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On the Problem of Kähler Convexity in the Bergman Metric

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1. Introduction

Let (M, ds^2) be a complete Kähler manifold of dimension n, and let $\mathcal{H}^{p,q}_{(2)}(M)$ be the space of square-integrable harmonic forms of bidegree (p,q). McNeal has studied the question: Under which reasonable conditions about the Kähler metric can one prove the vanishing of $\mathcal{H}^{p,q}_{(2)}(M)$ when $p+q \neq n$? As a sufficient condition he found that there should exist an exhausting function V for M that is at the same time a potential for ds^2 such that V dominates its gradient. We define this property as follows.

Definition. Assume that the Kähler metric ds^2 has a global potential $V \in C^2(M)$ on M. Then we say that V *dominates* its gradient if there exist constants $A, B \geq 0$ such that

$$|\partial V|^2_{ds^2} \leq A + BV \tag{1.1}$$

throughout M.

In [M2] such a Kähler manifold is called *Kähler convex*; if (1.1) holds with $B = 0$, it is called *Kähler hyperbolic.*

In complex analysis there is a case of special interest in which $M = D$ is a pseudoconvex bounded domain in $\mathbb{C}^n$ that is endowed with the Bergman metric. Let $K_D(z)$ denote the Bergman kernel function on the diagonal of $D \times D$. Then $V_D = \log K_D$ is a potential of the Bergman metric.

Donnelly and Fefferman [DoFe] proved the vanishing of $\mathcal{H}^{p,q}_{(2)}(D)$ when $p+q \neq n$ and D is strongly pseudoconvex. Later, Donnelly [Do1; Do2] gave a simpler proof of this by a method that applies also to the case of finite-type pseudoconvex domains in $\mathbb{C}^2$ and to certain classes of finite-type domains in $\mathbb{C}^n$ with $n \geq 3$ (see e.g. [M1]). In these cases he showed using results of [C; M1] that even Kähler hyperbolicity holds. Also in [Do2] it was shown that the domain $D = \{z \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^{10} + |z_3|^{10} + |z_2|^2|z_3|^2 < 1\}$ is not Kähler hyperbolic in the Bergman metric.

The purpose of this paper is to show (by means of an example) that, on a smooth bounded weakly pseudoconvex domain of finite type, the potential V_D in general will not dominate its gradient. We will do this using ideas from [Do2; M2]; the

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key point is that the estimate (1.1) for $V = \log K_D$ can be reformulated in terms of domain functionals from Bergman theory.

2. Certain Domain Functionals

Let Ω be a bounded domain in $\mathbb{C}^n$. By $\|\cdot\|$ we denote the usual L^2-norm for functions that are square-integrable over Ω with respect to the Lebesgue measure. The subspace $H^2(\Omega) = \mathcal{O}(\Omega) \cap L^2(\Omega)$ is closed and induces a Hermitian kernel $K_\Omega(\cdot,\cdot)$, the Bergman kernel function of Ω. The function $V(z) = \log K_\Omega(z,z)$ is smooth and strictly plurisubharmonic; hence it is the potential of a Kähler metric, the Bergman metric B^2_Ω on Ω.

For $X = (X_1,\dots,X_n) \in \mathbb{C}^n$ and a function $f \in C^1(\Omega)$, we denote by $X(f)$ the directional derivative

$$X(f)(z) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) X_j. \tag{2.1}$$

Besides the well-known representation of $K_\Omega(w,w)$,

$$K_\Omega(w,w) = \max\{|f(w)|^2 \mid f \in H^2(\Omega),\ \|f\| \le 1\}, \tag{2.2}$$

we also consider the following domain functional:

$$E_\Omega(w;X) := \max\{|f(w)|^2 \mid f \in H^2(\Omega),\ \|f\| \le 1,\ X(f)(w) = 0\}.$$

By means of Bergman's method [B] we obtain

$$E_\Omega(w;X) = \frac{K_\Omega(w,w)^2 B^2_\Omega(w;X)}{X\bar{X}(K_\Omega)(w,w)}. \tag{2.3}$$

