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Formulas for the Dimensions of Some Affine Deligne–Lusztig Varieties

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1. Introduction

Let F be $\mathbb{F}_q((t))$ with ring of integers $\mathcal{O}_F$, and let G be a split connected reductive group over F. Let L be the completion of the maximal unramified extension of F, $\bar{\mathbb{F}}_q((t))$. Let σ be the Frobenius automorphism of L over F. Let $\mathcal{B}_n$ be the affine building for $G(E)$ where E/F is the unramified extension of degree n in L (so $E = \mathbb{F}_{q^n}((t))$), and let $\mathcal{B}_\infty$ be the affine building for $G(L)$. Let T be a split torus in G, let $B = UT$ be a Borel subgroup, and let I be an Iwahori in $G(L)$ containing $T(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers of L. Let A_M and C_M be the correspondingly specified apartment and alcove, which we assume are in $\mathcal{B}_1$; we will call these the *main apartment* and the *main alcove,* respectively. We assume that C_M is in the positive Weyl chamber in A_M specified by B. Let $P \supseteq I$ be a parahoric subgroup of $G(L)$. If $b \in G(L)$ then the σ-conjugacy class of b is $\{x^{-1}b\sigma(x) : x \in G(L)\}$. Let $\tilde{W} = N(L)/T(\mathcal{O}_L)$ be the extended affine Weyl group, and let $\tilde{W}_P = N(L) \cap P/T(\mathcal{O}_L)$. Here N is the normalizer of T.

If $\tilde{w} \in \tilde{W}$, then we define (after Rapoport [12] and Kottwitz) the *generalized affine Deligne–Lusztig variety* $X^P_{\tilde{w}}(b\sigma) = \{x \in G(L)/P : \mathrm{inv}_P(x, b\sigma(x)) = \tilde{w}\}$. Here $\mathrm{inv}_P : G(L)/P \times G(L)/P \to P\backslash G(L)/P = \tilde{W}_P\backslash\tilde{W}/\tilde{W}_P$ is the relative position map associated to P. Rapoport [12] asked which pairs $(b, \tilde{w})$ give rise to non-empty sets and, for these pairs, what is $\dim(X^P_{\tilde{w}}(b\sigma))$. Kottwitz and Rapoport [9; 12] answered the emptiness/non-emptiness part of this question for $P = K$, the maximal bounded subgroup of $G(L)$ associated to some special vertex v_M of C_M.

In Section 3 we consider the case $G = \mathrm{SL}_3$ with $b = 1$ and $P = I$. Complete results on emptiness/non-emptiness and dimension are shown for this case in Figure 5. In Section 4 we consider $G = \mathrm{Sp}_4$, again with $b = 1$ and $P = I$. Emptiness/non-emptiness results and dimension results are shown in Figure 10. The case $G = \mathrm{SL}_2$ ($b = 1$, $P = I$) can be handled using an even simpler version of the same methods.

Rapoport showed in [12, Prop. 4.2] that, for general G, $X^K_{\tilde{w}}(\sigma)$ is non-empty for any $\tilde{w}$ corresponding to a dominant cocharacter in the coroot lattice. This is also shown in some special cases in [9]. Rapoport [13] conjectured a specific formula for the dimension of the $X^K_{\tilde{w}}(\sigma)$. The knowledge of the $X^I_{\tilde{w}}(\sigma)$ mentioned in the

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previous paragraph gives knowledge of the $X^P_{\tilde{w}}(\sigma)$, so the dimensions of the $X^K_{\tilde{w}}(\sigma)$ are computed in Section 5 for SL_2, SL_3, and Sp_4. The result is that $\dim(X^K_{\tilde{w}}(\sigma)) = \langle \mu, \rho \rangle$, where $\mu \in X_*(T)$ dominant corresponds to $\tilde{w} \in \tilde{W}_K \backslash \tilde{W} / \tilde{W}_K$ and where ρ is half the sum of the positive roots for G. This supports the conjecture of Rapoport in [13]. Preliminary work toward a proof of this conjecture in general has been done with Kottwitz.

In Section 6 we present a formula that encapsulates part of the results pictured in Figures 5 and 10. The formula also holds for SL_2. It is too soon to conjecture that this formula holds for general G. Some results on emptiness/non-emptiness for $b \neq 1$ when $G = \mathrm{SL}_2$, SL_3, or Sp_4 can be found in [14]. Section 7 contains a summary of these results.

This work has significance for the study of the reduction modulo p of Shimura varieties. Interested readers should see the survey article by Rapoport [13]. Other non-emptiness results for affine Deligne–Lusztig varieties can be found in [15].

