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Rotation of Wedges of Extendability for Tubelike CR Manifolds of CR Dimension 1

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1. Introduction

The concept of wedge extendability, as described for example in the work of Tumanov [T] and Baouendi and Rothschild [BaR], has been a major focus of research on CR extension. (See e.g. [BaER] for a general treatment of these and related topics.) We briefly describe this concept. Consider a smooth, generic CR submanifold M of $\mathbb{C}^n$ with codimension d and CR dimension $n-d$. This means that if $p \in M$, there exists an open neighborhood $\mathcal{U}$ of p in $\mathbb{C}^n$ and a smooth function $\rho = (\rho_1, \dots, \rho_d) \colon \mathcal{U} \to \mathbb{R}^d$ such that

$$M = \{z \in \mathcal{U} \mid \rho(z) = 0\}$$

and $\partial\rho_1 \wedge \dots \wedge \partial\rho_d \neq 0$. We then define a *wedge* as follows.

Definition 1.1. Let Γ be an open, convex cone in $\mathbb{R}^d$ with vertex at the origin, and let $\mathcal{V} \subset \mathbb{C}^n$ be an open neighborhood of the point p. Then the wedge $\mathcal{W}(\Gamma, \mathcal{V}, p)$ with edge M centered at p is

$$\{z \in \mathcal{V} \mid \rho(z) \in \Gamma\}. \tag{1.1}$$

A number of additional hypotheses on M will guarantee the existence of a common wedge to which all CR functions on a fixed open set on the manifold extend holomorphically. The weakest such hypothesis is minimality. The submanifold M is said to be *minimal* at p if there is no submanifold $S \subset M$ through p with strictly smaller real dimension but the same CR dimension as M. To be precise, the following theorem holds.

Theorem 1.2 [T; BaR]. *If M is minimal at p then, for every open neighborhood U of p in M, there exists an open neighborhood $\mathcal{V}$ of p in $\mathbb{C}^n$ with $M \cap \mathcal{V} \subset U$ and a wedge $\mathcal{W} = \mathcal{W}(\Gamma, \mathcal{V}, p)$ such that every continuous CR function on U extends holomorphically to $\mathcal{W}$. Conversely, if M is not minimal at p then there exists a continuous CR function defined in some neighborhood of p in M that does not extend holomorphically to any wedge with edge M centered at p.*

Sufficiency was established by Tumanov, and necessity was established by Baouendi and Rothschild.

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Wedges do not have some of the characteristics one might reasonably expect of the regions described by a general theory of CR extension. To begin with, if U and V are open sets in M and if $V \subseteq U$, then the region of extendability for CR functions on V should be contained in the region of extendability for CR functions on U. Boggess, Glenn, and Nagel [BoGN] show that wedges do not have this property. They give an example of a manifold for which, given any fixed wedge of extendability for CR functions on an open neighborhood U of the origin, one can find a sufficiently small open neighborhood V of the origin such that every wedge of extendability for V is disjoint from the fixed wedge for U. In this example, there is no fixed direction contained in the region of extendability associated with every neighborhood of the origin. We refer to this phenomenon as *wedge rotation*.

The goal of this paper is to characterize those points on a generic tubelike CR submanifold of $\mathbb{C}^n$ of CR dimension 1 at which wedge rotation occurs. After making precise what it means to say that the region of extendability for any neighborhood of a point contains a fixed direction, we present an alternative approach to the example considered by Boggess, Glenn, and Nagel that gives some insight into why wedges fail to have the desired containment property. We then give a necessary and sufficient condition for the region of extendability for a tubular neighborhood of a point of finite type to contain a fixed direction in terms of the Hörmander numbers at that point. This theorem thus leads to a necessary condition for an *arbitrary* neighborhood of a point of finite type to contain a fixed direction, and it shows that the phenomenon of wedge rotation is common. This provides a strong argument for the development of an alternative to wedges in the theory of CR extension.