This maximum is attained for the function

$$f_{(w;X)}(z) := \frac{\sqrt{E_\Omega(w;X)}}{K_\Omega(w,w)^2 B^2_\Omega(w;X)} \big(X\bar{X}(K_\Omega)(w,w) \cdot K_\Omega(z,w) \\ - X(K_\Omega(\cdot,w))|_w \cdot \bar{X}(K_\Omega(z,w)) \big). \tag{2.4}$$

Let

$$F_\Omega(w;X) := \frac{E_\Omega(w;X)}{K_\Omega(w,w)}.$$

We denote by $Q_\Omega(w)$ the length of the gradient

$$\left(\frac{\partial \log K_\Omega(z,z)}{\partial z_1}\bigg|_{z=w}, \dots, \frac{\partial \log K_\Omega(z,z)}{\partial z_n}\bigg|_{z=w} \right)$$

measured in the Bergman metric. Then, by the Cauchy–Schwarz inequality, we have

$$\frac{|X(K_\Omega(\cdot,w))|_w|^2}{K_\Omega(w,w)^2} \le Q_\Omega(w) B^2_\Omega(w;X) \tag{2.5}$$

and hence

$$F_\Omega(w;X) = \frac{B^2_\Omega(w;X)}{B^2_\Omega(w;X) + \frac{|X(K_\Omega(\cdot,w)|_w|^2}{K_\Omega(w,w)^2}} \ge \frac{1}{1+Q_\Omega(w)}. \tag{2.6}$$

This shows our first result, as follows.

LEMMA 2.1. *On the domain* Ω, *the potential* $\log K_\Omega$ *dominates its gradient in the Bergman metric if and only if there exist nonnegative constants* A, B *such that, for any* $X \in \mathbb{C}^n \setminus \{0\}$,

$$F_\Omega(w; X) \geq \frac{1}{A + B \cdot \log K_\Omega(w, w)}. \tag{2.7}$$

In [M2, Prop. 3.1] it is shown that this estimate is sufficient. Its necessity is a consequence of (2.5) and (2.6).

In the next section we study a class of bounded weakly pseudoconvex domains with real-analytic boundary yet in which (2.7) is violated.

3. A Series of Examples

Our examples are domains in $\mathbb{C}^3$. Let a, b, c, d, m be positive integers and let

$$P(z_2, z_3) := |z_2|^{2m} + |z_3|^{2m} + |z_2|^{2a}|z_3|^{2b} + |z_2|^{2c}|z_3|^{2d}.$$

We require that

$$a > b, \quad a > c, \quad d > c, \quad d > b;$$
$$ad - bc < m \cdot \min\{a - c, d - b\}.$$

Let us furthermore put

$$x_2 = \frac{d-b}{2(ad-bc)} \quad \text{and} \quad x_3 = \frac{a-c}{2(ad-bc)}.$$

Then

$$2ax_2 + 2bx_3 = 1, \qquad 2cx_2 + 2dx_3 = 1,$$

and also

$$x_2 > \frac{1}{2m}, \qquad x_3 > \frac{1}{2m}.$$

We shall prove the following theorem.

THEOREM 3.1. *Let*

$$r(z_1, z_2, z_3) := \operatorname{Re} z_1 + |z_1|^2 + P(z_2, z_3)$$

and

$$D = \{r < 0\}.$$

Assume that $a < 2b$. *If*

$$0 < \varepsilon < \frac{1}{2}\left(\frac{a}{b} - 1\right)\left(x_2 - \frac{1}{2m}\right),$$

then for sufficiently small $t > 0$ *we have*

$$F_D(w(t), e_2) \leq c_0 t^{2\varepsilon}$$

with an unimportant constant c_0. *Here* $e_2 = (0,1,0)$ *and* $w(t) = (-t, t^{(1/2m)+\varepsilon}, 0)$.

REMARKS. (i) Certainly $K_D(w(t), w(t)) \leq Ct^{-4}$, hence $\log K_D(w(t), w(t)) \leq 4\log(1/t) + C$ (with some constant $C > 0$). This proves that (2.7) cannot hold on D.

(ii) The theorem applies for example in the case $a = 7$, $b = 5$, $c = 6$, $d = 8$, and $m \geq 27$.

Proof of Theorem 3.1

We prove the theorem in three steps.

