, $\hat{G}_1$ cannot be conjugate to a finite-index subgroup of $\hat{G}_2$.

We thus have shown that $\tilde{L}_2 \subset \tilde{L}_1$. We shall now consider the two possibilities for $\tilde{L}_1$. In what follows we denote the map $\theta_1 = \theta_2$ by θ.

Let $\tilde{L}_1$ be hypersurface (1) and let $\Lambda_j^z := \theta \circ F_j \circ \Lambda^z \circ F_j^{-1} \circ \theta^{-1}$ and $\Lambda_j^w := \theta \circ F_j \circ \Lambda^w \circ F_j^{-1} \circ \theta^{-1}$ be generators of $\hat{G}_j$, $j = 1, 2$. Since $\Phi(\hat{G}_1) \subset \hat{G}_2$, it follows that

$$\begin{aligned} \varphi \circ \Lambda_1^z \circ \varphi^{-1} &= (\Lambda_2^z)^{a_1} \circ (\Lambda_2^w)^{a_2}, \\ \varphi \circ \Lambda_1^w \circ \varphi^{-1} &= (\Lambda_2^z)^{b_1} \circ (\Lambda_2^w)^{b_2} \end{aligned}$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that $a_1 b_2 - a_2 b_1 \neq 0$.

Consider the maps $\tilde{\Lambda}_1^z := \theta^{-1} \circ \varphi \circ \Lambda_1^z \circ \varphi^{-1} \circ \theta$ and $\tilde{\Lambda}_1^w := \theta^{-1} \circ \varphi \circ \Lambda_1^w \circ \varphi^{-1} \circ \theta$; they generate a subgroup of the group G_2. Let F be the linear map such that $F \circ \tilde{\Lambda}_1^z \circ F^{-1} = \Lambda^z$ and $F \circ \tilde{\Lambda}_1^w \circ F^{-1} = \Lambda^w$.

We shall now introduce an intermediate domain D through which the map f can be factored. Let $T := F(\tilde{T}_1)$ and $D := \Pi(T)$. Clearly, D is a Reinhardt domain. We define a biholomorphic map $\mathbf{f}$ from $\hat{D}_1 \setminus I$ onto D as follows: for $p \in \hat{D}_1 \setminus I$ consider a point $p' \in \Pi_1^{-1}(p)$ and let $\mathbf{f}(p) := (\Pi \circ F \circ \theta^{-1} \circ \varphi \circ \theta)(p')$. By the construction of F, this definition is independent of the choice of p'. It is straightforward to prove that $\mathbf{f}$ is a biholomorphic map between $\bar{\hat{D}}_1 \setminus I$ and $\bar{D}$. The domain D is Kobayashi-hyperbolic as a biholomorphic image of the bounded domain $\hat{D}_1 \setminus I$.

It is shown in [Kr] that a biholomorphic map between two hyperbolic Reinhardt domains in $\mathbb{C}^n$ can be represented as the composition of their automorphisms and an elementary biholomorphic map between them. Since $\hat{D}_1 \setminus I$ and D do not intersect I, it follows from [Kr] that all automorphisms of these domain are elementary. Therefore, $\mathbf{f}$ is an elementary map.

Further, $F_2^{-1} = \mathbf{G} \circ F$, where $\mathbf{G}$ is an affine transformation of the form (3.1) with

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Hence $V := \Pi(F(\tilde{L}_2))$ is mapped onto U_2 by an elementary map $\mathbf{g}$ of the form

$$z \mapsto \operatorname{const} z^{a_1} w^{b_1},$$

$$w \mapsto \operatorname{const} z^{a_2} w^{b_2}.$$

It is straightforward to verify that $f = \mathbf{g} \circ \mathbf{f}$ on $\mathbf{f}^{-1}(V) \setminus (C_f \cup J_f)$, and therefore f is an elementary map.

Assume now that $\tilde{L}_1$ is hypersurface (3). Then each of $\hat{G}_1$ and $\hat{G}_2$ has exactly two fixed points: 0 and p_∞. We thus have either that $\varphi(0) = 0$ and $\varphi(p_\infty) = p_\infty$ or that $\varphi(0) = p_\infty$ and $\varphi(p_\infty) = 0$. Hence φ preserves $B^2 \cap \mathcal{L}_z$ and so can be lifted to a holomorphic automorphism of $\tilde{T}_1$; that is, there exists a map $\tilde{\varphi} \in \operatorname{Aut}(\tilde{T}_1)$ such that $\theta \circ \tilde{\varphi} = \varphi \circ \theta$. The map $\tilde{\varphi}$ is also defined on $\tilde{L}_1$ and can be chosen to satisfy the condition $\tilde{\varphi}(p_1') = p_2'$, which yields $\tilde{\varphi} \circ G_1 \circ \tilde{\varphi}^{-1} \subset G_2$. Hence, as before, we can construct an intermediate hyperbolic domain D, a biholomorphic map $\mathbf{f}$ from $\hat{D}_1 \setminus I$ onto D (that, as before, turns out to be elementary), and an elementary map $\mathbf{g}$ from a portion of ∂D into U_2 such that $f = \mathbf{g} \circ \mathbf{f}$ on a portion of U_1. Thus, we again obtain that f is elementary.

The proof of the proposition is complete. □

Remark 3.3. The argument in the last paragraph of the proof of Proposition 3.2 shows that f can be represented as a composition of two elementary maps $\mathbf{f}$ and $\mathbf{g}$ (as described there) whenever the corresponding automorphism φ of B^2 can be lifted to an automorphism of the tube domain $\tilde{T}_1$.

It now remains to consider the cases when $\tilde{L}_2 \subset \tilde{L}_1$ and $\tilde{L}_1$ is either hypersurface (2) or hypersurface (4) of (3.2). As in the proof of Proposition 3.2, we denote the map $\theta_1 = \theta_2$ by θ.

Case I

Let $\tilde{L}_1$ be hypersurface (2). Then the only fixed point of each of $\hat{G}_1$ and $\hat{G}_2$ in $\overline{B^2}$ is p_∞ and so φ has the form (3.5). Assume that the group $\hat{G}_1$ contains an element $\hat{g}_1$ changing the z-coordinate. Then the only complex line preserved by $\hat{g}_1$ is $\mathcal{L}_z$ (see (3.4)). The map $\varphi \circ \hat{g}_1 \circ \varphi^{-1}$ also preserves a unique complex line, and it follows from (3.4) that this line is $\mathcal{L}_z$. Therefore, φ preserves $\mathcal{L}_z$ (i.e., we have $\zeta = 0$). Arguing as in the last paragraph of the proof of Proposition 3.2, we obtain that f in this case is elementary.