2. General Methodology

For this and the next two sections we let $P = I$, so $\tilde{w} \in \tilde{W}_P \backslash \tilde{W} / \tilde{W}_P = \tilde{W}$, and we let $X_{\tilde{w}}(\sigma) = X^I_{\tilde{w}}(\sigma)$. In this section we assume the group G to be simply connected, so that I is the stabilizer of C_M. First note that, if $\tilde{w}C_M \cap C_M$ is non-empty, then $X_{\tilde{w}}(\sigma)$ can be identified with a disjoint union of (non-affine) Deligne–Lusztig varieties whose structure and dimension are already known [3]. Let v_1 be a vertex in A_M and let v_2 be a vertex in $\mathcal{B}_1$ in the same $G(F)$ orbit as v_1. We require that $v_1 \notin C_M$. Let Q_1 be the last alcove in a minimal gallery from C_M to v_1, and let $\mathcal{Q}_2$ be the set of all alcoves Q_2 containing v_2 such that Q_2 and $\sigma(Q_2)$ have some fixed relative position, p_r. Note that Q_1 does not depend on the choice of minimal gallery from C_M to v_1. We require that p_r be such that $Q_2 \cap \mathcal{B}_1 = \{v_2\}$.

DEFINITION 2.1. The (v_1, v_2, p_r)*-piece of* $X_{\tilde{w}}(\sigma)$ (which may be empty) is the set of all alcoves $D \subset \mathcal{B}_\infty$ such that there exists a $y \in G(L)$ with $yC_M = D$, $yQ_1 = Q_2$ for some $Q_2 \in \mathcal{Q}_2$ (so $yv_1 = v_2$), and $\mathrm{inv}(D, \sigma(D)) = \tilde{w}$.

DEFINITION 2.2. The (v_1, v_2, p_r)*-superpiece* is the collection of all alcoves $D \subset \mathcal{B}_\infty$ such that there exists a $y \in G(L)$ with $yC_M = D$ and $yQ_1 = Q_2$ for some $Q_2 \in \mathcal{Q}_2$.

In other words, the (v_1, v_2, p_r)-superpiece is the disjoint union, over all $\tilde{w} \in \tilde{W}$, of the (v_1, v_2, p_r)-pieces of the $X_{\tilde{w}}(\sigma)$ (many of which will be empty).

LEMMA 2.1. *For $\tilde{w}$ with $\tilde{w}C_M \cap C_M = \emptyset$, every alcove of $X_{\tilde{w}}(\sigma)$ is in the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$ for some (v_1, v_2, p_r).*

Proof. Let $D \in X_{\tilde{w}}(\sigma)$. Consider the set of all galleries which start at D and which have an alcove containing some vertex in $\mathcal{B}_1$. Let Γ be a minimal length element of this set. Let the alcoves of Γ be $D = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$, where $\Gamma_n \cap \mathcal{B}_1 \neq \emptyset$. We have $\Gamma_i \cap \mathcal{B}_1 = \emptyset$ for any $i < n$ by minimality. If $n > 1$, then Γ_n must contain only one vertex in $\mathcal{B}_1$, since Γ_{n-1} and Γ_n share all but one vertex. Let $\Gamma_n \cap \mathcal{B}_1 = \{v_2\}$

and let $v_1 = \rho_{C_M}(y^{-1}v_2)$, where $y \in G(L)$ is such that $yC_M = D$ and ρ_{C_M} is the retraction of $\mathcal{B}_\infty$ onto A_M centered at C_M. Since $\Gamma_n \cap \mathcal{B}_1 = \{v_2\}$, it follows that $p_r = \mathrm{inv}(\Gamma_n, \sigma(\Gamma_n))$ is one of the allowed choices.

In the case that $n = 1$, we have $D \cap \mathcal{B}_1 \neq \emptyset$ and so $D \in X_{\tilde{w}}(\sigma)$ for some $\tilde{w}$ with $\tilde{w}C_M \cap C_M \neq \emptyset$. □

The approach outlined in this section so far was suggested by Kottwitz and is similar to that used in [6].

Recall that the set G/I can be given variety structure by writing it as an increasing union of sets $(G/I)_m$ $(m = 1, 2, 3, \dots)$ and that each of these sets has an $\tilde{m}$ such that any alcove $D \in (G/I)_m$ has $d(D, C_M) \leq \tilde{m}$, where $d(\cdot,\cdot)$ is the metric on $\mathcal{B}_\infty$. The sets $(G/K)_m$ are defined in [5] and [10], and the $(G/I)_m$ are defined similarly. It is also true that if P is the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$ then $P \subset (G/I)_m$ for large enough m. The sets $X_{\tilde{w}}(\sigma) \cap (G/I)_m$ and $\bar{P} \cap (G/I)_m$ are locally closed subsets of $(G/I)_m$; here $\bar{P}$ is the (v_1, v_2, p_r)-superpiece. Therefore, $\bar{P} \cap X_{\tilde{w}}(\sigma) \cap (G/I)_m = P \cap (G/I)_m$ is a locally closed subset of $(G/I)_m$. Using this together with Lemma 2.2, we can write $\dim X_{\tilde{w}}(\sigma) = \sup_m \dim X_{\tilde{w}}(\sigma) \cap (G/I)_m = \sup_{P,m} \dim P \cap (G/I)_m = \sup_P \dim P$, allowing us to compute the dimensions of $X_{\tilde{w}}(\sigma)$ from the dimensions of the pieces P.