First Step: Model Domains

For $t > 0$ let
$$\Omega_t = \left\{(z_2, z_3) \in \mathbb{C}^2 \mid P(z_2, z_3) < \tfrac{t}{4}\right\}$$
and
$$D_t = \Delta\left(-t, \tfrac{t}{2}\right) \times \Omega_t.$$
Then, for $t < 1/16$ we have
$$D_t \subset D$$
because, for such t,
$$r(z) \leq -\tfrac{t}{4} + 4t^2 < 0$$
for $z = (z_1, z_2, z_3) \in D_t$.

We claim that, with $\tilde{w}(t) = (t^{(1/2m)+\varepsilon}, 0)$ we have
$$E_D(w(t), e_2) \leq \frac{4}{\pi t^2} E_{\Omega_t}(\tilde{w}(t), (1, 0)). \tag{3.8}$$
For this we use
$$E_D(w(t), e_2) \leq E_{D_t}(w(t), e_2),$$
which is a well-known property of the domain functionals under consideration. Next we exploit the Cartesian product structure of D_t to derive
$$K_{D_t}(w(t), w(t)) = \frac{4}{\pi t^2} K_{\Omega_t}(\tilde{w}(t), \tilde{w}(t)),$$
$$\frac{\partial^2 K_{D_t}(w(t), w(t))}{\partial z_2 \partial \bar{z}_2} = \frac{4}{\pi t^2} \frac{\partial^2 K_{\Omega_t}(\tilde{w}(t), \tilde{w}(t))}{\partial z_2 \partial \bar{z}_2},$$
$$B^2_{D_t}(w(t), e_2) = B^2_{\Omega_t}(\tilde{w}(t); (1, 0)).$$
Substituting into (2.3) yields
$$E_{D_t}(w(t), e_2) = \frac{4}{\pi t^2} E_{\Omega_t}(\tilde{w}(t); (1, 0))$$
and hence (3.8).

Our next project is a good lower bound on the Bergman kernel of D at $w(t)$ by means of the Bergman kernel of a suitable model domain of dimension 2. We begin with a preparatory lemma.

LEMMA 3.1. *Let*
$$\Omega_t^* := \{(z_2, z_3) \in \mathbb{C}^2 \mid P(z_2, z_3) < t - t^2\}.$$
Then there exists a constant $C > 0$ (independent of t) such that
$$K_D(w(t), w(t)) \geq C t^{-2} K_{\Omega_t^*}(\tilde{w}(t), \tilde{w}(t)).$$

Proof. We will demonstrate the existence of a constant $C_1 > 0$ such that, given a function $f \in H^2(\Omega_t^*)$, one can find a function $f^t \in H^2(D)$ with the following properties:

$$f^t(-t, w') = \tfrac{1}{t} f(w') \quad \text{for } \ w' \in \Omega_t^*, \qquad \|f^t\| \le 2C_1 \|f\|_{L^2(\Omega_t^*)}.$$

By virtue of (2.2), this implies

$$K_D(-t, w') \ge \frac{|f^t(-t, w')|^2}{\|f^t\|^2} \ge \frac{1}{4C_1^2 t^2} \frac{|f(w')|^2}{\|f\|^2}$$

for any $f \in H^2(\Omega_t^*)$ and $w' \in \Omega_t^*$. From this the lemma will follow easily.

Let $f \in H^2(\Omega_t^*)$. Then we can view f as a function that is holomorphic on $D \cap \{z_1 = -t\} = \{(-t, z') : z' \in \Omega_t^*\}$. In order to find f^t, we use a result of Ohsawa [O]. Since $\operatorname{Re} z_1 < 0$ for $z \in D$, we have $\left|\frac{z_1+t}{z_1-t}\right| < 1$ on D. Hence the function

$$\psi(z) := -2 \log|z_1 - t|$$

satisfies

$$C_\psi := \sup\{\psi(z) + 2 \log|z_1 + t|, \, z \in D\} \le 0$$

and is a negligible weight (in the sense of [O]). Furthermore, the function $\frac{1}{t} f$ satisfies

$$\int_{D \cap \{z_1 = -t\}} \left| \frac{f(z')}{t} \right|^2 e^{-\psi(-t, z')} \, d^4 z' = 4\|f\|^2$$

and, by Ohsawa's result, there exists a holomorphic extension f^t of $\frac{1}{t} f$ to D such that