Assume now that none of the elements of $\hat{G}_1$ changes the z-coordinate—that is, assume that $\hat{G}_1$ consists of transformations of the form

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto w + i\alpha_1 n + i\beta_1 m, \quad n, m \in \mathbb{Z}, \end{aligned} \tag{3.6}$$

for some $\alpha_1, \beta_1 \geq 0$ with $\alpha_1 + \beta_1 > 0$. If in formula (3.5) we have $\zeta = 0$, then φ preserves $\mathcal{L}_z$ and we again obtain that f is elementary. Therefore, we shall assume that $\zeta \neq 0$.

We shall show first of all that the group $\hat{G}_1$ has only one generator.

PROPOSITION 3.4. *The group $\hat{G}_1$ consists of transformations of the form*

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto w + i\alpha_0 n, \quad n \in \mathbb{Z}, \end{aligned} \tag{3.7}$$

for some $\alpha_0 > 0$.

Proof. Let $\gamma := \mathcal{L}_z \cap S^3$ and $\gamma' := \varphi^{-1}(\gamma)$. Clearly, $\gamma' = \{z = -1/\lambda e^{-it}\zeta\} \cap S^3$. Let $\gamma'_k := \{z = \ln_0(-1/\lambda e^{-it}\zeta) + i2\pi k\} \cap \tilde{L}_1$ $(k \in \mathbb{Z})$ be the curves in $\tilde{L}_1$ forming the set $\theta^{-1}(\gamma')$ (here $\ln_0$ denotes the principal branch of the logarithm). For some $k_0 \in \mathbb{Z}$ and $c \in \mathbb{R}$ and for sufficiently small $\varepsilon > 0$, the circle $\tilde{\gamma} := \{|z - (\ln_0(-1/\lambda e^{-it}\zeta) + i2\pi k_0)| = \varepsilon\} \cap \tilde{L}_1 \cap \{\operatorname{Im} w = c\}$ lies in $\tilde{L}_1 \setminus (\Pi_1^{-1}(C_f \cup J_f) \cup_{k\in\mathbb{Z}} \gamma'_k)$. Recall that near $p'_1 \in \tilde{L}_1 \setminus \Pi_1^{-1}(C_f \cup J_f)$ we have

$$\Pi_2 \circ \eta \circ \varphi \circ \theta = f \circ \Pi_1,$$

where η is some analytic element of θ^{-1}. The map on the right-hand side is well-defined everywhere on $\tilde{L}_1$, so the analytic continuation of the map on the left-hand side along $\tilde{\gamma}$ produces a single-valued map. Clearly, after the analytic continuation of $\eta \circ \varphi \circ \theta$ along $\tilde{\gamma}$, the value of this map changes by $(\pm 2\pi, 0)$. Hence G_2 contains the map Λ^z (defined at the beginning of this section). Transformations in G_2 have the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w \end{pmatrix} + i\begin{pmatrix} \alpha'_2 \\ \alpha_2 \end{pmatrix} n + i\begin{pmatrix} \beta'_2 \\ \beta_2 \end{pmatrix} m, \quad n, m \in \mathbb{Z}, \tag{3.8}$$

for some linearly independent vectors $(\alpha'_2, \alpha_2), (\beta'_2, \beta_2) \in \mathbb{R}^2$. Since the map Λ^z is contained in G_2, for some $n_0, m_0 \in \mathbb{Z}$ it follows that

$$\begin{aligned} \alpha'_2 n_0 + \beta'_2 m_0 &= 2\pi, \\ \alpha_2 n_0 + \beta_2 m_0 &= 0. \end{aligned}$$

Hence α_2 and β_2 are rationally dependent.

Next, a straightforward calculation shows that the subgroup $\varphi \circ \hat{G}_1 \circ \varphi^{-1}$ of $\hat{G}_2$ consists of the maps

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto w + i\lambda^2\alpha_1 n + i\lambda^2\beta_1 m, \quad n, m \in \mathbb{Z}. \end{aligned} \tag{3.9}$$

Taking into account that the general form of an element of $\hat{G}_2$ is

$$\begin{aligned} z &\mapsto \exp(i\alpha_2' n + i\beta_2' m)z, \\ w &\mapsto w + i\alpha_2 n + i\beta_2 m, \quad n, m \in \mathbb{Z}, \end{aligned}$$

we see that α_1 and β_1 are rationally dependent; therefore, transformations from $\hat{G}_1$ have the form (3.7) for some $\alpha_0 > 0$ as required. $\square$

Let $D := \{(z, w) \in \mathbb{C}^2 : |w| > \exp[|z|^2]\}$. We shall now construct a locally biholomorphic map $\mathbf{h}$ from $\hat{D}_1 \setminus I$ onto $D \setminus I = D \setminus \mathcal{L}_z$. Obviously, the tube domain over the logarithmic diagram of D is precisely $\tilde{T}_1$ and so $\tilde{T}_1$ covers $D \setminus I$ by means of the map Π. We now use the map θ to construct a subgroup $\hat{G} \subset \mathrm{Aut}(B^2)$ from the group G acting on $\tilde{T}_1$ (in a manner similar to the way the groups $\hat{G}_1, \hat{G}_2$ were derived from G_1, G_2). Clearly, $\hat{G}$ consists of the transformations

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto w + i2\pi n, \quad n \in \mathbb{Z}. \end{aligned}$$

Consider the following automorphism of B^2:

$$\varphi_1 \colon z \mapsto \delta_0 z, \; w \mapsto \delta_0^2 w,$$

where $\delta_0 := \sqrt{2\pi/\alpha_0}$. From (3.7) we obtain $\varphi_1 \circ \hat{G}_1 \circ \varphi_1^{-1} \subset \hat{G}$. For $p \in \hat{D}_1 \setminus I$ consider a point $p' \in \Pi_1^{-1}(p)$, let $q \in \theta^{-1}((\varphi_1 \circ \theta)(p'))$, and set $\mathbf{h}(p) := \Pi(q)$. Clearly, this definition is independent of the choices of p' and q, and the map so defined is locally biholomorphic. Moreover, since φ_1 preserves $\mathcal{L}_z$, we can show that $\mathbf{h}$ is elementary by arguing as in the last paragraph of the proof of Proposition 3.2.

