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Surfaces with Degenerate CR Singularities That Are Locally Polynomially Convex

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1. Introduction and Statement of Results

A compact subset $K \subset \mathbb{C}^n$ is said to be *polynomially convex* if, for every point $\zeta \notin K$, there exists a holomorphic polynomial P such that $P(\zeta) = 1$ and $\sup_K |P| < 1$. A subset K is said to be *locally* polynomially convex at a point $p \in K$ if there exists a closed ball $\mathbb{B}(p)$ centered at p such that $K \cap \mathbb{B}(p)$ is polynomially convex. In general, it is difficult to determine whether a given compact $K \subset \mathbb{C}^n$ is polynomially convex when $n > 1$. In contrast, there is a considerable body of work concerning the (local) polynomial convexity of smooth surfaces in $\mathbb{C}^n$. Let $\mathfrak{S}$ be a smooth real surface in $\mathbb{C}^n$, $n > 1$. A point $p \in \mathfrak{S}$ is said to be *totally real* if the tangent plane $T_p(\mathfrak{S})$ at p is not a complex line. A point on $\mathfrak{S}$ that is not totally real will be called a *CR singularity*. At a totally real point $p \in \mathfrak{S}$, the surface $\mathfrak{S}$ is locally polynomially convex. This is not always the case if $p \in \mathfrak{S}$ is an isolated CR singularity. At a CR singularity $p \in \mathfrak{S}$, if the order of contact of $T_p(\mathfrak{S})$ with $\mathfrak{S}$ equals 2 then the situation is well understood. Suppose $\mathfrak{S} \subset \mathbb{C}^2$; then there exist local holomorphic coordinates (z, w) with respect to which $p = (0,0)$ and such that $\mathfrak{S}$ is locally given by an equation of the form $w = |z|^2 + \gamma(z^2 + \bar{z}^2) + F(z)$, where $\gamma > 0$ and $F(z) = O(|z|^3)$. In Bishop's terminology, the CR singularity $p = (0,0)$ is said to be elliptic if $0 < \gamma < 1/2$, parabolic if $\gamma = 1/2$, and hyperbolic if $\gamma > 1/2$. Bishop showed [1] that if p is elliptic then, given $\varepsilon_0 > 0$, there is a 1-parameter family of analytic discs whose boundaries are contained in $\mathfrak{S} \cap \mathbb{B}(p; \varepsilon_0)$, whence $\mathfrak{S}$ is *not* polynomially convex. Much later, Forstnerič and Stout showed [3] that if $p \in \mathfrak{S}$ is an isolated, hyperbolic CR singularity then $\mathfrak{S}$ is locally polynomially convex at p.

Very little is known when the order of contact of $T_p(\mathfrak{S})$ with $\mathfrak{S}$ at a CR singularity p is *greater* than 2. We will call such a CR singularity a *degenerate* CR singularity. The aim of this paper is to study when $\mathfrak{S}$ is locally polynomially convex at an isolated, degenerate CR singularity. Knowing so may be useful in function theory: for instance, if a surface $\mathfrak{S}$ had only isolated CR singularities and if one knew that $\mathfrak{S}$ was locally polynomially convex at each singularity, then $\mathfrak{S}$ would have a Stein neighborhood basis. Local polynomial convexity at a degenerate CR singularity may be inferred in some cases when $\mathfrak{S}$ is the graph of a function F^* for $F^*\colon \mathbb{C} \to \mathbb{C}$ a globally defined, finitely sheeted branched covering; see [7]

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for a precise statement. It would, however, be useful to deduce local polynomial convexity using merely local information. Toward this end, we provide certain sufficient conditions for a smooth surface $\mathfrak{S}$ to be locally polynomially convex at an isolated, degenerate CR singularity. Given a compact subset $K \subset \mathbb{C}^n$, let $\mathcal{P}(K)$ denote the function space of uniform limits of the holomorphic polynomials on K. Questions about the polynomial convexity of K are closely related to whether $\mathcal{P}(K) = \mathcal{C}(K)$. In particular, for any compact $K \subset \mathbb{C}^n$, $\mathcal{P}(K) = \mathcal{C}(K)$ implies that K is polynomially convex (we will justify this assertion in Section 3). Here and in what follows, $\mathcal{C}(K)$ will denote the class of *complex*-valued continuous functions on K. Our results provide sufficient conditions showing that, given a surface $\mathfrak{S}$ and an isolated CR singularity $p \in \mathfrak{S}$, there exists a small compact $\mathfrak{S}$-neighborhood of p that is polynomially convex and, moreover, that all continuous functions on this portion of $\mathfrak{S}$ can be approximated uniformly by holomorphic polynomials.

We now state our first result.

THEOREM 1.1. *Let $\mathfrak{S}$ be a smooth surface in $\mathbb{C}^2$ given by*

$$w = \sum_{\alpha+\beta=k} C_{\alpha,\beta} z^\alpha \bar{z}^\beta + F(z),$$

where (z, w) are holomorphic coordinates on $\mathbb{C}^2$, F is a smooth function satisfying $F(z) = o(|z|^k)$ as $z \to 0$, and $k > 2$. Assume that $\mathfrak{S}$ has an isolated CR singularity at the origin. Let us write

$$\sum_{\alpha+\beta=k} C_{\alpha,\beta} z^\alpha \bar{z}^\beta = C_{k,0} z^k + C_{0,k} \bar{z}^k + \Sigma(z), \quad C_{0,k} \neq 0.$$

If $|\Sigma(z)| \leq \kappa |z|^k$ for some κ satisfying

$$0 \leq \kappa < |C_{0,k}| \min\left\{\frac{\pi}{2k}, \frac{1}{2}\right\}, \tag{1.1}$$

then there exists a constant $\varepsilon_0 > 0$ such that $\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. Furthermore, $\mathcal{P}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

One may ask whether there is a purely geometric condition such as hyperbolicity—as opposed to the analytical condition given here—according to which a surface $\mathfrak{S}$ is locally polynomially convex at a degenerate CR singularity $p \in \mathfrak{S}$. The Maslov index (see [2] for a definition) is an invariant associated with an isolated CR singularity. Elliptic CR singularities have Maslov index 1, whereas hyperbolic points have Maslov index −1. In view of the definition of the Maslov index, it is reasonable to ask if a surface is locally polynomially convex at an isolated, degenerate CR singularity with *negative* Maslov index. However, this is not always true. An example of Wiegerinck [9, Ex. 4.3] shows that a surface can have a nontrivial polynomial hull near an isolated, degenerate CR singularity with negative Maslov index. It thus seems that additional conditions are necessary.

The analytical condition in Theorem 1.1 essentially states that, if a surface $\mathfrak{S}$ is (around a CR singularity taken to be the origin) presented as a graph of a function F^* with leading order $k > 2$, then $\mathfrak{S}$ is locally polynomially convex if the Taylor coefficients of all the leading terms of F^* *other than* the $\bar{z}^k$-term are in some sense small in comparison to that of the $\bar{z}^k$-term. However, by adapting the technique of Forstnerič and Stout to the case of degenerate CR singularities, one can also demonstrate local polynomial convexity in cases where some of the leading Taylor coefficients of the graphing function are not small in comparison to that of the $\bar{z}^k$-term. This is the situation addressed by the following theorem.