$$\|f^t\|^2 \le C_1 e^{C_\psi} \int_{D \cap \{z_1 = -t\}} \left| \frac{f(z')}{t} \right|^2 e^{-\psi(z')} \, d^4 z' \le 4C_1 \|f\|^2$$

with some unimportant constant $C_1 > 0$. $\square$

Hence, so far we have obtained (with some constant $C_* > 0$)

$$F_D(w(t), e_2) \le C_* \frac{E_{\Omega_t}(\tilde{w}(t), (1,0))}{K_{\Omega_t^*}(\tilde{w}(t), \tilde{w}(t))}, \tag{3.9}$$

and everything is reduced to the problem of giving a good upper bound for $E_{\Omega_t}(\tilde{w}(t), (1,0))$ and a suitable lower bound for $K_{\Omega_t^*}(\tilde{w}(t), \tilde{w}(t))$.

Second Step: Estimating the Domain Functionals of the Ω_t and Ω_t^*

We use the fact that Ω_t is a Reinhardt domain in $\mathbb{C}^2$ with center at 0. Therefore, its Bergman kernel can be represented as

$$K_{\Omega_t}(z', w') = \sum_{k,\ell=0}^{\infty} \frac{1}{a_{k\ell}} (z_2 \bar{w}_2)^k (z_3 \bar{w}_3)^\ell, \tag{3.10}$$

where $z' := (z_2, z_3)$ and $a_{k\ell}$ denotes the normalizing factor,

$$a_{k\ell} = \int_{\Omega_t} |\zeta_2^k \zeta_3^\ell|^2 \, d^2\zeta_2 \, d^2\zeta_3.$$

If now $w_3 = 0$ then the maximizing function $f_{(w_2,0),(1,0)}$ defined in (2.4) takes the form

$$\begin{aligned} &f_{(w_2,0),(1,0)}(z) \\ &:= \frac{\sqrt{E_\Omega(w;(1,0))}}{K_\Omega(w,w)^2 B_\Omega^2(w;X)} \left(\frac{\partial^2 K_\Omega}{\partial z_2 \partial \bar{z}_2}((w_2,0),(w_2,0)) \cdot K_\Omega(z',(w_2,0)) \right. \\ &\qquad \left. - \frac{\partial K_\Omega}{\partial z_2}((z_2,0),(w_2,0)) \bigg|_{z_2=w_2} \cdot \frac{K_\Omega}{\partial \bar{w}_2}(z',(w_2,0)) \right). \end{aligned}$$

By virtue of (3.10), only the terms with $\ell = 0$ will contribute to the function and hence it is independent of the variable z_3.

We now choose $w' = \tilde{w}(t)$ and write

$$f_{\tilde{w}(t),(1,0)}(z) = \sum_{k=0}^{\infty} b_k \frac{z_2^k}{\sqrt{a_k}},$$

where $a_k = a_{k0}$ and where $b_k = b_k(t)$ denotes the inner product between $f_{\tilde{w}(t),(1,0)}$ and $\zeta_2^k/\sqrt{a_k}$. By the Cauchy–Schwarz inequality we have

$$|b_k| \le 1$$

for all k. But the auxiliary condition $\frac{\partial f_{\tilde{w},(1,0)}}{\partial z_2}(\tilde{w}(t)) = 0$ requires

$$\frac{b_1}{\sqrt{a_1}} = -\sum_{k=2}^{\infty} k b_k \frac{(\tilde{w}(t))_2^{k-1}}{\sqrt{a_k}}, \tag{3.11}$$

which in turn implies that

$$\begin{aligned} f_{\tilde{w}(t),(1,0)}(\tilde{w}(t)) &= \frac{b_0}{\sqrt{a_0}} - (\tilde{w}(t)_2) \sum_{k=2}^{\infty} k b_k \frac{(\tilde{w}(t))_2^{k-1}}{\sqrt{a_k}} + \sum_{k=2}^{\infty} b_k \frac{(\tilde{w}(t))_2^k}{\sqrt{a_k}} \\ &= \frac{b_0}{\sqrt{a_0}} + \sum_{k=2}^{\infty} (1-k) b_k \frac{(\tilde{w}(t))_2^k}{\sqrt{a_k}}. \end{aligned}$$