We shall now pause to describe the general form of a bounded domain whose complement to I can be mapped onto $D \setminus I$ by means of an elementary map and to describe such elementary maps. For $a_1, b_1, c_1, d_1 \in \mathbb{Z}$, $a_1 > 0$, $b_1 > 0$, $c_1 \geq 0$, and $d_1 > 0$ such that $a_1 d_1 - b_1 c_1 > 0$ and for $C_1, E_1 > 0$, consider the domain

$$\begin{aligned} &R(a_1, b_1, c_1, d_1, C_1, E_1) \\ &\qquad := \{(z, w) \in \mathbb{C}^2 : C_1 |z|^{c_1} |w|^{-d_1} > \exp[E_1 |z|^{2a_1} |w|^{-2b_1}],\; w \neq 0\}. \end{aligned}$$

The general form of an elementary map from $R(a_1, b_1, c_1, d_1, C_1, E_1) \setminus I$ onto $D \setminus I$ is

$$\begin{aligned} z &\mapsto e^{i\tau_1}\sqrt{C_1} z^{a_1} w^{-b_1}, \\ w &\mapsto e^{i\tau_2}\sqrt{E_1} z^{c_1} w^{-d_1}, \end{aligned} \tag{3.10}$$

where $\tau_1, \tau_2 \in \mathbb{R}$. We observe that $R(a_1, b_1, c_1, d_1, C_1, E_1) \cap \mathcal{L}_z \neq \emptyset$ only if $c_1 = 0$. It is straightforward to show that a bounded domain whose complements to I can be mapped onto $D \setminus I$ by an elementary map, up to permutation of the variables, is some $R(a_1, b_1, c_1, d_1, C_1, E_1)$ minus a closed subset of

$\mathcal{L}_z$. Because $\hat{D}_1$ is pseudoconvex, up to permutation of the variables we have either $\hat{D}_1 = R(a_1, b_1, c_1, d_1, C_1, C_2)$ or $\hat{D}_1 = R(a_1, b_1, 0, d_1, C_1, C_2) \setminus \mathcal{L}_z$ for some $a_1, b_1, c_1, d_1, C_1, E_1$.

We shall now construct a biholomorphic map $\mathbf{f}\colon D \to D^\lambda$, where $D^\lambda := \{(z,w) \in \mathbb{C}^2 : |w|^{\lambda^2} > \exp[|z|^2]\}$. The tube domain

$$T^\lambda := \{(z,w) \in \mathbb{C}^2 : \lambda^2 \operatorname{Re} w > \exp[2 \operatorname{Re} z]\}$$

covers $D^\lambda \setminus I$ by means of Π and is mapped into B^2 by the map

$$\theta^\lambda\colon z \mapsto e^z,\ w \mapsto \lambda^2 w.$$

Denote by $\hat{G}^\lambda \subset \operatorname{Aut}(B^2)$ the subgroup obtained by means of θ^λ from the group G acting on T^λ. Clearly, $\hat{G}^\lambda$ consists of the transformations

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto w + i\lambda^2 2\pi n, \quad n \in \mathbb{Z}. \end{aligned}$$

Let φ_2 be the following automorphism of B^2:

$$\begin{aligned} z &\mapsto \lambda e^{it} z + \delta_0 \zeta, \\ w &\mapsto \lambda^2 w + 2\delta_0 \bar{\zeta} \lambda e^{it} z + \delta_0^2 |\zeta|^2 + i\delta_0 \mu. \end{aligned}$$

A straightforward calculation shows that $\varphi_2 \circ \hat{G} \circ \varphi_2^{-1} = \hat{G}^\lambda$. Let $\mathcal{L}' := \{z = -1/\lambda e^{-it} \delta_0 \zeta\}$. For $p \in D \setminus (\mathcal{L}_z \cup \mathcal{L}')$ consider a point $p' \in \Pi^{-1}(p)$, let $q \in \theta^{\lambda^{-1}}((\varphi_2 \circ \theta)(p'))$, and set $\mathbf{f}(p) := \Pi(q)$. This definition is obviously independent of the choices of p' and q. It is straightforward to verify that $\mathbf{f}$ maps $D \setminus (\mathcal{L}_z \cup \mathcal{L}')$ biholomorphically onto $D^\lambda \setminus (\mathcal{L}_z \cup \mathcal{L}'')$, where $\mathcal{L}'' := \{z = \delta_0 \zeta\}$. Since D and D^λ have bounded realizations, it follows that $\mathbf{f}$ extends to a map (also denoted by $\mathbf{f}$) from D onto D^λ. This map is biholomorphic, and we have $\mathbf{f}(D \cap \mathcal{L}_z) = D^\lambda \cap \mathcal{L}''$ and $\mathbf{f}(D \cap \mathcal{L}') = D^\lambda \cap \mathcal{L}_z$. Furthermore, $\mathbf{f}$ can be represented as $\mathbf{f} = \mathbf{f}_1 \circ \mathbf{f}_2$, with

$$\mathbf{f}_1\colon z \mapsto \lambda z,\ w \mapsto w$$

and $\mathbf{f}_2 \in \operatorname{Aut}(D)$. The map $\mathbf{f}_2$ has the form (see [Kr; Sh])

$$\begin{aligned} z &\mapsto e^{i\tau_1} z + s, \\ w &\mapsto e^{i\tau_2} \exp[2\bar{s} e^{i\tau_1} z + |s|^2] w, \end{aligned} \tag{3.11}$$

where $\tau_1, \tau_2 \in \mathbb{R}$ and $s \in \mathbb{C}^*$.

Finally, we define a locally biholomorphic map $\mathbf{g}$ from $D^\lambda \setminus I = D^\lambda \setminus \mathcal{L}_z$ onto $\Omega := \Pi_2(\tilde{T}_1)$; it is constructed similarly to the map $\mathbf{h}$. Consider the following automorphism of B^2:

$$\varphi_3\colon z \mapsto \frac{1}{\delta_0} z,\ w \mapsto \frac{1}{\delta_0^2} w.$$

It follows from (3.7) and (3.9) that $\varphi_3 \circ \hat{G}^\lambda \circ \varphi_3^{-1} \subset \hat{G}_2$. For $p \in D^\lambda \setminus I$ let $p' \in \Pi^{-1}(p)$, let $q \in \theta^{-1}((\varphi_3 \circ \theta^\lambda)(p'))$, and set $\mathbf{g}(p) := \Pi_2(q)$. As before, this definition is independent of the choices of p' and q (recall that G_2 contains the transformation Λ^z), and the map so defined is locally biholomorphic. Since φ_3

preserves $\mathcal{L}_z$, we obtain—arguing again as in the last paragraph of the proof of Proposition 3.2—that $\mathbf{g}$ is elementary.