THEOREM 1.2. *Let $\mathfrak{S}$ be a smooth surface in $\mathbb{C}^2$ given by*

$$w = \sum_{\alpha+\beta=2k} C_{\alpha,\beta} z^\alpha \bar{z}^\beta + \tilde{F}(z),$$

where (z, w) are holomorphic coordinates on $\mathbb{C}^2$, $\tilde{F}$ is a smooth function satisfying $\tilde{F}(z) = o(|z|^{2k})$ as $z \to 0$, and $k > 1$. Assume that $\mathfrak{S}$ has an isolated CR singularity at the origin. Let us write

$$\sum_{\alpha+\beta=2k} C_{\alpha,\beta} z^\alpha \bar{z}^\beta = C_{2k,0} z^{2k} + C_{k,k}|z|^{2k} + C_{0,2k}\bar{z}^{2k} + \tilde{\Sigma}(z),$$

$$\gamma = \frac{|C_{0,2k}|}{|C_{k,k}|}.$$

If $\gamma > 1/2$ and $|\tilde{\Sigma}(z)| \leq \kappa(2\gamma - 1)|z|^{2k}$ for some κ satisfying

$$0 \leq \kappa < \frac{|C_{k,k}|}{2} \min\left\{\frac{\pi}{2k}, \frac{2\gamma - 1}{2\gamma(3\gamma + 2)}\right\}, \tag{1.2}$$

then there exists a constant $\varepsilon_0 > 0$ such that $\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. Furthermore,

$$\mathcal{P}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}).$$

We do not claim that condition (1.2) is the best possible condition that guarantees local polynomial convexity. On the other hand, if $\tilde{\Sigma}(z) = 0$, then an obvious modification of the arguments in [1] shows that $\mathfrak{S}$ would *not* be polynomially convex if $\gamma < 1/2$. The case $\gamma = 1/2$ leads to varying phenomena, as in the case when $0 \in \mathbb{C}^2$ is a nondegenerate CR singularity (see [5]).

2. Some Notation and Remarks

The primary purpose of this section is to state Kallin's lemma [6], which is instrumental in demonstrating (local) polynomial convexity of various configurations in $\mathbb{C}^n$, $n > 1$, and to remark upon its connection with our results. We state a form of Kallin's lemma that we shall use in Sections 3 and 4; the reader is referred to [6] for the original result.

LEMMA 2.1 (Kallin). *Suppose X_1 and X_2 are compact subsets of $\mathbb{C}^n$ such that $\mathcal{P}(X_j) = \mathcal{C}(X_j)$, $j = 1, 2$. Let $\phi\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic polynomial such that $\phi(X_j) \subset W_j$, $j = 1, 2$, where W_1 and W_2 are polynomially convex compact sets in $\mathbb{C}$ and $W_1 \cap W_2 = \{0\}$. Assume that $\phi^{-1}\{0\} \cap (X_1 \cup X_2) = X_1 \cap X_2$. Then $\mathcal{P}(X_1 \cup X_2) = \mathcal{C}(X_1 \cup X_2)$.*

The version of Kallin's lemma stated here is implicit in the proof of [3, Thm. IV], but for the reader's convenience we provide the following.

Sketch of Proof of Lemma 2.1. The conditions on W_j $(j = 1, 2)$ imply that W_1 and W_2 are peak sets for $\mathcal{P}(W_1 \cup W_2)$. Since $\mathcal{P}(X_1 \cup X_2)$ is a closed subspace of $\mathcal{C}(X_1 \cup X_2)$, there is a regular Borel measure μ on $X_1 \cup X_2$ that annihilates $\mathcal{P}(X_1 \cup X_2)$. Define $\mu_j := \mu|_{X_j}$, $j = 1, 2$. Let $f \in \mathcal{P}(W_1 \cup W_2)$ peak on W_1. Then, for every holomorphic polynomial P on $\mathbb{C}^n$,

$$0 = \lim_{\nu\to\infty} \int_{X_1 \cup X_2} (f \circ \phi)^\nu P \, d\mu = \int_{X_1} P \, d\mu,$$

whence μ_1 annihilates $\mathcal{P}(X_1) = \mathcal{C}(X_1)$, which implies that $\mu_1 = 0$. Similarly, $\mu_2 = 0$. We have shown that $\mu = 0$. Therefore $\mathcal{P}(X_1 \cup X_2) = \mathcal{C}(X_1 \cup X_2)$. □

In our proofs, we will extend a technique presented in [3, Thm. IV]. In the proofs of both theorems, we will find an appropriate proper polynomial mapping of $\mathbb{C}^2$ onto $\mathbb{C}^2$ such that the preimage of an appropriately small compact $\mathfrak{S}$-neighborhood of the origin under this proper mapping is a finite union of compact subsets $X_1, \dots, X_N$ that satisfy $\mathcal{P}(X_j) = \mathcal{C}(X_j)$, $j = 1, 2, \dots, N$. In Theorem 1.1 $N = k$, and in Theorem 1.2 $N = 2$. We will then show that the sets $X_1, \dots, X_N$ are mapped by a polynomial into distinct sectors in $\mathbb{C}$, which intersect only at the origin. It is at this stage that one needs Lemma 2.1, and one infers that $\mathcal{P}(X_1 \cup \dots \cup X_N) = \mathcal{C}(X_1 \cup \dots \cup X_N)$. The desired conclusions follow from the last statement by appealing to the theory of analytic covers. These proofs are presented in the next section. The proof of Theorem 1.2 incorporates the use of certain technical lemmas whose proofs are deferred to Section 4.

Before presenting the proofs of our results, we define a couple of concepts that will be used in Section 3. First, if K is a compact subset of $\mathbb{C}^n$ then the *polynomially convex hull* of K, written $\hat{K}$, is defined by

$$\hat{K} := \{\zeta \in \mathbb{C}^n \mid |P(\zeta)| < \sup_K |P| \text{ for every holomorphic polynomial } P\}.$$

Second, given a uniform algebra $\mathcal{A}$, the *maximal ideal space* of $\mathcal{A}$ is the space of all algebra homomorphisms of $\mathcal{A}$ to $\mathbb{C}$, viewed as a subspace of the dual space $\mathcal{A}^*$ with the weak* topology (it is a standard fact that every complex homomorphism of $\mathcal{A}$ is in fact continuous). Recall that, for a compact subset K, the maximal ideal space of $\mathcal{C}(K)$ is homeomorphically identified with K. We will need this fact in Section 3.

3. Proof of the Main Results

Proof of Theorem 1.1

To begin we note that, without loss of generality, we may assume $C_{k,0} = 0$. For when this is not the case, we can simply choose new holomorphic coordinates (z^*, w^*) given by

$$z^* := z, \qquad w^* := w - C_{k,0}\, z^k,$$

whereby the coefficient of the $(z^*)^k$-term of the function defining $\mathfrak{S}$ vanishes. For this reason, we shall assume in the following argument that $C_{k,0} = 0$. Let $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Psi(z,w) := (z, w^k)$. This is a proper map of $\mathbb{C}^2$ onto itself having multiplicity k. Notice that there exists a $\delta > 0$ such that $\Psi^{-1}(\{(z,w) : |z| \le \delta\} \cap \mathfrak{S}) = \bigcup_{j=1}^{k} \mathsf{S}_j(\delta)$, where $\mathsf{S}_1(\delta), \dots, \mathsf{S}_k(\delta)$ are compact sets given by the equations

$$\mathsf{S}_j(\delta) : w = e^{2\pi i (j-1)/k} c^* \bar{z}\{1 + H(z)\}, \quad |z| \le \delta,$$

where $c^* := |C_{0,k}| e^{i \operatorname{Arg}(C_{0,k})/k}$ and where H is a continuous function satisfying a useful estimate. To justify this statement, we introduce the function F^* and view the surface $\mathfrak{S}$ as the graph of F^*. The function F^* may be written as

$$F^*(z) = C_{0,k}\bar{z}^k \left\{ 1 + \frac{\Sigma(z)}{C_{0,k}\bar{z}^k} + \frac{F(z)}{C_{0,k}\bar{z}^k} \right\}.$$

Observe that:

(a) owing to the estimate for $\Sigma(z)$ and condition (1.1),

$$\left| \frac{\Sigma(z)}{C_{0,k}\bar{z}^k} \right| \le \frac{\kappa}{|C_{0,k}|} < \frac{1}{2};$$

and

(b) $\lim_{z \to 0} F(z)/C_{0,k}\bar{z}^k = 0$.