2. Definitions and Statement of the Theorem

Let M be a $(2n-d)$-dimensional smooth (C^∞) real submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$. Let $p \in M$; denote by $T_p(M)$ the real tangent space to M at p and by $H_p(M)$ the maximal complex subspace of $T_p(M)$. Then

$$H_p(M) = T_p(M) \cap J(T_p(M)),$$

where J is the usual complex structure map on $\mathbb{R}^{2n}$. We say that M is a *CR submanifold* of $\mathbb{C}^n$ of *CR dimension k* if $\dim_{\mathbb{R}} H_p(M) = 2k$ for all $p \in M$. Then

$$2n - 2d \le 2k \le 2n - d.$$

We say that M is *generic* if $2k = 2n - 2d$, that is, if k is minimal. In this paper we consider the special case in which M is a tubelike, generic CR submanifold of $\mathbb{C}^n$ of CR dimension 1. For such a manifold, one can find local coordinates in which p is the origin and M is given near the origin by

$$\{(z = x + iy, w_1, \dots, w_{n-1}) \in \mathbb{C}^n \mid \operatorname{Re}(w_j) = \phi_j(x),\ 1 \le j \le n-1\}. \tag{2.1}$$

Since each ϕ_j is C^∞ it follows that

$$\phi_j(x) = \sum_{\ell=2}^{m} a_{j,\ell} x^\ell + E_m^j(x), \tag{2.2}$$

where $E^j_m(x) = x^{m+1}c_j(x)$ with each c_j a C^∞-function in a neighborhood of the origin.

Denote the complexifications of $T_0(M)$ and $H_0(M)$ by $T_0^{\mathbb{C}}(M)$ and $H_0^{\mathbb{C}}(M)$, respectively. The complexified holomorphc tangent bundle $H^{\mathbb{C}}(M)$ is spanned near the origin by the vector fields

$$\bar{L} = \frac{\partial}{\partial \bar{z}} + \sum_{j=1}^{n-1} \phi_j'(x) \frac{\partial}{\partial \bar{w}_j},$$

$$L = \frac{\partial}{\partial z} + \sum_{j=1}^{n-1} \phi_j'(x) \frac{\partial}{\partial w_j}.$$

Let $w_j = u_j + iv_j$. We may write $L = \frac{1}{2}(X - iY)$ and $\bar{L} = \frac{1}{2}(X + iY)$, where

$$X = \frac{\partial}{\partial x} + \sum_{j=1}^{n-1} \phi_j'(x) \frac{\partial}{\partial u_j},$$

$$Y = \frac{\partial}{\partial y} + \sum_{j=1}^{n-1} \phi_j'(x) \frac{\partial}{\partial v_j}.$$

Then $H_0^{\mathbb{C}}(M)$ is spanned by $\left\{X_0 = \frac{\partial}{\partial x}, Y_0 = \frac{\partial}{\partial y}\right\}$. We consider commutators of X and Y at the origin as follows:

$$\begin{aligned}[X, Y]_0 &= \sum_{j=1}^{n-1} \phi_j''(0) \frac{\partial}{\partial v_j} \\ &= 2 \sum_{j=1}^{n-1} a_{j,2} \frac{\partial}{\partial v_j}.\end{aligned}$$

More generally, define

$$[X, Y]^{(p,0)} = [X, [X, \ldots, [X, Y]]], \quad p \geq 0,$$

to be the $(p+1)$th-order commutator involving $[X, Y]$ and p additional copies of the vector field X. It is clear that

$$\begin{aligned}[X, Y]_0^{(p,0)} &= \sum_{j=1}^{n-1} \phi_j^{(p+2)}(0) \frac{\partial}{\partial v_j} \\ &= (p+2)! \sum_{j=1}^{n-1} a_{j,p+2} \frac{\partial}{\partial v_j}. \end{aligned} \tag{2.3}$$

Thus we may identify $[X, Y]_0^{(p,0)}$ with the vector $(a_{1,p+2}, \ldots, a_{n-1,p+2}) \in \mathbb{R}^{n-1}$. Observe that, for any $p \geq 0$, $[Y, [X, Y]^{(p,0)}] = 0$ and so the only nonvanishing commutators of X and Y at the origin are the $[X, Y]_0^{(p,0)}$.