The composition $\mathbf{g} \circ \mathbf{f} \circ \mathbf{h}$ maps $V := \mathbf{h}^{-1}((\mathbf{g} \circ \mathbf{f})^{-1}(U_2) \setminus I)$ into $U_2 \subset \partial\Omega$. Since $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$, it follows that $f = \mathbf{g} \circ \mathbf{f} \circ \mathbf{h}$ on $V \setminus (C_f \cup J_f)$. Therefore, $f = \mathbf{g} \circ \mathbf{f} \circ \mathbf{h}$ on $\hat{D}_1 \setminus \mathbf{h}^{-1}(\mathcal{L}')$. Clearly, f maps $\hat{D}_1 \setminus \mathbf{h}^{-1}(\mathcal{L}')$ onto a set of the form $\Omega \setminus U$, where either $U = \emptyset$ (if $\hat{D}_1 \cap I \neq \emptyset$) or $U = \mathbf{g}(\mathcal{L}'' \cap D^\lambda)$ (if $\hat{D}_1 \cap I = \emptyset$).

If $U \neq \emptyset$, then $f(\hat{D}_1)$ is not a Reinhardt domain because $s \neq 0$ in formula (3.11). This shows that $U = \emptyset$ (i.e., $\hat{D}_1 \cap I \neq \emptyset$), which implies that, up to permutation of the variables, $\hat{D}_1 = R(a_1, b_1, 0, d_1, C_1, E_1)$ for some $a_1, b_1, c_1, d_1, C_1, E_1$ and that $\mathbf{h}$ has the form (3.10).

Further, Ω is a bounded Reinhardt domain not intersecting I, and $D^\lambda \setminus I$ is mapped onto Ω by an elementary map. It is not difficult to describe all such domains and the corresponding elementary maps. A domain of this kind has the form

$$\{(z,w) \in \mathbb{C}^2 : C_2|z|^{c_2/\Delta}|w|^{a_2/\Delta} > \exp[E_2|z|^{2d_2/\Delta}|w|^{2b_2/\Delta}],\ z \neq 0,\ w \neq 0\},$$

where $\Delta := a_2d_2 - b_2c_2$, $\Delta \neq 0$, $a_2, b_2, c_2, d_2 \in \mathbb{Z}$, $a_2 \geq 0$, $b_2 > 0$, $c_2 \leq 0$, $d_2 < 0$, and $C_2, E_2 > 0$. The general form of an elementary map from $D^\lambda \setminus I$ onto this domain is

$$\begin{aligned} z &\mapsto \operatorname{const} z^{a_2} w^{-b_2}, \\ w &\mapsto \operatorname{const} z^{-c_2} w^{d_2}. \end{aligned} \tag{3.12}$$

In particular, Ω and $\mathbf{g}$ must have these forms.

Since $U = \emptyset$, we obtain $\hat{D}_2 = \Omega \cup \mathbf{g}(\mathcal{L}_z \cap D^\lambda)$. If $a_2 > 0$ and $c_2 < 0$, then it follows from (3.12) that $\mathbf{g}(\mathcal{L}_z \cap D^\lambda) = \{0\}$. However, $\Omega \cup \{0\}$ is not an open set in this case, and therefore either $a_2 = 0$ or $c_2 = 0$. If $a_2 = 0$, then $c_2 < 0$ and we have

$$\hat{D}_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z| < \exp[-E_2'|z|^{-2d_2/b_2c_2}|w|^{-2/c_2}],\ z \neq 0\}$$

for some $E_2' > 0$; if $c_2 = 0$, then $a_2 > 0$ and we have

$$\hat{D}_2 = \{(z,w) \in \mathbb{C}^2 : C_2|w| < \exp[-E_2''|z|^{2/a_2}|w|^{2b_2/a_2d_2}],\ w \neq 0\}$$

for some $E_2'' > 0$. These two classes of domains are obtained from one another by permutation of the variables.

It is clear that every subdomain of $\hat{D}_1$ mapped properly by f onto a bounded Reinhardt domain—and whose envelope of holomorphy coincides with $\hat{D}_1$ up to permutation of the variables—has the form

$$\left\{(z,w) \in \mathbb{C}^2 : C_1^{*(1/d_1)} \exp\left[-\frac{E_1}{d_1}|z|^{2a_1}|w|^{-2b_1}\right] < |w| < C_1^{1/d_1} \exp\left[-\frac{E_1}{d_1}|z|^{2a_1}|w|^{-2b_1}\right]\right\}$$

for some $0 \leq C_1^* < C_1$, and hence D_1 is of this form. We thus have obtained (iv) of Theorem 0.1.

Case II

Let $\tilde{L}_1$ be hypersurface (4) of (3.2). For the purposes of this case we realize S^3 as $\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ and B^2 as $\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$. Then we have $\theta = \Pi$ and $\theta(\tilde{T}_1) = B^2 \setminus I$ and that each of $\hat{G}_1, \hat{G}_2$ consists of transformations of the form

$$z \mapsto e^{i\alpha_1} z,$$
$$w \mapsto e^{i\alpha_2} w,$$

where the vector (α_1, α_2) varies over a lattice in $\mathbb{R}^2$. We shall now consider three subcases.

CASE II.A. Assume first that $\varphi(B^2 \cap I) = B^2 \cap I$. In this case φ can be lifted to an automorphism $\tilde{\varphi}$ of $\tilde{T}_1$ such that $\tilde{\varphi} \circ G_1 \circ \tilde{\varphi}^{-1} \subset G_2$. Then, arguing as in the last paragraph of the proof of Proposition 3.2, we see that f is elementary.

CASE II.B. Assume now that $\varphi(B^2 \cap I) \neq B^2 \cap I$ and that φ maps a coordinate complex line into a coordinate complex line. Suppose that $\varphi(B^2 \cap \mathcal{L}_w) = B^2 \cap \mathcal{L}_w$. Then φ has the form

$$z \mapsto e^{it_1} \frac{z-a}{1-\bar{a}z},$$
$$w \mapsto e^{it_2} \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} w,$$

where $|a| < 1, a \neq 0$, and $t_1, t_2 \in \mathbb{R}$. It is now clear from the inclusion $\varphi \circ \hat{G}_1 \circ \varphi^{-1} \subset \hat{G}_2$ that none of the elements of $\hat{G}_1$ changes the z-coordinate.

Consider the tube domain

$$T := \{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > \exp[2 \operatorname{Re} w]\}.$$

This domain covers $B^2 \setminus \mathcal{L}_w$ by means of the map

$$\hat{\theta} \colon z \mapsto \frac{z-1}{z+1}, \ w \mapsto -\frac{2e^w}{z+1},$$

and $\tilde{T}_1$ covers $T \setminus \{z = 1\}$ by means of the map

$$\check{\theta} \colon z \mapsto -\frac{e^z+1}{e^z-1}, \ w \mapsto w - \ln_0(e^z - 1).$$

Clearly, $\theta = \hat{\theta} \circ \check{\theta}$. Since the groups $\hat{G}_1, \hat{G}_2$ preserve $\mathcal{L}_w$, their elements can be lifted to automorphisms of T. Similarly, φ can be lifted to an automorphism of T. The general form of a lift of φ is

$$\begin{aligned} z &\mapsto -\frac{(e^{it_1}(1-a)+1-\bar{a})z - e^{it_1}(1+a)+1+\bar{a}}{(e^{it_1}(1-a)-1+\bar{a})z - e^{it_1}(1+a)-1-\bar{a}}, \\ w &\mapsto w + \ln \frac{-2e^{it_2}\sqrt{1-|a|^2}}{(e^{it_1}(1-a)-1+\bar{a})z - e^{it_1}(1+a)-1-\bar{a}}, \end{aligned} \tag{3.13}$$

where ln is a branch of the logarithm.