Taking absolute values, we find (since $|b_k| \le 1$) that

$$\sqrt{E_{\Omega_t}(\tilde{w}(t),(1,0))} = |f_{\tilde{w}(t),(1,0)}(\tilde{w}(t))| \le \frac{1}{\sqrt{a_0}} + \sum_{k=2}^{\infty} (k+1) \frac{(\tilde{w}(t))_2^k}{\sqrt{a_k}}.$$

In the same way, we treat the Bergman kernel of Ω_t^* at $\tilde{w}(t)$:

$$K_{\Omega_t^*}(\tilde{w}(t),\tilde{w}(t)) = \sum_{k=0}^{\infty} \frac{\tilde{w}(t)_2^{2k}}{a_k^*} \ge \frac{\tilde{w}(t)_2^2}{a_1^*},$$

where

$$a_k^* = \int_{\Omega_t^*} |\zeta_2|^{2k} \, d^2\zeta_2 \, d^2\zeta_3$$

for all $k \ge 0$.

Third Step: Bounds on the Coefficients a_k and a_1^*

In the following lemma we describe the lower bound for the a_k and the suitable upper bound on a_1^* that is needed.

LEMMA 3.2. *For $k \geq 1$,*

$$a_k \geq c_* \frac{1}{k+1} 36^{-2(k+1)} t^{(k+1)/m} t^{2x_3+(2a/b)(x_2-1/2m)}.$$

Moreover,

$$a_0 \geq c_* t^{2x_2+2x_3} \quad \text{and} \quad a_1^* \leq \frac{1}{c_*} t^{2x_3+(2a/b)(x_2-1/2m)+2/m},$$

where c_* denotes some unimportant constant.

Proof. (i) We first carry out the details for the coefficients a_k with $k \geq 1$. Let

$$\phi(y) := \left[\left(\frac{1}{t}\right)^{1/2m} + \left(\frac{y^{2b}}{t}\right)^{1/2a} + \left(\frac{y^{2d}}{t}\right)^{1/2c} \right]^{-1}.$$

Then we have

$$\left\{ z' \;\middle|\; |z_3| < \tfrac{1}{2} t^{1/2m},\ |z_2| < \tfrac{1}{12}\phi(|z_3|) \right\} \subset \Omega_t.$$

Using polar coordinates and then the scaled variable $\eta = t^{-x_3} y$, we obtain

$$\begin{aligned}
a_k &\geq \int_{|z_3|<t^{1/2m}/2} \left(\int_{|z_2|<\phi(|z_3|)/12} |z_2|^{2k}\, d^2 z_2 \right) d^2 z_3 \\
&= 4\pi^2 \int_0^{t^{1/2m}/2} y \left(\int_0^{\phi(y)/12} x^{2k+1}\, dx \right) dy \\
&= \frac{2\pi^2}{k+1} 12^{-2(k+1)} \int_0^{t^{1/2m}/2} y\phi(y)^{2k+2}\, dy \\
&= \frac{2\pi^2}{k+1} 12^{-2(k+1)} t^{2x_3} \int_0^{t^{-(x_3-1/2m)}/2} \eta(\phi(t^{x_3}\eta))^{2k+2}\, d\eta.
\end{aligned}$$

But we observe that

$$\phi(t^{x_3}\eta) = \left[\left(\frac{1}{t}\right)^{1/2m} + \frac{\eta^{b/a}}{t^{x_2}} + \frac{\eta^{d/c}}{t^{x_2}} \right]^{-1} = t^{x_2}\psi(\eta)$$

with

$$\psi(\eta) = \frac{1}{t^{x_2-1/2m} + \eta^{b/a} + \eta^{d/c}}.$$

This gives us

$$\begin{aligned}
a_k &\geq \frac{2\pi^2}{k+1} 12^{-2(k+1)} t^{2(k+1)x_2+2x_3} \int_0^{t^{-(x_3-1/2m)}/2} \eta\psi(\eta)^{2k+2}\, d\eta \\
&\geq \frac{2\pi^2}{k+1} 12^{-2(k+1)} t^{2(k+1)x_2+2x_3} \int_0^1 \eta\psi(\eta)^{2k+2}\, d\eta
\end{aligned}$$

for small enough t. Here we use that $x_3 > 1/2m$.