Let $\mathcal{L}_0^1(M)$ be the subspace of $T_0^{\mathbb{C}}(M)$ spanned by X_0 and Y_0, and for $p \geq 2$ let $\mathcal{L}_0^p(M)$ be the subspace of $T_0^{\mathbb{C}}(M)$ spanned by X_0, Y_0, and the $[X, Y]_0^{(q,0)}$ for $0 \leq q \leq p-2$. We define the Hörmander numbers $(m_1, \dots, m_{n-1})$ at the origin by

$$m_r = \inf\{p \mid \dim_{\mathbb{C}} \mathcal{L}_0^p(M) \geq r+2\}.$$

We say that M is of *finite type* m at the origin if $m_{n-1} = m$. This is equivalent to saying that $\mathcal{L}_0^m(M) = T_0^{\mathbb{C}}(M)$ but $\mathcal{L}_0^{m-1}(M)$ is a proper subspace of $T_0^{\mathbb{C}}(M)$. It is clear that M is of type m if $\{(a_{1,\ell}, \dots, a_{n-1,\ell}) \mid 2 \leq \ell \leq m\}$ spans $\mathbb{R}^{n-1}$.

Let

$$M^\varepsilon = \{(z, w_1, \dots, w_{n-1}) \in M \mid |\mathrm{Re}(z)| < \varepsilon\}$$

and

$$\gamma_M^\varepsilon = \{(x, \phi_1(x), \dots, \phi_{n-1}(x)) \mid |x| < \varepsilon\}.$$

Denote the convex hull of γ_M^ε by $\mathrm{ch}(\gamma_M^\varepsilon)$. At the origin, the tangent space to γ_M^ε is spanned by $(1, 0, \dots, 0)$. We call the orthogonal complement in $\mathbb{R}^n$ to this tangent space the space *normal* to the curve, denoting the slice of $\mathrm{Int}(\mathrm{ch}(\gamma_M^\varepsilon))$ in the normal space at the origin by $N_0(\gamma_M^\varepsilon)$.

We then have the following special case of the theorem of Boivin and Dwilewicz.

THEOREM 2.1 [BoiD]. *Every continuous CR function on $M^\varepsilon = \gamma_M^\varepsilon + i\mathbb{R}^n$ can be continuously extended to a function on* $\mathrm{Int}(\mathrm{ch}(\gamma_M^\varepsilon)) + i\mathbb{R}^n$ *that is holomorphic on* $\mathrm{Int}(\mathrm{ch}(\gamma_M^\varepsilon)) + i\mathbb{R}^n$.

We conclude this section with a definition and the statement of our main theorem.

DEFINITION 2.2. We say that $N_0(\gamma_M^\varepsilon)$ contains a *fixed direction* for every $\varepsilon > 0$ if there is a vector $\mathbf{v}$ in $\{(x, u_1, \dots, u_{n-1}) \mid x = 0\}$ such that, for every $\varepsilon > 0$, there exists a $c = c(\varepsilon) > 0$ such that $c\mathbf{v} \in N_0(\gamma_M^\varepsilon)$.

THEOREM 2.3. *For every $\varepsilon > 0$, $N_0(\gamma_M^\varepsilon)$ contains a fixed direction if and only if M has at most one even Hörmander number. More specifically*:

(i) *if all of the Hörmander numbers are odd, then* $\mathrm{Int}(\mathrm{ch}(\gamma_M^\varepsilon))$ *contains a full neighborhood of the origin*;

(ii) *if there is exactly one even Hörmander number, m_K, then there exists a $c = c(\varepsilon) > 0$ such that $-cJ([X, Y]_0^{(m_K-2,0)}) \in N_0(\gamma_M^\varepsilon)$; and*

(iii) *if there are two or more even Hörmander numbers at the origin, then $N_0(\gamma_M^\varepsilon)$ does not contain a fixed direction for every $\varepsilon > 0$.*

The theorem of Boivin and Dwilewicz (our Theorem 2.1) describes exactly the regions of extendability for tubular neighborhoods of a point. In general, it is much harder to describe precisely the region of extendability for an arbitrary neighborhood of a point. However, an arbitrary open neighborhood of a point is contained in a tubular neighborhood, so if the regions of extendability for tubular neighborhoods of a point fail to contain a fixed direction as the neighborhoods shrink then the same will be true of arbitrary neighborhoods. Hence this theorem immediately implies that, if p is a point of finite type on M and if $\{U^\varepsilon\}$ is *any* family of open neighborhoods of p that shrink about p as $\varepsilon \to 0$, then the regions of extendability for the U^ε contain a fixed direction for every ε only if at most one Hörmander number at p is even. The phenomenon of wedge rotation observed by Boggess, Glenn, and Nagel is thus common.