For arbitrary $g_1 \in G_1$, consider the locally defined self-map $\tilde{g}_1 = \check{\theta} \circ g_1 \circ \check{\theta}^{-1}$ of T. Clearly, it coincides with a lift of $\hat{g}_1 \in \hat{G}_1$ and hence extends to an automorphism of T. Let $\tilde{G}_1 := \{\tilde{g}_1, g_1 \in G_1\}$, and let $\tilde{G}_2$ be the group constructed from G_2 in the same way. The groups $\tilde{G}_1, \tilde{G}_2$ are abelian and have at most two generators. Observe that a lift $\tilde{\varphi}$ of φ to an automorphism of T can be chosen so that $\tilde{\varphi} \circ \tilde{G}_1 \circ \tilde{\varphi}^{-1} \subset \tilde{G}_2$.

The group G_1 consist of transformations of the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w \end{pmatrix} + i\begin{pmatrix} \alpha_1' \\ \alpha_1 \end{pmatrix} n + i\begin{pmatrix} \beta_1' \\ \beta_1 \end{pmatrix} m, \quad n, m \in \mathbb{Z}, \tag{3.14}$$

for some linearly independent vectors $(\alpha_1', \alpha_1), (\beta_1', \beta_1) \in \mathbb{R}^2$. Then $\hat{G}_1$ consists of the maps

$$z \mapsto \exp[i\alpha_1' n + i\beta_1' m] z,$$
$$w \mapsto \exp[i\alpha_1 n + i\beta_1 m] w, \quad n, m \in \mathbb{Z}.$$

Since no map in $\hat{G}_1$ changes the z-coordinate, it follows that $\alpha_1', \beta_1' \in 2\pi \cdot \mathbb{Z}$. Therefore, elements of $\tilde{G}_1$ have the form (3.6). By (3.13) we then have that every element of $\tilde{G}_1$ commutes with every lift of φ to an automorphism of T. Hence $\tilde{G}_1 \subset \tilde{G}_2$.

Next, if the group G_2 is given by (3.8) then, arguing as in the proof of Proposition 3.4, we obtain that G_2 contains the map Λ^z. As before, this yields that α_2 and β_2 are rationally dependent. Furthermore, transformations from $\tilde{G}_2$ have the form

$$z \mapsto \frac{(1 + C(n,m))z + 1 - C(n,m)}{(1 - C(n,m))z + 1 + C(n,m)},$$
$$w \mapsto w + \ln \frac{2}{(1 - C(n,m))z + 1 + C(n,m)} + i\alpha_2 n + i\beta_2 m, \quad n, m \in \mathbb{Z},$$

where $C(n,m) := \exp[i\alpha_2' n + i\beta_2' m]$. For elements of $\tilde{G}_1$, the corresponding constants $C(n,m)$ are necessarily equal to 1, which implies that α_1 and β_1 are also rationally dependent. Therefore, transformations from $\tilde{G}_1$ have the form (3.7) for some $\alpha_0 > 0$.

Let $D^{\alpha_0} := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^{\alpha_0/\pi} < 1\}$. We shall now construct a locally biholomorphic map $\mathbf{h}$ from $\hat{D}_1 \setminus I$ onto $D^{\alpha_0} \setminus I$. Clearly, $\tilde{T}_1$ covers $D^{\alpha_0} \setminus I$ by means of the map

$$\Pi^{\alpha_0} : z \mapsto e^z,\ w \mapsto \exp[(2\pi/\alpha_0)w].$$

The group G^{α_0} constructed from D^{α_0}, in the same way as G_1 and G_2 were constructed from $\hat{D}_1$ and $\hat{D}_2$, consists of the following transformations:

$$z \mapsto z + i2\pi n,$$
$$w \mapsto w + i\alpha_0 m, \quad n, m \in \mathbb{Z}.$$

For $p \in \hat{D}_1 \setminus I$, consider a point $p' \in \Pi_1^{-1}(p)$ and set $\mathbf{h}(p) := \Pi^{\alpha_0}(p')$. Clearly, this definition is independent of the choice of p', and the map so defined is locally biholomorphic. The automorphism of B^2 that corresponds to $\mathbf{h}$ is the identity and therefore preserves I. Arguing as in the last paragraph of the proof of Proposition 3.2, we see that $\mathbf{h}$ is elementary.

It is straightforward to describe the general form of a bounded domain not intersecting I that can be mapped onto $D^{\alpha_0} \setminus I$ by an elementary map and to describe such elementary maps. Such a domain must have the form

$$\{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1}|w|^{2b_1} + E_1|z|^{\alpha_0 c_1/\pi}|w|^{\alpha_0 d_1/\pi} < 1,\ z \neq 0,\ w \neq 0\}, \tag{3.15}$$

where $a_1, b_1, c_1, d_1 \in \mathbb{Z}$, with either $a_1 d_1 - b_1 c_1 > 0$, $a_1 \geq 0$, $b_1 \leq 0$, $c_1 \leq 0$, and $d_1 \geq 0$ or $a_1 d_1 - b_1 c_1 < 0$, $a_1 \leq 0$, $b_1 \geq 0$, $c_1 \geq 0$, $d_1 \leq 0$, and $C_1, E_1 > 0$. An elementary map that takes domain (3.15) onto $D^{\alpha_0} \setminus I$ has the form

$$z \mapsto e^{i\tau_1}\sqrt{C_1} z^{a_1} w^{b_1},$$
$$w \mapsto e^{i\tau_2} E_1^{\pi/\alpha_0} z^{c_1} w^{d_1},$$

where $\tau_1, \tau_2 \in \mathbb{R}$. Thus, $\hat{D}_1 \setminus I$ and $\mathbf{h}$ must have these forms. Since $\hat{D}_1$ is pseudoconvex, it is either domain (3.15) or, up to permutation of the variables, one of the following domains:

$$\{(z,w) \in \mathbb{C}^2 : C_1|w|^{2b_1} + E_1|z|^{\alpha_0 c_1/\pi}|w|^{\alpha_0 d_1/\pi} < 1,\ w \neq 0\}$$

(here $a_1 = 0$, $b_1 > 0$, $c_1 > 0$, and $d_1 \leq 0$);

$$\{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1}|w|^{2b_1} + E_1|w|^{\alpha_0 d_1/\pi} < 1,\ w \neq 0\} \tag{3.16}$$

(here $a_1 > 0$, $b_1 \leq 0$, $c_1 = 0$, and $d_1 > 0$); or

$$\{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1} + E_1|w|^{\alpha_0 d_1/\pi} < 1\} \tag{3.17}$$

(here $a_1 > 0$, $b_1 = 0$, $c_1 = 0$, and $d_1 > 0$) for some $a_1, b_1, c_1, d_1, C_1, E_1$.