We split the interval $[0,1]$ into I_1 and I_2, where

$$I_1 = [0, t^{(a/b)(x_2-1/2m)}] \quad \text{and} \quad I_2 = [t^{(a/b)(x_2-1/2m)}, 1].$$

On I_1 we have

$$\psi(\eta) \geq \tfrac{1}{3} t^{-(x_2-1/2m)};$$

hence

$$\begin{aligned}\int_{I_1} \eta\psi(\eta)^{2k+2}\,d\eta &\geq 3^{-2(k+1)} t^{-2(k+1)(x_2-1/2m)} \int_{I_1} \eta\,d\eta \\ &= \frac{1}{2}\cdot 3^{-2(k+1)} t^{(2a/b)(x_2-1/2m)} \cdot t^{-2(k+1)(x_2-1/2m)}.\end{aligned}$$

Thus we obtain the estimate

$$\begin{aligned}a_k &\geq \frac{\pi^2}{k+1} 12^{-2(k+1)} t^{2(k+1)x_2+2x_3} 3^{-2(k+1)} t^{(2a/b)(x_2-1/2m)} \cdot t^{-2(k+1)(x_2-1/2m)} \\ &= \frac{\pi^2}{k+1} 36^{-2(k+1)} t^{(k+1)/m} t^{2x_3+(2a/b)(x_2-1/2m)}. \qquad (3.12)\end{aligned}$$

(ii) For the case $k = 0$, we also use the interval I_2:

$$a_0 \geq \frac{\pi^2}{72} t^{2x_2+2x_3} \int_{t^{(a/b)(x_2-1/2m)}}^{1} \eta\psi(\eta)^2\,d\eta.$$

On this interval we have

$$\psi(\eta) \geq \tfrac{1}{3}\eta^{-b/a}$$

and

$$\begin{aligned}\int_{I_2} \eta\psi(\eta)^2\,d\eta &\geq \frac{1}{9}\int_{t^{(a/b)(x_2-1/2m)}}^{1} \eta^{1-2b/a}\,d\eta \\ &= \frac{1}{18(1-b/a)}\big(1 - t^{2(1-b/a)(a/b)(x_2-1/2m)}\big).\end{aligned}$$

For small enough t, this will give us

$$a_0 \geq c_* t^{2x_2+2x_3}.$$

(iii) We now estimate a_1^* from above in a similar way, starting with

$$\begin{aligned}a_1^* &\leq \int_{|z_3|<t^{1/2m}} \bigg(\int_{|z_2|<\phi(|z_3|)} |z_2|^2\,d^2z_2\bigg)\,d^2z_3 \\ &= 4\pi^2 \int_0^{t^{1/2m}} y \bigg(\int_0^{\phi(y)} x^3\,dx\bigg)\,dy \\ &= \pi^2 t^{2x_3} \int_0^{t^{-(x_3-1/2m)}} \eta(\phi(t^{x_3}\eta))^4\,d\eta \\ &= \pi^2 t^{4x_2+2x_3} \int_0^{t^{-(x_3-1/2m)}} \eta\psi(\eta)^4\,d\eta \\ &= \pi^2 t^{4x_2+2x_3} \bigg(\int_{I_1} \eta\psi(\eta)^4\,d\eta + \int_{I_2} \eta\psi(\eta)^4\,d\eta + \int_{I_3} \eta(\psi(\eta))^4\,d\eta\bigg), \qquad (3.13)\end{aligned}$$

where $I_3 = [1, t^{-(x_3-1/2m)}]$.

Now we estimate from above:

$$\int_{I_1} \eta\psi(\eta)^4 \, d\eta \le t^{-4(x_2-1/2m)} \int_0^{t^{(a/b)(x_2-1/2m)}} \eta \, d\eta$$
$$\le t^{-4(x_2-1/2m)+(2a/b)(x_2-1/2m)} = t^{(2a/b-4)(x_2-1/2m)};$$

$$\int_{I_2} \eta\psi(\eta)^4 \, d\eta \le \int_{t^{(a/b)(x_2-1/2m)}}^1 \eta^{1-4b/a} \, d\eta.$$

Because $2b > a$, we have

$$\int_{I_2} \eta\psi(\eta)^4 \, d\eta \le \frac{1}{2(2b/a-1)} t^{(2a/b-4)(x_2-1/2m)}.$$