3. An Example

The following example illustrates what can happen to wedges as the open set U shrinks about the origin.

EXAMPLE. Let

$$M = \{(z = x + iy,\ w_2 = u_2 + iv_2,\ w_4 = u_4 + iv_4) \in \mathbb{C}^3 \mid u_2 = x^2,\ u_4 = x^4\}.$$

The Hörmander numbers at the origin are $(2, 4)$. Let

$$\gamma_M^\varepsilon = \{(x, x^2, x^4) \mid |x| < \varepsilon\}$$

and

$$\gamma_4^\varepsilon = \{(x, x^2, x^3, x^4) \mid |x| < \varepsilon\}.$$

If Π is the projection map

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_4)$$

then, since Π is linear,

$$\mathrm{ch}(\gamma_M^\varepsilon) = \Pi(\mathrm{ch}(\gamma_4^\varepsilon)).$$

The classical theory of power moment sequences gives a precise description of $\mathrm{ch}(\gamma_4^\varepsilon)$ (see for example [KNu, Chap. III, Thm. 2.3]). A point (u_1, u_2, u_3, u_4) is in $\mathrm{Int}(\mathrm{ch}(\gamma_4^\varepsilon))$ if and only if the quadratic forms

$$\sum_{j,k=0}^{2} u_{j+k}\xi_j\xi_k \quad \text{and} \quad \sum_{j,k=0}^{1} (\varepsilon^2 u_{j+k} - u_{j+k+2})\xi_j\xi_k, \quad u_0 = 1, \tag{3.1}$$

are positive. Thus we see that $(0, u_2, u_3, u_4) \in N_0(\gamma_4^\varepsilon)$ if and only if the matrices

$$\begin{bmatrix} 1 & 0 & u_2 \\ 0 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \varepsilon^2 - u_2 & -u_3 \\ -u_3 & \varepsilon^2 u_2 - u_4 \end{bmatrix}$$

are positive definite. By the signature rule of Jacobi, this is equivalent to the condition that the successive principal minors are positive. That is, $(0, u_2, u_3, u_4) \in N_0(\gamma_4^\varepsilon)$ if and only if the u_j satisfy

$$\begin{aligned} u_2 &> 0, \\ u_2(u_4 - u_2^2) - u_3^2 &> 0, \\ \varepsilon^2 - u_2 &> 0, \\ (\varepsilon^2 - u_2)(\varepsilon^2 u_2 - u_4) - u_3^2 &> 0. \end{aligned}$$

Since Π maps the space normal to γ_4^ε at the origin onto the space normal to γ_M^ε at the origin, we need only consider the image of this set under Π. This image certainly contains

$$\{(0, u_2, u_4) \mid 0 < u_2 < \varepsilon^2,\ u_2^2 < u_4 < \varepsilon^2 u_2\}, \tag{3.2}$$

and if $c \neq 0$ then the projection under Π of the level set where $u_3 = c$ is contained in the set in (3.2). Therefore, this set is equal to $N_0(\gamma_M^\varepsilon)$.

Suppose now that $\mathcal{W}$ is a fixed wedge to which CR functions on M^ε extend. (See Definition 1.1 for the precise definition of a wedge.) The existence of such a wedge follows from Tumanov's theorem (our Theorem 1.2) because M is of finite type—and hence minimal—at every point. As a result, there exist δ_1 and δ_2 positive such that if $(x + iy, u_2 + iv_2, u_4 + iv_4) \in \mathcal{W}$ then

$$\delta_1 u_2 < u_4 < \delta_2 u_2,$$

where $\delta_2 \leq \varepsilon^2$. Consider now any wedge of extendability for CR functions on M^{ε_0}, where $\varepsilon_0 = \sqrt{\delta_1/2}$. There exist η_1 and η_2 positive such that a point $(x + iy, u_2 + iv_2, u_4 + iv_4)$ in this wedge satisfies

$$\eta_1 u_2 < u_4 < \eta_2 u_2 \leq \varepsilon_0^2 u_2 = \frac{\delta_1}{2} u_2.$$

Therefore, any wedge of extendability for M^{ε_0} is disjoint from $\mathcal{W}$. Observe that in this example there is no fixed direction contained in $N_0(\gamma_M^\varepsilon)$ for every $\varepsilon > 0$.