We shall now construct $\mathbf{f} \in \mathrm{Aut}(D^{\alpha_0})$. Let $\mathcal{L}' := \{z = a\}$ and $\mathcal{L}'' := \{z = -e^{it_1}a\}$. It is straightforward to observe that $\tilde{G}^{\alpha_0} = \tilde{G}_1$. In particular, elements of $\tilde{G}^{\alpha_0}$ commute with $\tilde{\varphi}$, which yields $\tilde{\varphi} \circ \tilde{G}^{\alpha_0} \circ \tilde{\varphi}^{-1} = \tilde{G}^{\alpha_0}$. For $p \in D^{\alpha_0} \setminus (I \cup \mathcal{L}')$ consider a point $p' \in {\Pi^{\alpha_0}}^{-1}(p)$, let $q \in \check{\theta}^{-1}((\tilde{\varphi} \circ \check{\theta})(p'))$, and set $\mathbf{f}(p) := \Pi^{\alpha_0}(q)$; this definition is independent of the choices of p' and q. It is straightforward to verify that $\mathbf{f}$ maps $D^{\alpha_0} \setminus (I \cup \mathcal{L}')$ biholomorphically onto $D^{\alpha_0} \setminus (I \cup \mathcal{L}'')$. Since D^{α_0} is bounded, $\mathbf{f}$ extends to a map (that we also denote by $\mathbf{f}$) from D^{α_0} onto itself. This map is biholomorphic, and we have $\mathbf{f}(D^{\alpha_0} \cap \mathcal{L}') \subset D^{\alpha_0} \cap I$ and $\mathbf{f}(D^{\alpha_0} \cap I) \subset D^{\alpha_0} \cap (I \cup \mathcal{L}'')$. It is now clear that $\mathbf{f}$ has the form

$$\begin{aligned} z &\mapsto e^{it_1}\frac{z-a}{1-\bar{a}z}, \\ w &\mapsto e^{it}\frac{(1-|a|^2)^{\pi/\alpha_0}}{(1-\bar{a}z)^{2\pi/\alpha_0}}w, \end{aligned} \tag{3.18}$$

where $t \in \mathbb{R}$.

Finally, we define a locally biholomorphic map $\mathbf{g}$ from $D^{\alpha_0} \setminus I$ onto $\Omega := \Pi_2(\tilde{T}_1)$; it is constructed similarly to the map $\mathbf{h}$. For $p \in \hat{D}^{\alpha_0} \setminus I$, consider a point $p' \in {\Pi^{\alpha_0}_0}^{-1}(p)$ and set $\mathbf{g}(p) := \Pi_2(p')$. Since $\tilde{G}_1 \subset \tilde{G}_2$ and since the map Λ^z belongs to G_2, the map

$$z \mapsto z,$$
$$w \mapsto w + i\alpha_0$$

belongs to G_2 as well. Hence, the definition of $\mathbf{g}$ is independent of the choice of p'. The map so defined is locally biholomorphic. The automorphism of B^2 corresponding to $\mathbf{g}$ is the identity and therefore preserves I. Arguing as in the last paragraph of the proof of Proposition 3.2, we see that $\mathbf{g}$ is elementary.

The composition $\mathbf{g}\circ\mathbf{f}\circ\mathbf{h}$ maps $V := \mathbf{h}^{-1}((\mathbf{g}\circ\mathbf{f})^{-1}(U_2)\setminus I)$ into $U_2 \subset \partial\Omega$. It is straightforward to verify that $f = \mathbf{g}\circ\mathbf{f}\circ\mathbf{h}$ on $V\setminus(C_f \cup J_f)$. Thus, $f = \mathbf{g}\circ\mathbf{f}\circ\mathbf{h}$ on $\hat{D}_1\setminus\mathbf{h}^{-1}(\mathcal{L}'\cup\mathcal{L}_w)$. Clearly, f maps $\hat{D}_1\setminus\mathbf{h}^{-1}(\mathcal{L}'\cup\mathcal{L}_w)$ onto $\Omega\setminus U$, where either $U = \emptyset$ or $U = \mathbf{g}(D^{\alpha_0}\cap\mathcal{L}'')$.

If $U \neq \emptyset$, then $f(\hat{D}_1)$ is not a Reinhardt domain because $a \neq 0$ in formula (3.18). Hence, in fact, $U = \emptyset$; that is, $\mathbf{h}(\hat{D}_1)\cap\mathcal{L}_z$ contains the punctured disc $\{z = 0,\ 0 < |w| < 1\}$ and therefore, up to permutation of the variables, $\hat{D}_1$ has one of the forms (3.16) or (3.17).

Next, Ω is a bounded Reinhardt domain not intersecting I, and $D^{\alpha_0}\setminus I$ can be mapped onto Ω by an elementary map. It is not hard to describe the general form of such domains and elementary maps. A domain of this kind has the form

$$\{(z,w)\in\mathbb{C}^2 : C_2|z|^{2d_2/\Delta}|w|^{-2b_2/\Delta} + E_2|z|^{-\alpha_0 c_2/\pi\Delta}|w|^{\alpha_0 a_2/\pi\Delta} < 1,\quad z\neq 0,\ w\neq 0\}, \tag{3.19}$$

where $\Delta := a_2d_2 - b_2c_2$, $\Delta \neq 0$, $a_2, b_2, c_2, d_2 \in \mathbb{Z}$, $a_2 \geq 0$, $b_2 \geq 0$, $c_2 \geq 0$, $d_2 \geq 0$, and $C_2, E_2 > 0$. An elementary map taking $D^{\alpha_0}\setminus I$ onto this domain is of the form

$$\begin{aligned} z &\mapsto \operatorname{const} z^{a_2} w^{b_2},\\ w &\mapsto \operatorname{const} z^{c_2} w^{d_2}. \end{aligned} \tag{3.20}$$

In particular, Ω and $\mathbf{g}$ must have these forms.