Finally,

$$\int_{I_3} \eta\psi(\eta)^4 \, d\eta \le \int_1^{t^{-(x_3-1/2m)}} \eta^{1-4d/c} \, d\eta \le \int_1^\infty \eta^{1-4d/c} \, d\eta = \frac{1}{2(2d/c-1)}.$$

In conjunction with (3.13), this yields

$$a_1^* \le c' t^{4x_2+2x_3+(2a/b-4)(x_2-1/2m)} = c' t^{2x_3+(2a/b)(x_2-1/2m)+2/m},$$

which proves the lemma. □

We now can finish our proof of Theorem 3.1:

$$\begin{aligned}
&\sqrt{E_{\Omega_t}(\tilde{w}(t),(1,0))} \\
&\quad \le \frac{1}{\sqrt{a_0}} + \sum_{k=2}^\infty (k+1) \frac{t^{((1/2m)+\varepsilon)k}}{\sqrt{a_k}} \\
&\quad \le \frac{1}{t^{x_2+x_3}} + \frac{1}{t^{x_3+(a/b)(x_2-1/2m)+1/2m}} \sum_{k=2}^\infty (k+1)^{3/2} (36t^{-1/2m})^k t^{((1/2m)+\varepsilon)k} \\
&\quad = \frac{1}{t^{x_2+x_3}} + \frac{36^2 t^{2\varepsilon}}{t^{x_3+(a/b)(x_2-1/2m)+1/2m}} \sum_{k=2}^\infty (k+1)^{3/2} (36t^\varepsilon)^{k-2} \\
&\quad \le \frac{1}{t^{x_2+x_3}} + c'' \frac{t^{2\varepsilon}}{t^{x_3+(a/b)(x_2-1/2m)+1/2m}} \quad (\text{for } t < 72^{-1/\varepsilon}) \\
&\quad \le c^* \frac{t^{2\varepsilon}}{t^{x_3+(a/b)(x_2-1/2m)+1/2m}},
\end{aligned}$$

with some constants $c'', c^* > 0$ (independent of t). In the second inequality we have used (3.12); and in the next-to-last line, the second member dominates the first one. This follows from the choice of ε.

On the other hand,

$$\sqrt{K_{\Omega_t^*}(\tilde{w}(t),\tilde{w}(t))} \ge \sqrt{c_*} \frac{t^\varepsilon}{t^{x_3+(a/b)(x_2-1/2m)+1/2m}}.$$

Hence we obtain
$$\frac{E_{\Omega_t}(\tilde{w}(t),(1,0))}{K_{\Omega_t^*}(\tilde{w}(t),\tilde{w}(t))} \le \hat{c}t^{2\varepsilon};$$
together with (3.9), this proves the theorem.

References

[B] S. Bergman, *The kernel function and conformal mapping,* 2nd ed., Mathematical Surveys, 5, Amer. Math. Soc., Providence, RI, 1970.

[C] D. W. Catlin, *Estimates of invariant metrics on pseudoconvex domains of dimension two,* Math. Z. 200 (1989), 429–466.

[DFH] K. Diederich, J. E. Fornæss, and G. Herbort, *Boundary behavior of the Bergman metric,* Complex analysis of several variables (Madison, WI, 1982), Proc. Sympos. Pure Math., 41, pp. 59–67, Amer. Math. Soc., Providence, RI, 1984.

[Do1] H. Donnelly, *L^2-cohomology of pseudoconvex domains with complete Kähler metric,* Michigan Math. J. 41 (1994), 433–442.

[Do2] ———, *L^2-cohomology of the Bergman metric for weakly pseudoconvex domains,* Illinois J. Math. 41 (1997), 151–160.

[DoFe] H. Donnelly and Ch. Fefferman, *L^2-cohomology and index theorem for the Bergman metric,* Ann. of Math. (2) 118 (1983), 593–618.

[M1] J. D. McNeal, *Estimates on the Bergman kernels of convex domains,* Adv. Math. 109 (1994), 108–139.

[M2] ———, *L^2 harmonic forms of some complete Kähler manifolds,* Math. Ann. 323 (2002), 319–349.

[O] T. Ohsawa, *On the extension of L^2 holomorphic functions III, Negligible weights,* Math. Z. 219 (1995), 215–225.

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