By (a) and (b), we can find a $\delta > 0$ so small that

$$\left| \frac{\Sigma(z)}{C_{0,k}\bar{z}^k} + \frac{F(z)}{C_{0,k}\bar{z}^k} \right| \le \frac{1}{2} \quad \forall |z| \le \delta.$$

Given this fact, it follows that $F^*(z)$ has k distinct kth roots $f_1^*(z), \dots, f_k^*(z)$ when $0 < |z| \le \delta$, each $\mathsf{S}_j(\delta)$ is the graph of f_j^* $(j = 1, \dots, k)$, and

$$f_j^*(z) = e^{2\pi i (j-1)/k} c^* \bar{z} \left[1 + \sum_{m=1}^{\infty} \alpha_m \left\{ \frac{\Sigma(z)}{C_{0,k}\bar{z}^k} \right\}^m + o(1) \right], \tag{3.1}$$

where the α_m are coefficients occurring in the Taylor expansion of $(1+x)^{1/k}$ around $x = 0$; that is,

$$(1+x)^{1/k} = 1 + \sum_{m=1}^{\infty} \alpha_m x^m, \quad |x| < 1.$$

The infinite series in (3.1) represents a function h that is homogeneous of degree 0, and we write $H(z) := h(z) + R(z)$, where $R(z) = o(1)$ as $z \to 0$. Notice that

$$\sup_{|z|=1}|h(z)| \leq \sum_{m=1}^{\infty}|\alpha_m|\left\{\min\left(\frac{1}{2}, \frac{\pi}{2k}\right)\right\}^m$$
$$= 1 - \left\{1 - \min\left(\frac{1}{2}, \frac{\pi}{2k}\right)\right\}^{1/k} < \min\left(\frac{1}{2}, \frac{\pi}{2k}\right). \tag{3.2}$$

The last inequality follows because $k > 2$ and because the term in braces is smaller than 1. In view of (3.1) and (3.2), it is possible to find a small constant $\varepsilon_1 > 0$ such that

$$|\bar{z}H(z) - \bar{\zeta}H(\zeta)| \leq \sup_{|\xi|=1}|h(\xi)||z-\zeta| + |z||h(z) - h(\zeta)| + |\bar{z}R(z) - \bar{\zeta}R(\zeta)|$$
$$< |z-\zeta| \quad \forall z \neq \zeta : |z|, |\zeta| \leq \delta \text{ and } \forall \delta \in (0, \varepsilon_1]. \tag{3.3}$$

We use standard estimates to bound $|z||h(z) - h(\zeta)|$ by $|z-\zeta|/2$, which is enabled by the bound (3.2).

The estimate (3.3) allows us to use a result of Wermer [8, Thm. 1] to conclude that, for each $\mathsf{S}_j(\delta)$, we have $\mathcal{P}(\mathsf{S}_j(\delta)) = \mathcal{C}(\mathsf{S}_j(\delta))$ for $j = 1, \ldots, k$, assuming, of course, that $0 < \delta \leq \varepsilon_1$.

Consider the polynomial $p(z, w) = zw/c^*$. For any $(z, w) \in \mathsf{S}_1(\delta)$,

$$\mathrm{Re}\{p(z,w)\} = |z|^2 + \mathrm{Re}\{|z|^2(h(z) + R(z))\} \geq |z|^2 - |z|^2|h(z)| - |z|^2|R(z)|,$$
$$|\mathrm{Im}\{p(z,w)\}| \leq |z|^2\{|h(z)| + |R(z)|\}.$$

In view of the estimate (3.2) and the fact that $R(z) = o(1)$ as $z \to 0$, we can find a number M satisfying $1/2 < M < 1 - \kappa/|C_{0,k}|$ and a small constant $\varepsilon_2 > 0$ such that

$$\mathrm{Re}\{p(z,w)\} \geq M|z|^2, \quad |\mathrm{Im}\{p(z,w)\}| < \frac{\pi}{2k}|z|^2 \quad \forall |z| \leq \delta,$$
$$p(\mathsf{S}_1(\delta)) \subsetneq \{x + iy \in \mathbb{C} : |y| \leq (\pi/2kM)x\} \quad \text{where } \delta \in (0, \varepsilon_2]. \tag{3.4}$$

Expression (3.4) states that (a) $p(\mathsf{S}_1(\delta))$ is a proper subset of the sector W_1 that is centered on the positive x-axis and (b) $p(\mathsf{S}_1(\delta))$ has an aperture of (π/kM). Note that $p(\mathsf{S}_j(\delta))$ is therefore a proper subset of the sector W_j, which is simply a copy of W_1 rotated by $(2\pi(j-1)/k)$, $j = 1, \ldots, k$.

We have shown so far that:

- for each $\mathsf{S}_j(\delta)$, $\mathcal{P}(\mathsf{S}_j(\delta)) = \mathcal{C}(\mathsf{S}_j(\delta))$, $j = 1, \ldots, k$, where $0 < \delta \leq \varepsilon_1$;
- $p(\mathsf{S}_j(\delta)) \subsetneq W_j$, $j = 1, \ldots, k$, where $0 < \delta \leq \varepsilon_2$;
- $W_\mu \cap W_\nu = \{0\}$ for all $\mu \neq \nu$, because the aperture of each W_j, $\pi/kM < 2\pi/k$; and
- $p^{-1}\{0\} \cap \left\{\bigcup_{j=1}^{k} \mathsf{S}_j(\delta)\right\} = \{(0,0)\}$, where $0 < \delta \leq \varepsilon_2$.

We define $\varepsilon_0 := \min(\varepsilon_1, \varepsilon_2)$; the facts just summarized allow us to apply Lemma 2.1 repeatedly to show that

$$\mathcal{P}\left(\bigcup_{j=1}^{k} \mathsf{S}_j(\varepsilon_0)\right) = \mathcal{C}\left(\bigcup_{j=1}^{k} \mathsf{S}_j(\varepsilon_0)\right). \tag{3.5}$$

Now let $f \in \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$. Define $\hat{f} := f \circ \Psi : \Psi^{-1}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) \to \mathbb{C}$. Since $\Psi^{-1}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \bigcup_{j=1}^{k} \mathsf{S}_j(\varepsilon_0)$, it follows that $\hat{f} \in \mathcal{C}\big(\bigcup_{j=1}^{k} \mathsf{S}_j(\varepsilon_0)\big)$. We can paraphrase (3.5) in the following way: for each $\epsilon > 0$, there exists a polynomial g_ϵ such that

$$|\hat{f}(z, e^{2\pi i(j-1)/k} w) - g_\epsilon(z, e^{2\pi i(j-1)/k} w)| < \epsilon \qquad \forall (z,w) \in \mathsf{S}_1(\varepsilon_0), \quad j = 1, \dots, k. \tag{3.6}$$