Assume now that $\hat{D}_1$ is of the form (3.17). Since $U = \emptyset$, it follows that $\hat{D}_2 = \Omega\cup\mathbf{g}(D^{\alpha_0}\cap I)$. If $a_2 > 0$ and $c_2 > 0$ or if $b_2 > 0$ and $d_2 > 0$, then from (3.20) we obtain that $\mathbf{g}(D^{\alpha_0}\cap I) = \{0\}$. However, $\Omega\cup\{0\}$ is not an open set in this case, and hence we have either $a_2 = 0$ and $d_2 = 0$ or $b_2 = 0$ and $c_2 = 0$. In the first case, $b_2, c_2 > 0$ and

$$\hat{D}_2 = \{(z,w)\in\mathbb{C}^2 : C_2|w|^{2/c_2} + E_2|z|^{\alpha_0/\pi b_2} < 1\}. \tag{3.21}$$

We then have either $D_1 = \hat{D}_1$, $D_2 = \hat{D}_2$, or

$$D_1 = \{(z,w)\in\mathbb{C}^2 : C_1|z|^{2a_1} < 1,\ E_1^{*-\pi/\alpha_0 d_1}(1 - C_1|z|^{2a_1})^{\pi/\alpha_0 d_1} < |w| < E_1^{-\pi/\alpha_0 d_1}(1 - C_1|z|^{2a_1})^{\pi/\alpha_0 d_1}\}, \tag{3.22}$$

$$D_2 = \{(z,w)\in\mathbb{C}^2 : C_2|w|^{2/c_2} < 1,\ E_2^{*-\pi b_2/\alpha_0}(1 - C_2|w|^{2/c_2})^{\pi b_2/\alpha_0} < |z| < E_2^{-\pi b_2/\alpha_0}(1 - C_2|w|^{2/c_2})^{\pi b_2/\alpha_0}\}$$

for some $E_1 < E_1^* \leq \infty$ and $E_2 < E_2^* \leq \infty$. Similarly, in the second case we have $a_2, d_2 > 0$ and

$$\hat{D}_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{2/a_2} + E_2|w|^{\alpha_0/\pi d_2} < 1\}. \tag{3.23}$$

We then have either $D_1 = \hat{D}_1$, $D_2 = \hat{D}_2$, or that D_1 has the form (3.22) and

$$D_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{2/a_2} < 1,\ E_2^{*-\pi d_2/\alpha_0}(1 - C_2|z|^{2/a_2})^{\pi d_2/\alpha_0} < |w| \\ < E_2^{-\pi d_2/\alpha_0}(1 - C_2|z|^{2/a_2})^{\pi d_2/\alpha_0}\}$$

for some $E_2 < E_2^* \le \infty$. The two forms of $\hat{D}_2$ just described are obtained from one another by permutation of the variables.

Assume now that $\hat{D}_1$ has the form (3.16). Then, as before, we have either $a_2 = 0$ and $c_2 > 0$ or $a_2 > 0$ and $c_2 = 0$. In the first case, $b_2 > 0$ and

$$\hat{D}_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{-2d_2/b_2c_2}|w|^{2/c_2} + E_2|z|^{\alpha_0/\pi b_2} < 1,\ z \ne 0\};$$

in the second case, $d_2 > 0$ and

$$\hat{D}_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{2/a_2}|w|^{-2b_2/a_2d_2} + E_2|w|^{\alpha_0/\pi d_2} < 1,\ w \ne 0\}.$$

These two types of domains are obtained from one another by permutation of the variables. In the first case we obtain

$$D_1 = \{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1}|w|^{2b_1} < 1, \\ E_1^{*-\pi/\alpha_0 d_1}(1 - C_1|z|^{2a_1}|w|^{2b_1})^{\pi/\alpha_0 d_1} < |w| \\ < E_1^{-\pi/\alpha_0 d_1}(1 - C_1|z|^{2a_1}|w|^{2b_1})^{\pi/\alpha_0 d_1}\}, \tag{3.24}$$

$$D_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{-2d_2/b_2c_2}|w|^{2/c_2} < 1, \\ E_2^{*-\pi b_2/\alpha_0}(1 - C_2|z|^{-2d_2/b_2c_2}|w|^{2/c_2})^{\pi b_2/\alpha_0} < |z| \\ < E_2^{-\pi b_2/\alpha_0}(1 - C_2|z|^{-2d_2/b_2c_2}|w|^{2/c_2})^{\pi b_2/\alpha_0}\}$$

for some $E_1 < E_1^* \le \infty$ and $E_2 < E_2^* \le \infty$. Similarly, in the second case D_1 has the form (3.24) and

$$D_2 = \{(z,w) \in \mathbb{C}^2 : C_2|z|^{2/a_2}|w|^{-2b_2/a_2d_2} < 1, \\ E_2^{*-\pi d_2/\alpha_0}(1 - C_2|z|^{2/a_2}|w|^{-2b_2/a_2d_2})^{\pi d_2/\alpha_0} < |w| \\ < E_2^{-\pi d_2/\alpha_0}(1 - C_2|z|^{2/a_2}|w|^{-2b_2/a_2d_2})^{\pi d_2/\alpha_0}\}$$

for some $E_2 < E_2^* \le \infty$.

Similar considerations in the cases where $\varphi(B^2 \cap I) \ne B^2 \cap I$ and either $\varphi(B^2 \cap \mathcal{L}_z) = B^2 \cap \mathcal{L}_z$ or $\varphi(B^2 \cap \mathcal{L}_z) = B^2 \cap \mathcal{L}_w$ or $\varphi(B^2 \cap \mathcal{L}_w) = B^2 \cap \mathcal{L}_z$ lead to the same descriptions of D_1, D_2, and f. We thus have obtained (v) of Theorem 0.1.

CASE II.C. Assume finally that $\varphi(B^2 \cap \mathcal{L}_z) \not\subset I$ and $\varphi(B^2 \cap \mathcal{L}_w) \not\subset I$. Arguing as in the proof of Proposition 3.4, we can prove that G_2 contains the map Λ^z as well as the map Λ^w (see the beginning of this section for definitions). Therefore, all elements of the inverse of the matrix A corresponding to the map F_2 (see (3.1)) are integers, and the locally defined map $\Pi_2 \circ \theta^{-1}$ from $S^3 \setminus I$ into U_2 extends to an elementary map $\mathbf{g}$ from $B^2 \setminus I$ onto the Reinhardt domain $\Omega := \Pi_2(\tilde{T}_1)$.