4. Proof of Theorem 2.3

We first make a change of coordinates. Let $\tilde{A}$ be the nonsingular linear transformation that reduces the matrix $(a_{j,\ell})$ for $1 \leq j \leq n-1$ and $2 \leq \ell \leq m$ to its corresponding row-echelon form. Define a biholomorphic change of variables on $\mathbb{C}^n$ by setting

$$z = z \quad \text{and} \quad \zeta = \tilde{A} w,$$

where $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{C}^{n-1}$ and $w = (w_1, \ldots, w_{n-1})$. As before, let $z = x + iy$ and set $\zeta_j = t_j + is_j$, $1 \leq j \leq n-1$. In terms of these new coordinates our manifold becomes

$$\hat{M} = \{(z, \zeta) \in \mathbb{C}^n \mid t_j = \psi_j(x) = x^{m_j} + \mathcal{E}_j(x),\ 1 \leq j \leq n-1\}, \tag{4.1}$$

where $\mathcal{E}_j(x) = x^{m_j+1} c_j(x)$ for a function c_j that is bounded in a neighborhood of the origin. Observe that, if $\{\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}\}$ denotes the standard basis for $\mathbb{R}^n$, then the transformation $\tilde{A}$ maps $-J([X, Y]_0^{(m_j-2,0)})$ to a vector in the $\mathbf{e}_j$ direction. Also, in the new coordinates, the lowest-degree term in the jth defining function for $\hat{M}$ is x^{m_j}, where m_j is the jth Hörmander number at the origin.

To prove the theorem, we establish three lemmas. Lemma 4.3 implies that the slice of the convex hull of γ in the normal space at the origin is contained in the convex hull of the projection of γ onto this same space. Although we do not have the reverse containment, Lemma 4.2 can be interpreted roughly as stating that the slice of the convex hull in the normal space to the curve includes (the projections of) all directions realized in the convex hull itself. We begin with several definitions.

DEFINITION 4.1. The hyperplane

$$H = \left\{(\xi_0, \xi_1, \dots, \xi_{n-1}) \,\middle|\, \sum_{j=0}^{n-1} \alpha_j \xi_j = a\right\}$$

cuts the set E if there exist y and z in E such that $\sum_{j=0}^{n-1} \alpha_j y_j < a$ and $\sum_{j=0}^{n-1} \alpha_j z_j > a$. We say that H *supports* E if, for all $x \in E$, either $\sum_{j=0}^{n-1} \alpha_j x_j \geq a$ or $\sum_{j=0}^{n-1} \alpha_j x_j \leq a$.

LEMMA 4.2. *If the plane*

$$\left\{(\xi_0, \xi_1, \dots, \xi_{n-1}) \,\middle|\, \sum_{j=1}^{n-1} \alpha_j \xi_j = 0\right\}$$

cuts $\gamma_{\hat{M}}$ *at* $(x_0, \psi_1(x_0), \dots, \psi_{n-1}(x_0))$, *then it also cuts* $N_0(\gamma_{\hat{M}}^{\varepsilon})$ *for* $\varepsilon > |x_0|$.

Proof. We show first that if $(x_0, \psi_1(x_0), \dots, \psi_{n-1}(x_0)) \in \gamma_{\hat{M}}$ then, given $\eta > 0$, there exists a $\lambda > 0$ such that

$$\{(0, t_1, \dots, t_{n-1}) \mid |t_j - \lambda\psi_j(x_0)| < \lambda\eta\}$$

contains a point of $N_0(\gamma_{\hat{M}}^{|x_0|})$. Geometrically, this means that every neighborhood of the projection onto the space normal to the curve of a point on the curve contains a dilate of some point in $N_0(\gamma_{\hat{M}})$.