We define

$$Q_\epsilon(z,w) := \frac{1}{k} \sum_{j=1}^{k} g_\epsilon(z, e^{2\pi i(j-1)/k} w).$$

Notice that if $g_\epsilon(z,w) = \sum_{0 \leq \mu+\nu \leq N} A_{\mu,\nu} z^\mu w^\nu$, then $Q_\epsilon(z,w)$ has the form

$$\begin{aligned} Q_\epsilon(z,w) &= \sum_{(\mu,\nu):\nu=kj} A_{\mu,kj} z^\mu w^{kj} \\ &\equiv P_\epsilon(z, w^k), \end{aligned}$$

where P_ϵ is itself a polynomial. Let us write $w = |w| e^{i\theta}$, $\theta \in [0, 2\pi)$. For $(z,w) \in \{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$, we compute

$$\begin{aligned} &|f(z,w) - P_\epsilon(z,w)| \\ &\quad = \left| \frac{1}{k} \sum_{j=1}^{k} \hat{f}(z, |w|^{1/k} e^{i(2\pi(j-1)+\theta)/k}) - Q_\epsilon(z, |w|^{1/k} e^{i\theta/k}) \right| \\ &\quad \leq \sum_{j=1}^{k} \frac{|\hat{f}(z, |w|^{1/k} e^{i(2\pi(j-1)+\theta)/k}) - g_\epsilon(z, |w|^{1/k} e^{i(2\pi(j-1)+\theta)/k})|}{k} \\ &\quad < k\left(\frac{\epsilon}{k}\right). \end{aligned}$$

The last inequality follows from the estimate (3.6). This establishes that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

Now we need only show that $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex. This follows from general abstract considerations. For this purpose, given a compact $K \Subset \mathbb{C}^n$, we define

$$\begin{aligned} \hat{K} &:= \text{the polynomially convex hull of } K, \\ \mathcal{A}(K; \mathbb{C}^n) &:= \text{the uniform algebra generated by the class } \{f|_K : f \in \mathcal{O}(\mathbb{C}^n)\}, \\ \mathcal{M}[\mathcal{A}(K; \mathbb{C}^n)] &:= \text{the maximal ideal space of the uniform algebra } \mathcal{A}(K; \mathbb{C}^n). \end{aligned}$$

We know that $\mathcal{M}[\mathcal{A}(K; \mathbb{C}^n)] = \hat{K}$ (see e.g. [4, Cor. VII.A(6)]). Thus, in our situation, $\mathcal{M}[\mathcal{A}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}; \mathbb{C}^2)] = \widehat{\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}}$. But since $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$, we have

$$\{(z,w) : |z| \le \varepsilon_0\} \widehat{\cap} \mathfrak{S} = \mathcal{M}[\mathcal{C}(\{(z,w) : |z| \le \varepsilon_0\} \cap \mathfrak{S})]$$
$$= \{(z,w) : |z| \le \varepsilon_0\} \cap \mathfrak{S}.$$

This concludes our proof of Theorem 1.

Before proving Theorem 1.2 we remark that, with respect to a new system of holomorphic coordinates (z^*, w^*) defined by

$$z^* := e^{-i\phi_0} z \quad \text{and}$$
$$w^* := \frac{w - (C_{2k,0} - C_{0,2k})z^{2k}}{C_{k,k}}$$

(where $\phi_0 := \mathrm{Arg}(C_{0,2k}/C_{k,k})/2k$), $\mathfrak{S}$ is expressed by the equation

$$\mathfrak{S} : w^* = |z^*|^{2k} + \gamma((z^*)^{2k} + (\bar{z}^*)^{2k}) + \frac{\tilde{\Sigma}(e^{i\phi_0} z^*)}{C_{k,k}} + \frac{\tilde{F}(e^{i\phi_0} z^*)}{C_{k,k}}.$$

For simplicity of notation we will denote the new coordinates by (z,w), and we then assert that the surface $\mathfrak{S}$ is expressed by an equation of the form

$$\mathfrak{S} : w = |z|^{2k} + \gamma(z^{2k} + \bar{z}^{2k}) + \Sigma(z) + F(z) \tag{3.7}$$

with respect to these new coordinates. In (3.7), $\Sigma(z)$ is homogeneous of degree $2k$, $F(z) = o(|z|^{2k})$ as $z \to 0$, and γ is precisely as defined in Theorem 1.2. Let us define a function $\psi(z,w) := |z|^{2k} + \gamma(z^{2k} + \bar{z}^{2k}) + \Sigma(z) + F(z) - w$. With respect to the new coordinates, $\mathfrak{S}$ is thus the zero set of ψ. Under this change of coordinates, $|\Sigma(z)| \le \kappa(2\gamma - 1)|z|^{2k}$, and the estimate (1.2) transforms to

$$0 \le \kappa < \frac{1}{2} \min\left\{\frac{\pi}{2k}, \frac{2\gamma - 1}{2\gamma(3\gamma + 2)}\right\}. \tag{3.8}$$

In the remainder of this paper, we will assume that $\mathfrak{S}$ is the zero set of ψ or (equivalently) is defined by (3.7).

Proof of Theorem 1.2

Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Phi(z,w) = (z, z^k w + \gamma(z^{2k} + w^2))$. This is a proper mapping of $\mathbb{C}^2$ onto itself having multiplicity 2. We first show that there is a small constant $\delta > 0$ such that $\Phi^{-1}(\{(z,w) : |z| \le \delta\} \cap \mathfrak{S}) = \mathcal{S}_1(\delta) \cup \mathcal{S}_2(\delta)$, where $\mathcal{S}_j(\delta)$, $j = 1, 2$, are compact sets given by

$$\begin{aligned} \mathcal{S}_1(\delta) &: w = \bar{z}^k + \mathcal{H}(z) + f_1(z), \quad |z| \le \delta, \\ \mathcal{S}_2(\delta) &: w = -\left(\frac{1}{\gamma} z^k + \bar{z}^k\right) - \mathcal{H}(z) + f_2(z), \quad |z| \le \delta, \end{aligned} \tag{3.9}$$

where

- f_1 and f_2 are continuous functions satisfying $f_j(z) = o(|z|^k)$ as $z \to 0$, $j = 1, 2$; and
- $\mathcal{H}(z)$ is a continuous function that is homogeneous of degree k and satisfies $|\mathcal{H}(z)| \le 2\kappa|z|^k$.