Let $\mathcal{L}'_z := \varphi^{-1}(B^2 \cap \mathcal{L}_z)$, $\mathcal{L}'_w := \varphi^{-1}(B^2 \cap \mathcal{L}_w)$, $\mathcal{L}''_z := \varphi(B^2 \cap \mathcal{L}_z)$, and $\mathcal{L}''_w := \varphi(B^2 \cap \mathcal{L}_w)$. Then the map $\hat{\varphi} := \mathbf{g} \circ \varphi \circ \theta$ takes $\tilde{T}'_1 := \tilde{T}_1 \setminus \theta^{-1}((\mathcal{L}'_z \cup \mathcal{L}'_w) \setminus I)$ onto $\Omega \setminus \mathbf{g}((\mathcal{L}''_z \cup \mathcal{L}''_w) \setminus I)$. Recall that, on an open subset of $\partial \tilde{T}'_1$, the map $\hat{\varphi}$ coincides with $f \circ \Pi_1$ and thus extends to all of $\tilde{T}_1$. Therefore, $\mathbf{g}$ extends to $B^2 \cap I$ and $f \circ \Pi_1$ maps $\tilde{T}_1$ onto $(\Omega \cup \mathbf{g}(B^2 \cap I)) \setminus \mathbf{g}(\mathcal{L}''_z \cup \mathcal{L}''_w)$. Thus, $\hat{D}_2 = \big((\Omega \cup \mathbf{g}(B^2 \cap I)) \setminus \mathbf{g}(\mathcal{L}''_z \cup \mathcal{L}''_w)\big) \cup f(\hat{D}_1 \cap I)$. Since $\hat{D}_2$ is a Reinhardt domain, it follows that $\mathbf{g}((\mathcal{L}''_z \cup \mathcal{L}''_w) \setminus I) \subset f(\hat{D}_1 \cap I)$ and therefore $\hat{D}_2 = \Omega \cup \mathbf{g}(B^2 \cap I)$. In particular, $\hat{D}_1 \cap I \neq \emptyset$.

Further, Ω is a bounded Reinhardt domain not intersecting I such that there exists an elementary map from $B^2 \setminus I$ onto Ω. Hence Ω has the form (3.19) with $\alpha_0 = 2\pi$ and $\mathbf{g}$ has the form (3.20). If either $a_2, c_2 > 0$ or $b_2, d_2 > 0$, then it follows from (3.20) that $\mathbf{g}(B^2 \cap I) = \{0\}$. However, $\Omega \cup \{0\}$ is not an open set in this case, and therefore either $a_2 = d_2 = 0$ or $b_2 = c_2 = 0$. Thus

$$\hat{D}_2 = \{(z,w) \in \mathbb{C}^2 : C'_2|z|^{2/a'_2} + E'_2|w|^{2/b'_2} < 1\},$$

where $a'_2, b'_2 \in \mathbb{N}$, $C'_2, E'_2 > 0$ (cf. (3.21) and (3.23)), and $\mathbf{g}$ (up to permutation of the variables) has the form

$$z \mapsto \operatorname{const} z^{a'_2},$$
$$w \mapsto \operatorname{const} w^{b'_2}.$$

This description shows that transformations in G_2 have the form (3.8) with $\alpha'_2 = 2\pi/a'_2$, $\alpha_2 = 0$, $\beta'_2 = 0$, and $\beta_2 = 2\pi/b'_2$.

It is straightforward to observe that, since $\varphi \circ \hat{G}_1 \circ \varphi^{-1} \subset \hat{G}_2$ and since

$$\varphi(B^2 \cap \mathcal{L}_z) \not\subset I \quad \text{and} \quad \varphi(B^2 \cap \mathcal{L}_w) \not\subset I,$$

every transformation in $\hat{G}_1$ has the form

$$\begin{aligned} z &\mapsto e^{i\alpha} z, \\ w &\mapsto e^{i\alpha} w, \end{aligned} \tag{3.25}$$

for $\alpha \in \mathbb{R}$. Since $\hat{D}_1$ is a bounded Reinhardt domain intersecting I with logarithmic diagram affinely equivalent to that of B^2, it follows that either up to permutation of the variables $\hat{D}_1$ has the form

$$\{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1}|w|^{2c_1} + E_1|w|^{2b_1} < 1,\ w \neq 0\} \tag{3.26}$$

or it has the form

$$\{(z,w) \in \mathbb{C}^2 : C_1|z|^{2a_1} + E_1|w|^{2b_1} < 1\}, \tag{3.27}$$

where $a_1, b_1, c_1 \in \mathbb{R}$, $a_1 > 0$, $b_1 > 0$, $c_1 \leq 0$, and $C_1, E_1 > 0$ (cf. (3.15)).

Assume first that $\hat{D}_1$ is a domain of the form (3.26). Then the group G_1 consists of transformations (3.14) with $\alpha'_1 = 2\pi a_1$, $\alpha_1 = 0$, $\beta'_1 = 2\pi c_1$, and $\beta_1 = 2\pi b_1$. Since all transformations in $\hat{G}_1$ are of the form (3.25), it follows that $a_1 \in \mathbb{N}$. Therefore, the matrix A corresponding to the map F_1 (see (3.1)) up to permutation of the rows is

$$\begin{pmatrix} a_1 & c_1 \\ 0 & b_1 \end{pmatrix}.$$

For an appropriate choice of an element of θ^{-1}, the locally defined map $\hat{\varphi} \circ F_1 \circ \theta^{-1}$ from $\hat{D}_1 \setminus I$ into $\hat{D}_2$ coincides with f and hence extends to all of $\hat{D}_1$. It then follows from this representation of f that either $f(\hat{D}_1 \cap I) = \mathbf{g}(\mathcal{L}_z'')$ or $f(\hat{D}_1 \cap I) = \mathbf{g}(\mathcal{L}_w'')$ and thus $\mathbf{g}((\mathcal{L}_z'' \cup \mathcal{L}_w'') \setminus I) \not\subset f(\hat{D}_1 \cap I)$, in contradiction to what we have already established. This shows that $\hat{D}_1$ does have the form (3.27).

Since all transformations in $\hat{G}_1$ are of the form (3.25), it follows that $a_1, b_1 \in \mathbb{N}$. Hence the locally defined map $\theta \circ \Pi_1^{-1}$ from $\hat{D}_1 \setminus I$ into $B^2 \setminus I$ extends to an elementary map $\mathbf{h}$ from $\hat{D}_1$ onto B^2. Clearly, up to permutation of its components, the map $\mathbf{h}$ has the form

$$z \mapsto \operatorname{const} z^{a_1},$$

$$w \mapsto \operatorname{const} w^{b_1},$$

and we have $f = \mathbf{g} \circ \varphi \circ \mathbf{h}$. It is straightforward to see that there exists no proper subdomain of $\hat{D}_1$ mapped properly by f onto a bounded Reinhardt domain and whose envelope of holomorphy coincides with $\hat{D}_1$. Therefore, $D_1 = \hat{D}_1$ and hence $D_2 = \hat{D}_2$.

We thus have obtained (vi) of Theorem 0.1, and the proof of the theorem is now complete. □

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