To prove the claim, let $(x_0, \psi_1(x_0), \dots, \psi_{n-1}(x_0))$ and $(y, \psi_1(y), \dots, \psi_{n-1}(y))$ be two points on $\gamma_{\hat{M}}$, and let $0 < \lambda < 1$. A convex linear combination of the two points on the curve is in $N_0(\gamma_{\hat{M}})$ if $\lambda x_0 + (1-\lambda)y = 0$ or (equivalently)

$$y = -\frac{\lambda}{1-\lambda} x_0,$$

and if $\lambda < 1/2$ then the point corresponding to y is in $\gamma_{\hat{M}}^{|x_0|}$. The jth coordinate of the corresponding point in $N_0(\gamma_{\hat{M}}^{|x_0|})$ is

$$t_j = \lambda[x_0^{m_j} + \mathcal{E}_j(x_0)] + (1-\lambda)\left[\left(-\frac{\lambda}{1-\lambda}x_0\right)^{m_j} + \mathcal{E}_j\left(-\frac{\lambda}{1-\lambda}x_0\right)\right].$$

Then

$$\begin{aligned} |t_j - \lambda\psi_j(x_0)| &= (1-\lambda)\left|\left(-\frac{\lambda}{1-\lambda}x_0\right)^{m_j} + \mathcal{E}_j\left(-\frac{\lambda}{1-\lambda}x_0\right)\right| \\ &\leq \lambda\left[|x_0|^{m_j}\left(\frac{\lambda}{1-\lambda}\right)^{m_j-1} + C|x_0|^{m_j+1}\left(\frac{\lambda}{1-\lambda}\right)^{m_j}\right]. \end{aligned} \tag{4.2}$$

Since $\lambda < 1/2$ and $m_j > 2$, it follows from (4.2) that

$$|t_j - \lambda\psi_j(x_0)| \leq \lambda\{\lambda 2^m[|x_0|^{m_j} + C|x_0|^{m_j+1}]\}. \tag{4.3}$$

The quantity in braces can be made arbitrarily small by taking λ sufficiently small. This establishes the claim.

Now, if the plane cuts $\gamma_{\hat{M}}$ at $(x_0, \psi_1(x_0), \ldots, \psi_{n-1}(x_0))$, then given $\delta > 0$ there exist x_1 and x_2 in $(x_0 - \delta, x_0 + \delta)$ such that $\sum_{j=1}^{n-1} \alpha_j \psi_j(x_1) < 0$ and $\sum_{j=1}^{n-1} \alpha_j \psi_j(x_2) > 0$. Then, by (4.3), given $\eta > 0$ there exist a $\lambda > 0$ and a point $(0, t_1, \ldots, t_{n-1}) \in N_0(\gamma_{\hat{M}}^{|x_1|})$ with

$$|t_j - \lambda \psi_j(x_1)| < \lambda \eta,$$

and thus

$$|\alpha_j t_j - \lambda \alpha_j \psi_j(x_1)| < \lambda \eta |\alpha_j|.$$

Hence

$$\alpha_j t_j < \lambda \alpha_j \psi_j(x_1) + \lambda \eta |\alpha_j|.$$

Summing over all j, we obtain

$$\sum_{j=1}^{n-1} \alpha_j t_j < \lambda \left(\sum_{j=1}^{n-1} \alpha_j \psi_j(x_1) + \eta \sum_{j=1}^{n-1} |\alpha_j| \right).$$

Since $\sum_{j=1}^{n-1} \alpha_j \psi_j(x_1) < 0$, by taking η sufficiently small we can find a point in $N_0(\gamma_{\hat{M}}^{|x_1|})$ for which $\sum \alpha_j t_j < 0$. A similar argument shows that we can find $(0, t_1', \ldots, t_{n-1}')$ in $N_0(\gamma_{\hat{M}}^{|x_2|})$ for which $\sum \alpha_j t_j' > 0$. The plane $\sum_{j=1}^{n-1} \alpha_j \xi_j = 0$ thus cuts $N_0(\gamma_{\hat{M}}^{\varepsilon})$ for any $\varepsilon \geq \max\{|x_1|, |x_2|\}$. Because x_1 and x_2 can be taken to be as close to x_0 as we like, this proves the lemma. $\square$