In order to justify the preceding statement, we need to analyze how the equations describing $\mathcal{S}_1(\delta)$ and $\mathcal{S}_2(\delta)$ arise. We first study the set $\mathcal{S}_1(\delta)$. For this purpose, we introduce the quantity $g_1(z)$ such that

$$\mathcal{S}_1(\delta) = \{(z, w) : w = \bar{z}^k + g_1(z),\ |z| \text{ small}\}$$

and then demand that $\psi[\Phi(z, \bar{z}^k + g_1(z))] = 0$. Thus g_1 satisfies the quadratic equation

$$\gamma g_1^2 + (2\gamma \bar{z}^k + z^k) g_1 - \{\Sigma(z) + F(z)\} = 0.$$

By the quadratic formula we have

$$\begin{aligned} g_1(z) &= -\frac{2\gamma\bar{z}^k + z^k}{2\gamma} + \frac{\sqrt{(2\gamma\bar{z}^k + z^k)^2 + 4\gamma\{\Sigma(z) + F(z)\}}}{2\gamma} \\ &= -\frac{2\gamma\bar{z}^k + z^k}{2\gamma} + \frac{2\gamma\bar{z}^k + z^k}{2\gamma}\left\{1 + \frac{4\gamma\{\Sigma(z) + F(z)\}}{(2\gamma\bar{z}^k + z^k)^2}\right\}^{1/2} \end{aligned} \tag{3.10}$$

in a small neighborhood of $z = 0$, where the square root is unambiguously defined. We choose the positive square root in equation (3.10) because it is this branch of the square root that ensures $f_1(z)$ will decay in the desired manner as $z \to 0$. To see this, observe that if we write $z = |z|e^{i\theta}$ then $2\gamma\bar{z}^{2k} + |z|^{2k} = |z|^{2k}\{(2\gamma\cos(2k\theta) + 1) - 2i\gamma\sin(2k\theta)\}$. Therefore,

$$\begin{aligned} |2\gamma\bar{z}^{2k} + |z|^{2k}| &= |z|^{2k}\sqrt{1 + 4\gamma^2 + 4\gamma\cos(2k\theta)} \\ &\geq |z|^{2k}\sqrt{1 + 4\gamma(\gamma - 1)} = (2\gamma - 1)|z|^{2k}. \end{aligned} \tag{3.11}$$

Since $\gamma > 1/2$, the quantity on the extreme right of estimate (3.11) is strictly positive when $z \neq 0$. Thus,

$$\left|\frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2}\right| = \left|\frac{4\gamma\Sigma(z)\bar{z}^{2k}}{(2\gamma\bar{z}^{2k} + |z|^{2k})^2}\right| \leq \frac{4\kappa\gamma(2\gamma - 1)|z|^{2k}}{(2\gamma - 1)^2|z|^{2k}} \leq \frac{2}{7}. \tag{3.12}$$

The last inequality is a consequence of the estimate (3.8) for κ. Since $|F(z)| = o(|z|^{2k})$ as $z \to 0$, it follows that there exists a $\delta > 0$ sufficiently small that $|4\gamma\{\Sigma(z) + F(z)\}/(2\gamma\bar{z}^k + z^k)^2| < 1$ for all $|z| \leq \delta$. Therefore, we can write

$$\begin{aligned} g_1(z) = {} & \frac{2\gamma\bar{z}^k + z^k}{2\gamma}\sum_{m=1}^{\infty}\beta_m\left\{\frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2}\right\}^m \\ & + \frac{F(z)}{2\gamma\bar{z}^k + z^k} + O(|z|^{k+1}) \quad \forall |z| \leq \delta, \end{aligned} \tag{3.13}$$

where the coefficients β_m are the coefficients occurring in the Taylor expansion of $(1 + x)^{1/2}$ around $x = 0$. The smallness of the quantity on the extreme left of (3.12) allows us to make the following estimate:

$$\begin{aligned} \sum_{m=1}^{\infty}|\beta_m|\left|\frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2}\right|^m &= 1 - \left\{1 - \left|\frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2}\right|\right\}^{1/2} \\ &< \left|\frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2}\right|. \end{aligned} \tag{3.14}$$

Using the inequalities (3.13), (3.12), and (3.14), we can write

$$\begin{aligned} g_1(z) &\equiv \mathcal{H}(z) + \frac{F(z)}{2\gamma\bar{z}^k + z^k} + O(|z|^{k+1}) \\ &\equiv \mathcal{H}(z) + f_1(z) \quad \forall |z| \le \delta, \end{aligned}$$

provided $\delta > 0$ is sufficiently small and where

$$\mathcal{H}(z) := \frac{2\gamma\bar{z}^k + z^k}{2\gamma} \sum_{m=1}^{\infty} \beta_m \left\{ \frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2} \right\}^m,$$

$$|\mathcal{H}(z)| \le 2\left| \frac{\Sigma(z)}{(2\gamma\bar{z}^k + z^k)} \right| \le 2\kappa |z|^k.$$

By this last estimate, we see that $\mathcal{H}(z)$ and $f_1(z)$ satisfy the desired properties. This completes the analysis of the compact $\mathcal{S}_1(\delta)$.

Next, we study the set $\mathcal{S}_2(\delta)$. Let $g_2(z)$ be such that

$$\mathcal{S}_2(\delta) = \left\{ (z, w) : w = -\left(\frac{z^k}{\gamma} + \bar{z}^k \right) + g_2(z),\ |z| \text{ small} \right\}.$$

Once more we require that $\psi[\Phi(z, -(z^k/\gamma + \bar{z}^k) + g_2(z))] = 0$. Thus, g_2 satisfies the quadratic equation

$$\gamma g_2^2 - (2\gamma\bar{z}^k + z^k) g_2 - \{\Sigma(z) + F(z)\} = 0.$$

By the quadratic formula,

$$\begin{aligned} g_2(z) &= \frac{2\gamma\bar{z}^k + z^k}{2\gamma} - \frac{\sqrt{(2\gamma\bar{z}^k + z^k)^2 + 4\gamma\{\Sigma(z) + F(z)\}}}{2\gamma} \\ &= \frac{2\gamma\bar{z}^k + z^k}{2\gamma} - \frac{2\gamma\bar{z}^k + z^k}{2\gamma} \left\{ 1 + \frac{4\gamma\{\Sigma(z) + F(z)\}}{(2\gamma\bar{z}^k + z^k)^2} \right\}^{1/2} \end{aligned} \tag{3.15}$$

for each $|z| \le \delta$ with $\delta > 0$ appropriately small. Unlike the case of equation (3.10), in (3.15) we choose the negative branch of the square root. We make this choice because it ensures that $f_2(z)$ decays in the desired manner as $z \to 0$. This is shown in exactly the same manner as in the case of $\mathcal{S}_1(\delta)$. Here it turns out that

$$\begin{aligned} g_2(z) &= -\frac{2\gamma\bar{z}^k + z^k}{2\gamma} \sum_{m=1}^{\infty} \beta_m \left\{ \frac{4\gamma\Sigma(z)}{(2\gamma\bar{z}^k + z^k)^2} \right\}^m - \frac{F(z)}{2\gamma\bar{z}^k + z^k} + O(|z|^{k+1}) \\ &\equiv -\mathcal{H}(z) + f_2(z) \quad \forall |z| \le \delta, \end{aligned} \tag{3.16}$$

and exactly the same $\delta > 0$ as the δ produced in the analysis on $\mathcal{S}_1(\delta)$ works. Exactly as in the preceding paragraph, from the expressions (3.13), (3.12), and (3.16) we establish the desired conclusion about the structure of $\mathcal{S}_2$.

Finally, owing to (a) the estimate $|\mathcal{H}(z)| \le 2\kappa|z|^k$ whereby

$$2|\mathcal{H}(z)| < \left(2 - \frac{1}{\gamma} \right) |z|^k$$

and (b) the fact that $\gamma > 1/2$, we have

$$\bar{z}^k + \mathcal{H}(z) \neq -\left(\frac{1}{\gamma} z^k + \bar{z}^k\right) - \mathcal{H}(z) \quad \forall z \neq 0.$$

Since $f_1(z), f_2(z) = o(|z|^k)$ as $z \to 0$, this inequality implies—lowering the value of $\delta > 0$ if necessary—that $\mathcal{S}_1(\delta) \cap \mathcal{S}_2(\delta) = \{0\}$. From this we conclude, since Φ is a mapping of multiplicity 2 and since $\Phi[\mathcal{S}_j(\delta)] \subset \mathfrak{S}$ $(j = 1, 2)$, that $\Phi^{-1}(\{(z, w) : |z| \leq \delta\} \cap \mathfrak{S}) = \mathcal{S}_1(\delta) \cup \mathcal{S}_2(\delta)$.