Let $\Pi \colon \mathbb{R}^n \to S_p$ be the orthogonal projection of $\mathbb{R}^n$ with standard basis $\{\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}\}$ onto the subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{e}_{\ell_1}, \ldots, \mathbf{e}_{\ell_p}\}$, where $p \leq n$ and $\ell_1 < \ell_2 < \cdots < \ell_p$. Then

$$[\Pi(\xi_0, \ldots, \xi_{n-1})]_j = \begin{cases} \xi_j & \text{if } j = \ell_k, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 4.3. *Let γ be a curve in $\mathbb{R}^n$. Then*

$$\mathrm{ch}(\gamma) \cap S_p \subseteq \mathrm{ch}(\Pi(\gamma)).$$

Proof. Let $H = \{\xi \in \mathbb{R}^p \mid \sum_{k=1}^{p} \alpha_{\ell_k} \xi_{\ell_k} = a\}$ be a hyperplane in S_p supporting $\Pi(\gamma)$. Define $\tilde{H}$ to be the corresponding hyperplane in $\mathbb{R}^n$ given by $\sum_{j=0}^{n-1} \tilde{\alpha}_j \xi_j = a$, where $\tilde{\alpha}_j = \alpha_{\ell_k}$ if $j = \ell_k$ and $\tilde{\alpha}_j = 0$ otherwise. Then $\tilde{H}$ supports γ, hence $\mathrm{ch}(\gamma)$, and therefore $\mathrm{ch}(\gamma) \cap S_p$ lies on the same side of H as $\Pi(\gamma)$. That is: H supports $\mathrm{ch}(\gamma) \cap S_p$; since $\mathrm{ch}(\Pi(\gamma))$ is the intersection of all such supporting half-spaces, the lemma is established. $\square$

LEMMA 4.4. *If $\{m_{j_1}, \ldots, m_{j_p}\}$ are odd Hörmander numbers for $\hat{M}$ at the origin, then any hyperplane of the form*

$$\alpha_0 \xi_0 + \sum_{k=1}^{p} \alpha_{j_k} \xi_{j_k} = 0$$

cuts $\gamma_{\hat{M}}^{\varepsilon}$ for any $\varepsilon > 0$.

Proof. Let $x \in (-\varepsilon, \varepsilon)$. Then

$$\alpha_0 x + \sum_{k=1}^{p} \alpha_{j_k} \psi_{j_k}(x) = \alpha_0 x + \sum_{k=1}^{p} \alpha_{j_k} x^{m_{j_k}} [1 + c_{j_k}(x) x]$$
$$= x \left\{ \alpha_0 + \sum_{k=1}^{p} \alpha_{j_k} x^{m_{j_k} - 1} [1 + c_{j_k}(x) x] \right\}, \tag{4.4}$$

and if $\alpha_0 \neq 0$ then the expression on the right changes sign at the origin. If $\alpha_0 = 0$, suppose K is the first integer for which $\alpha_{j_K} \neq 0$. Then the right-hand side of (4.4) reduces to

$$\sum_{k=K}^{p} \alpha_{j_k} x^{m_{j_k}} [1 + c_{j_k}(x) x] = x^{m_{j_K}} \sum_{k=K}^{p} \alpha_{j_k} x^{m_{j_k} - m_{j_K}} [1 + c_{j_k}(x) x].$$

Since m_{j_K} is odd and $\alpha_{j_K} \neq 0$, this expression changes sign at the origin. Therefore, the hyperplane cuts the curve $\gamma_{\hat{M}}^{\varepsilon}$ for any $\varepsilon > 0$. □

With these lemmas established, we now complete the proof of Theorem 2.3.

Proof. Statement (i) follows immediately from Lemma 4.4, for if all of the Hörmander numbers are odd then the lemma states that, for any $\varepsilon > 0$, any plane through the origin cuts $\gamma_{\hat{M}}^{\varepsilon}$ and hence the origin is an interior point of the convex hull of $\gamma_{\hat{M}}^{\varepsilon}$.