We shall need the following lemma, whose proof is deferred to Section 4.

Lemma 4.1. *Let $\mathcal{S}$ be a surface in $\mathbb{C}^2$ that, in a neighborhood of the origin, is defined by the equation*

$$w = \bar{z}^k + \sigma(z) + G(z),$$

where

- $\gamma > 1/2$,
- *σ is a continuous function that is homogeneous of degree k such that $|\sigma(z)| \leq 2\kappa|z|^k$ for some κ satisfying condition* (3.8), *and*
- *G is a continuous function satisfying $G(z) = o(|z|^k)$ as $z \to 0$.*

Then there exists a small constant $\varepsilon_0 > 0$ such that $\mathcal{P}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}) = \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})$.

By a direct application of Lemma 4.1, we see that there exists an $\varepsilon_1 > 0$ such that $\mathcal{P}(\mathcal{S}_1(\varepsilon)) = \mathcal{C}(\mathcal{S}_1(\varepsilon))$ for all $\varepsilon \leq \varepsilon_1$. The image of $\mathcal{S}_2(\delta)$ under the biholomorphic map $(z, w) \mapsto (z, -w - z^k/\gamma)$ is of the same form as $\mathcal{S}$ in Lemma 4.1. Thus, $\mathcal{P}(\mathcal{S}_2(\varepsilon)) = \mathcal{C}(\mathcal{S}_2(\varepsilon))$ for all $\varepsilon \leq \varepsilon_1$.

Let $\phi_{\epsilon(\gamma)}$ denote the polynomial

$$\phi_{\epsilon(\gamma)}(z, w) = \frac{z^{2k} - w^2}{4} + \epsilon(\gamma) z^k w, \tag{3.17}$$

where

$$\epsilon(\gamma) := \begin{cases} \frac{3}{16} & \text{if } \gamma \geq 1, \\ \min\left\{\frac{3}{16}, \frac{2\gamma - 1}{8\gamma(1-\gamma)}\right\} & \text{if } \frac{1}{2} < \gamma < 1. \end{cases}$$

We now refer the reader to Lemmas 4.2 and 4.3 in the next section. These lemmas tell us that there exists a small constant $\varepsilon_2 > 0$ such that, for every $\delta < \varepsilon_2$, $\phi_{\epsilon(\gamma)}$ maps $\mathcal{S}_1(\delta)$ into a closed sector that is symmetric with respect to the x-axis and is strictly contained in $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$, while $\phi_{\epsilon(\gamma)}(\mathcal{S}_2(\delta)) \setminus \{0\}$ is contained in $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Let $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. At this stage in the proof, we have all the elements needed to invoke Kallin's lemma (i.e., Lemma 2.1) to conclude that $\mathcal{P}(\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0)) = \mathcal{C}(\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0))$. We want to deduce from this that $\mathcal{P}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$. The deduction is achieved by the following argument of Forstnerič and Stout given in [3]; we present it here for the reader's convenience. Let $f \in \mathcal{C}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$. Note that, since $\Phi^{-1}(\{(z, w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0)$, it follows that $f \circ \Phi \in \mathcal{C}(\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0))$. From what we have shown, there is a sequence of polynomials $\{Q_n\}_{n \in \mathbb{N}}$ such that $Q_n \to f \circ \Phi$ uniformly on $\mathcal{S}_1(\varepsilon_0) \cup \mathcal{S}_2(\varepsilon_0)$. Let U be the open, dense set in $\mathbb{C}^2$

such that, for each $\zeta \in U$, $\Phi^{-1}\{\zeta\}$ consists of two distinct points $\zeta^{(+)}$ and $\zeta^{(-)}$. By the standard theory of analytic covers, the function

$$P_n(\zeta) := \frac{Q_n(\zeta^{(+)}) + Q_n(\zeta^{(-)})}{2}, \quad \zeta \in U,$$

which is holomorphic in U, extends to an entire function. By construction, the sequence of entire functions $\{P_n\}_{n\in\mathbb{N}}$ converges uniformly on $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ to f. Hence $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S})$.

In view of this last fact, it follows that $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathfrak{S}$ is polynomially convex, proving Theorem 1.2. The abstract uniform algebras argument needed for showing this is precisely the one given in the last paragraph of the proof of Theorem 1.1.

4. Technical Lemmas

In this section, we prove the three technical lemmas used in the proof of Theorem 1.2. But first we make the following observation, easily verified, about the quantity κ arising in the condition (3.8):

$$\kappa < \frac{1}{2}\min\left\{\frac{\pi}{2k}, \frac{2\gamma - 1}{2\gamma(3\gamma + 2)}\right\} \implies \kappa < \frac{1}{2}\min\left(\frac{1}{4}, \frac{\pi}{2k}\right), \tag{4.1}$$

which we shall use in several instances below.

LEMMA 4.1. *Let $\mathcal{S}$ be the surface in $\mathbb{C}^2$ that, in a neighborhood of the origin, is defined by the equation*

$$w = \bar{z}^k + \sigma(z) + G(z),$$

where

- $\gamma > 1/2$,
- *σ is a continuous function that is homogeneous of degree k such that $|\sigma(z)| \leq 2\kappa|z|^k$ for some κ satisfying condition* (3.8), *and*
- *G is a continuous function satisfying $G(z) = o(|z|^k)$ as $z \to 0$.*

Then there exists a small constant $\varepsilon_0 > 0$ such that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})$.

Proof. We follow closely the techniques used in the proof of Theorem 1.1. As before, let $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Psi(z,w) := (z, w^k)$. We first show that there exists a $\delta > 0$ such that $\Psi^{-1}(\{(z,w) : |z| \leq \delta\} \cap \mathcal{S}) = \bigcup_{j=1}^k S_j(\delta)$, where $S_1(\delta), \ldots, S_k(\delta)$ are compact sets given by

$$S_j(\delta) : w = e^{2\pi i(j-1)/k}\bar{z}\{1 + H(z)\}, \quad |z| \leq \delta,$$

where H satisfies certain useful size estimates. Recall from (3.11) that

$$|2\gamma\bar{z}^{2k} + |z|^{2k}| \geq (2\gamma - 1)|z|^{2k}$$

and that—because $\gamma > 1/2$—the quantity on the right is strictly positive when $z \neq 0$. Now, $\mathcal{S}$ is the graph of a function G^* that may be written as

$$G^*(z) = \bar{z}^k \left\{ 1 + \frac{\sigma(z)}{\bar{z}^k} + \frac{G(z)}{\bar{z}^k} \right\}.$$

Observe that:

(a) owing to the estimate for $\sigma(z)$ and to (4.1), we have

$$\left| \frac{\sigma(z)}{\bar{z}^k} \right| \le 2\kappa < \min\left(\frac{1}{4}, \frac{\pi}{2k} \right); \tag{4.2}$$

and

(b) $\lim_{z\to 0} G(z)/\bar{z}^k = 0$.