We now establish (ii). Toward this end, suppose that $\hat{M}$ has exactly one even Hörmander number m_K at the origin. By Lemma 4.4, any plane of the form

$$\sum_{j=1}^{n-1} \alpha_j \xi_j = 0$$

with $\alpha_K = 0$ cuts $\gamma_{\hat{M}}^{\varepsilon}$ for every $\varepsilon > 0$. Thus, by Lemma 4.2, any such hyperplane cuts the convex set $N_0(\gamma_{\hat{M}}^{\varepsilon})$. Since m_K is even, there exists an $\varepsilon_1 > 0$ such that, for all $x \in (-\varepsilon_1, \varepsilon_1)$ with $x \neq 0$, we have $\psi_K(x) > 0$. Let $\mathcal{R}^{\varepsilon_1} = N_0(\gamma_{\hat{M}}^{\varepsilon_1}) \cap \{\xi \in \mathbb{R}^n \mid \xi_K > 0\}$. This set is convex because it is the intersection of two convex sets, and the only point possibly in $N_0(\gamma_{\hat{M}}^{\varepsilon_1})$ that is not in $\mathcal{R}^{\varepsilon_1}$ is the origin. Consider $\Pi(\mathcal{R}^{\varepsilon_1})$, the projection of $\mathcal{R}^{\varepsilon_1}$ onto the plane on which $\xi_K = 0$. Every hyperplane in this linear subspace passing through the origin cuts $\Pi(\mathcal{R}^{\varepsilon_1})$ and so the origin is an interior point of this set. But the origin is the projection of some point on the ξ_K axis, so there is some point on this axis in $\mathcal{R}^{\varepsilon_1}$ and hence in $N_0(\gamma_{\hat{M}}^{\varepsilon_1})$, as desired.

Finally, we prove (iii). Suppose there are two even Hörmander numbers, m_K and m_J, with $K < J$. Then there exists an $\varepsilon_1 > 0$ such that, if $x \in (-\varepsilon_1, \varepsilon_1)$ and $x \neq 0$, then $\psi_K(x) > 0$. Consider the projection of the curve $\gamma_{\hat{M}}^{\varepsilon_1}$ onto the subspace S_p spanned by $\{\mathbf{e}_K, \ldots, \mathbf{e}_{n-1}\}$. Then consider the angle made with $\mathbf{e}_K$ by the vector from the origin to the projected point. This angle satisfies

$$\cos\theta = \frac{\psi_K(x)}{\sqrt{[\psi_K(x)]^2 + \cdots + [\psi_{n-1}(x)]^2}} = \frac{x^{m_K}(1 + xc_K(x))}{x^{m_K}\sqrt{1 + [xd(x)]}}.$$

Let $\eta(\varepsilon_1)$ be the maximum angle that the projection of a point of $\gamma_{\hat{M}}^{\varepsilon_1}$ onto S_p makes with $\mathbf{e}_{j_0}$. Since $\cos\theta \to 1$ as $x \to 0$, it follows that $\eta(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Let $E^{\eta(\varepsilon_1)}$ be the set of all points in S_p that make an angle of less than $\eta(\varepsilon_1)$ with the ξ_K-axis. This set is convex, and since it contains $\Pi(\gamma_{\hat{M}}^{\varepsilon_1})$ it also contains $\mathrm{ch}[\Pi(\gamma_{\hat{M}}^{\varepsilon_1})]$. Hence, by Lemma 4.3,

$$\mathrm{ch}(\gamma_{\hat{M}}^{\varepsilon_1}) \cap S_p \subseteq \mathrm{ch}[\Pi(\gamma_{\hat{M}}^{\varepsilon_1})] \subseteq E^{\eta(\varepsilon_1)}.$$

The only points contained in $E^{\eta(\varepsilon)}$ for every $\varepsilon > 0$ are those on the ξ_K-axis. Thus, if there is a scalar multiple of a fixed vector $(0, a_1, \ldots, a_{n-1})$ in the normal space for all ε, then the vector must satisfy $a_{K+1} = \cdots = a_{n-1} = 0$.

On the other hand, since m_J is even there exist $0 < \varepsilon_2 < \varepsilon_1$ such that, if $x \in (-\varepsilon_2, \varepsilon_2)$ and $x \neq 0$, then $\psi_J(x) > 0$. Hence, the normal space contains no point in the plane $\xi_J = 0$ other than the origin and so contains no vector in that direction. Since we have already shown that any vector in $N_0(\gamma_{\hat{M}}^{\varepsilon})$ for all ε, must satisfy $a_j = 0$ for $j \geq K + 1$ and since $J > K$, it follows that $a_J = 0$. This shows that no such vector exists, completing our proof of the theorem. $\square$

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