By (a) and (b), we can choose the $\delta > 0$ introduced at the beginning of this proof to be so small that

$$\left| \frac{\sigma(z)}{\bar{z}^k} + \frac{G(z)}{\bar{z}^k} \right| \le \frac{1}{2} \quad \forall |z| \le \delta.$$

Given this fact, $G^*(z)$ has k distinct kth roots $g_1^*(z), \dots, g_k^*(z)$ when $z \ne 0$, each $S_j(\delta)$ is the graph of g_j^* ($j = 1, \dots, k$), and

$$g_j^*(z) = e^{2\pi i(j-1)/k} \bar{z} \left[1 + \sum_{m=1}^{\infty} \alpha_m \left\{ \frac{\sigma(z)}{\bar{z}^k} \right\}^m + o(1) \right], \tag{4.3}$$

where the α_m are exactly as in expression (3.1). The infinite series in equation (4.3) represents a function h that is homogeneous of degree 0, and we write $H(z) := h(z) + R(z)$, where $R(z) = o(1)$ as $z \to 0$. Arguing as before yields

$$\begin{aligned} \sup_{|z|=1} |h(z)| &\le \sum_{m=1}^{\infty} |\alpha_m| \left\{ \min\left(\frac{1}{4}, \frac{\pi}{2k} \right) \right\}^m \\ &= 1 - \left\{ 1 - \min\left(\frac{1}{4}, \frac{\pi}{2k} \right) \right\}^{1/k} < \min\left(\frac{1}{4}, \frac{\pi}{2k} \right). \end{aligned} \tag{4.4}$$

In view of (4.3) and (4.4), it is possible to find a small constant $\varepsilon_1 > 0$ such that

$$\begin{aligned} |\bar{z}H(z) - \bar{\zeta}H(\zeta)| &\le \sup_{|\xi|=1} |h(\xi)||z - \zeta| + |z||h(z) - h(\zeta)| + |\bar{z}R(z) - \bar{\zeta}R(\zeta)| \\ &< |z - \zeta| \quad \forall z \ne \zeta : |z|, |\zeta| \le \delta \text{ and } \forall \delta \in (0, \varepsilon_1]. \end{aligned} \tag{4.5}$$

The estimate (4.5) allows us—as in the proof of Theorem 1.1—to use a result of Wermer [8, Thm. 1] to conclude that for each $S_j(\delta)$ we have $\mathcal{P}(S_j(\delta)) = \mathcal{C}(S_j(\delta))$, $j = 1, \dots, k$, assuming, of course, that $0 < \delta \le \varepsilon_1$.

Consider the polynomial $q(z, w) = zw$. For any $(z, w) \in S_1(\delta)$,

$$\operatorname{Re}\{q(z,w)\} = |z|^2 + \operatorname{Re}\{|z|^2(h(z) + R(z))\} \ge |z|^2 - |z|^2|h(z)| - |z|^2|R(z)|,$$

$$|\operatorname{Im}\{q(z,w)\}| \le |z|^2\{|h(z)| + |R(z)|\}.$$

In view of the estimate (4.4) and the fact that $R(z) = o(1)$ as $z \to 0$, we can find a small constant $\varepsilon_2 > 0$ such that

$$\operatorname{Re}\{q(z,w)\} \ge \frac{3}{4}|z|^2, \quad |\operatorname{Im}\{q(z,w)\}| < \frac{\pi}{2k}|z|^2 \quad \forall |z| \le \delta,$$

$$q(S_1(\delta)) \subsetneq \{x + iy \in \mathbb{C} : |y| \le (2\pi/3k)x\} \quad \text{where } \delta \in (0, \varepsilon_2].$$

In other words, $q(S_1(\delta))$ is a proper subset of the sector W_1 that is centered on the positive x-axis, and it has an aperture of $(4\pi/3k)$. Therefore, $q(S_j(\delta))$ is a proper subset of the sector W_j, which is simply a copy of W_1 rotated by $(2\pi(j-1)/k)$, $j = 1, \dots, k$. Furthermore, $W_\mu \cap W_\nu = \{0\}$ for all $\mu \neq \nu$. The details for showing that there exists an $\varepsilon_0 > 0$ such that $\mathcal{P}\big(\bigcup_{j=1}^k S_j(\varepsilon_0)\big) = \mathcal{C}\big(\bigcup_{j=1}^k S_j(\varepsilon_0)\big)$ are no different from those given in the proof of Theorem 1.1. We omit these details. Finally, from $\mathcal{P}\big(\bigcup_{j=1}^k S_j(\varepsilon_0)\big) = \mathcal{C}\big(\bigcup_{j=1}^k S_j(\varepsilon_0)\big)$ we conclude, exactly as in Theorem 1.1, that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})$. □

Next, we prove the requisite lemmas for showing that the polynomial $\phi_{\epsilon(\gamma)} : \mathbb{C}^2 \to \mathbb{C}$, as defined by equation (3.17), maps the surfaces $\mathcal{S}_1(\delta)$ and $\mathcal{S}_2(\delta)$—encountered in the proof of Theorem 1.2—into two sectors in $\mathbb{C}$ that intersect only at the origin.

LEMMA 4.2. *Let Σ_1 be the surface in $\mathbb{C}^2$ given by the equation*

$$w = \bar{z}^k + \sigma(z) + G(z),$$

where γ, $\sigma(z)$, and $G(z)$ are as in Lemma 4.1. Then there exist small constants $\delta_1, R_1 > 0$ such that $\phi_{\epsilon(\gamma)}(\{(z,w) : |z| \leq R_1\} \cap \Sigma_1)$ is contained in the sector $W(\delta_1) = \{x + iy \in \mathbb{C} : |y| \leq (1/\delta_1)x\}$.

Proof. For $(z,w) \in \Sigma_1$ we compute

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &= -\operatorname{Re}\left\{\frac{2\bar{z}^k\sigma(z)}{4} + \frac{\sigma(z)^2}{4} - \epsilon(\gamma)z^k\sigma(z)\right\} \\ &\quad + \epsilon(\gamma)|z|^{2k} + o(|z|^{2k}), \\ \operatorname{Im}\{\phi_{\epsilon(\gamma)}(z,w)\} &= \frac{1}{2}\operatorname{Im}(z^{2k}) - \operatorname{Im}\left\{\frac{2\bar{z}^k\sigma(z)}{4} + \frac{\sigma(z)^2}{4} - \epsilon(\gamma)z^k\sigma(z)\right\} \\ &\quad + o(|z|^{2k}).\end{aligned}$$

We consider the following two cases.

Case (i): $\gamma \geq 1$ and $\epsilon(\gamma) = 3/16$. In view of the estimates on $\sigma(z)$—including the upper bound (4.2)—and of (4.1), we can find a $R_1 > 0$ sufficiently small that, if $(z,w) \in \Sigma_1$, we have

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &> \frac{3}{16}(1-2\kappa)|z|^{2k} - \left\{\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4}\right\}2\kappa|z|^{2k} + o(|z|^{2k}) \\ &\geq \frac{3}{32}(1-8\kappa)|z|^{2k} \quad \forall |z| \leq R_1. \end{aligned} \tag{4.6}$$

Notice that, as $\kappa < 1/8$, $(1-8\kappa) > 0$ in (4.6).

Case (ii): $1/2 < \gamma < 1$ and $\epsilon(\gamma) = \min\{(2\gamma-1)/8\gamma(1-\gamma), 3/16\}$. In this case we first compute that, for all $(z,w) \in \Sigma_1$,

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &> \frac{2\gamma-1}{8\gamma(1-\gamma)}(1-2\kappa)|z|^{2k} - \left\{\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}\right\}2\kappa|z|^{2k} + o(|z|^{2k}) \\ &= \frac{2\gamma-1}{8\gamma(1-\gamma)}|z|^{2k} + \frac{2\kappa}{8}|z|^{2k} - 2\kappa\left\{\frac{2\gamma-1}{8\gamma(1-\gamma)}+\frac{6}{8}\right\}|z|^{2k} \\ &\quad + o(|z|^{2k}). \end{aligned} \tag{4.7}$$

At this point we observe that

$$2\kappa \le \frac{2\gamma-1}{2\gamma(3\gamma+2)} < \frac{2\gamma-1}{8\gamma-6\gamma^2-1} \quad \forall \gamma \in (1/2, 1).$$

In view of this fact and the estimate (4.7), we can find a constant $R_1 > 0$ such that, if $(z,w) \in \Sigma_1$, then

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &> \frac{2\gamma-1}{8\gamma(1-\gamma)}|z|^{2k} + \frac{2\kappa}{8}|z|^{2k} \\ &\quad - \frac{2\gamma-1}{8\gamma-6\gamma^2-1}\left\{\frac{2\gamma-1}{8\gamma(1-\gamma)}+\frac{6}{8}\right\}|z|^{2k} + o(|z|^{2k}) \\ &> \frac{\kappa}{4}|z|^{2k} + o(|z|^{2k}) > \frac{\kappa}{8}|z|^{2k} \quad \forall |z| \le R_1. \end{aligned} \tag{4.8}$$

It is evident from these expressions that there exists a $C > 0$ such that

$$|\operatorname{Im}\{\phi_{\epsilon(\gamma)}(z,w)\}| \le C|z|^{2k} \quad \forall (z,w) \in \{(z,w) : |z| \le R_1\} \cap \Sigma_1. \tag{4.9}$$

The result then follows from (4.6), (4.8), and (4.9). □

LEMMA 4.3. *Let Σ_2 be the surface in $\mathbb{C}^2$ given by the equation*

$$w = -\left(\frac{1}{\gamma}z^k + \bar{z}^k\right) + \sigma(z) + G(z),$$

where $\gamma, \sigma(z)$, and $G(z)$ are as in Lemma 4.1. Then there exists a constant $R_2 > 0$ such that $\phi(\{(z,w) : |z| \le R_2\} \cap \Sigma_2) \setminus \{0\}$ is contained in the open left half-plane.

Proof. Once again, we analyze the problem into two cases.

Case (i): $\gamma \ge 1$ and $\epsilon(\gamma) = 3/16$. We compute to find that, for $(z,w) \in \Sigma_2$,

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &\le -\left\{\frac{1}{2\gamma}+\epsilon(\gamma)\right\}|z|^{2k} + \left\{\epsilon(\gamma)\left(\frac{1}{\gamma}\right)+\frac{1}{4\gamma^2}\right\}|\operatorname{Re}(z^{2k})| \\ &\quad + \frac{|z|^k|\sigma(z)|}{4}\left\{\frac{2}{\gamma}+2+4\epsilon(\gamma)+\frac{|\sigma(z)|}{|z|^k}\right\} + o(|z|^{2k}) \\ &< -\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k} - \frac{3}{16}\left(1-\frac{1}{\gamma}\right)|z|^{2k} \\ &\quad + \frac{|z|^k|\sigma(z)|}{4}\frac{2+3\gamma}{\gamma} + o(|z|^{2k}).\end{aligned}$$

The last inequality is the consequence of the estimate (4.2) and the fact that $\epsilon(\gamma) = 3/16$. Now, exploiting the bounds on $|\sigma(z)|$ and κ, we obtain

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &< -\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k}-\frac{3}{16}\left(1-\frac{1}{\gamma}\right)|z|^{2k}\\ &\quad+\frac{2\gamma-1}{8\gamma^2}|z|^{2k}+o(|z|^{2k})\\ &< -\frac{3}{16}\left(1-\frac{1}{\gamma}\right)|z|^{2k}+o(|z|^{2k})\\ &< 0 \quad \forall (z,w)\in\{(z,w):|z|\le R_2\}\cap\Sigma_2, \end{aligned}\tag{4.10}$$

given some sufficiently small $R_2 > 0$. We can find such an R_2 because in this case $(1-1/\gamma) > 0$.

Case (ii): $1/2 < \gamma < 1$ *and* $\epsilon(\gamma) = \min\{(2\gamma-1)/8\gamma(1-\gamma),\ 3/16\}$. Following the computation performed in Case (i) yields

$$\begin{aligned}&\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\}\\ &\quad< -\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k}-\epsilon(\gamma)\left(1-\frac{1}{\gamma}\right)|z|^{2k}+\frac{|z|^k|\sigma(z)|}{4}\frac{2+3\gamma}{\gamma}+o(|z|^{2k}).\end{aligned}$$

This estimate is derived from the very first line of the estimate on $\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\}$ under Case (i), coupled with the fact that $\epsilon(\gamma)\le 3/16$. We now use the fact that $\epsilon(\gamma)\le(2\gamma-1)/8\gamma(1-\gamma)$ to get

$$\begin{aligned}\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} &< -\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k}-\frac{2\gamma-1}{8\gamma(1-\gamma)}\left(1-\frac{1}{\gamma}\right)|z|^{2k}\\ &\quad+\frac{|z|^k|\sigma(z)|}{4}\frac{2+3\gamma}{\gamma}+o(|z|^{2k})\\ &\le -\frac{1}{2}\left(\frac{1}{2\gamma}-\frac{1}{4\gamma^2}\right)|z|^{2k}+\frac{2\kappa}{4}\left(\frac{2+3\gamma}{\gamma}\right)|z|^{2k}+o(|z|^{2k}).\end{aligned}$$

Applying the fact that $2\kappa < (2\gamma-1)/2\gamma(3\gamma+2)$ to the last inequality, we see that there exists a constant $R_2 > 0$ such that $\operatorname{Re}\{\phi_{\epsilon(\gamma)}(z,w)\} < 0$ for all $(z,w)\in\{(z,w):|z|\le R_2\}\cap\Sigma_2$.

Given the last conclusion and inequality (4.10), the result is established. □

References

[1] E. Bishop, *Differentiable manifolds in complex Euclidean space,* Duke Math. J. 32 (1965), 1–21.

[2] F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of* $\mathbb{C}^2$, Ann. Inst. Fourier (Grenoble) 37 (1987), 1–44.

[3] F. Forstnerič and E. L. Stout, *A new class of polynomially convex sets,* Ark. Mat. 29 (1991), 51–62.

[4] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables,* Prentice-Hall, Englewood Cliffs, NJ, 1965.

[5] B. Jöricke, *Local polynomial hulls of discs near isolated parabolic points,* Indiana Univ. Math. J. 46 (1997), 789–826.

[6] E. Kallin, *Fat polynomially convex sets,* Function algebras (Proc. Internat. Sympos. on Function Algebras, Tulane University, 1965), pp. 149–152, Scott-Foresman, Chicago, 1966.

[7] A. G. O'Farrell and K. J. Preskenis, *Uniform approximation by polynomials in two functions,* Math. Ann. 284 (1989), 529–535.

[8] J. Wermer, *Approximation on a disk,* Math. Ann. 155 (1964), 331–333.

[9] J. Wiegerinck, *Local polynomially convex hulls at degenerated CR singularities of surfaces in* $\mathbb{C}^2$, Indiana Univ. Math. J. 44 (1995), 897–915.

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