LEMMA 2.2. *$P \cap (G/I)_m \neq \emptyset$ for only finitely many pieces P of $X_{\tilde{w}}(\sigma)$.*

In order to prove Lemma 2.2, we need the following.

DEFINITION 2.3. Let A be an apartment in $\mathcal{B}_\infty$, D an alcove in A, and v a vertex in D. Define the *barely neighboring alcoves of D in A through v* to be all alcoves $E \subset A$ such that $E \cap D = \{v\}$. Let the *barely neighboring cone of D in A through v* be all points $a \in A$ for which the geodesic from v to a passes through the barely neighboring alcoves of D in A through v.

LEMMA 2.3. *Let D_1, D_2 be two alcoves in $\mathcal{B}_\infty$, let v be a vertex in $\mathcal{B}_\infty$, and let P_i be the intersection of all apartments containing D_i and v. Let E_i be the alcove in P_i containing v (there is only one such). Assume $E_1 \cap E_2 = \{v\}$. Then there exists a positive constant l (depending only on the group G) such that $d(D_1, D_2) \geq l d(D_i, v)$ for either i.*

Proof. By symmetry, it suffices to find $l > 0$ such that $d(D_1, D_2) \geq l d(D_1, v)$. Let $x_i \in D_i$ and let $z_t = tv + (1-t)x_2$. By the negative curvature inequality, $(1-t)d^2(x_2, x_1) \geq d^2(z_t, x_1) - td^2(v, x_1) + t(1-t)d^2(x_2, v)$ [4, p. 225]. Choose $t \neq 1$ such that $z_t \in E_2$. Now let A be an apartment containing D_1 and E_2. We have $v \in A$, so $E_1 \subset A$. Let $\tilde{D}_2 \subset A$ be an alcove such that $\mathrm{inv}(\tilde{D}_2, E_2) = \mathrm{inv}(D_2, E_2)$, and let $\tilde{x}_2 \in \tilde{D}_2$ be such that $\mathrm{inv}(\tilde{x}_2, E_2) = \mathrm{inv}(x_2, E_2)$. Since $\tilde{x}_2$, v, and x_1 are in an apartment, it follows that

$$(1-\tilde{t})d^2(\tilde{x}_2, x_1) = d^2(\tilde{z}_{\tilde{t}}, x_1) - \tilde{t}d^2(v, x_1) + \tilde{t}(1-\tilde{t})d^2(\tilde{x}_2, v),$$

where $\tilde{z}_{\tilde{t}} = \tilde{t}v + (1-\tilde{t})\tilde{x}_2$ [4, p. 226]. Choose $t = \tilde{t}$. Since $z_t \in E_2 \subset A$, $\tilde{z}_{\tilde{t}} = z_t$. Since $d^2(\tilde{x}_2, v) = d^2(x_2, v)$, we have $d(x_2, x_1) \geq d(\tilde{x}_2, x_1)$. Let $\mathcal{C}$ be the barely neighboring cone of E_1 in A through v. We know $\tilde{x}_2 \in \mathcal{C}$, so $d(x_2, x_1) \geq d(\mathcal{C}, x_1)$.

Let $B_\alpha = \{x \in A : d(x, v) = \alpha\}$. Let W_1 be the set of all $x \in A$ such that the geodesic from x to v passes through E_1, and let $\bar{W}_1$ be the closure. Define $\varphi_t : A \to A$ by $\varphi_t(x) = tx + (1-t)v_1$. Then the invertible map $\varphi_t \times \varphi_t : B_\alpha \cap \bar{W}_1 \times \bar{\mathcal{C}} \to B_{t\alpha} \cap \bar{W}_1 \times \bar{\mathcal{C}}$ and the relationship $td(x, y) = d(\varphi_t(x), \varphi_t(y))$ (for $x, y \in A$) together imply that $tm_\alpha = m_{t\alpha}$, where m_α is the minimum of d on $B_\alpha \cap \bar{W}_1 \times \bar{\mathcal{C}}$. We now have $d(x_2, x_1) \geq d(\mathcal{C}, x_1) \geq d(x_1, v)m_1$, which implies the desired result because $m_1 > 0$. □

Proof of Lemma 2.2. Let D_2 be an alcove in P, the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$. We now apply Lemma 2.3 with $D_1 = C_M$ and $v = v_2$. Since the elements of $(G/I)_m$ are all within a fixed distance of C_M, we must have v_1 and v_2 within a fixed distance of C_M. There are only finitely many such v_1 and v_2. □

Note that the structure of the (v_1, v_2, p_r)-superpiece does not depend on v_2, provided v_2 is some vertex in $\mathcal{B}_1$ in the same $G(F)$ orbit as v_1. Hence, for each (v_1, p_r)-pair, we fix an arbitrary vertex $v_2 \in A_M$ in the same $G(F)$-orbit as v_1 and then compute the possible values of $\operatorname{inv}(D, \sigma(D))$ for D in the (v_1, v_2, p_r)-superpiece. We will discuss how this computation is carried out for SL_3 and Sp_4. The results tell us for which $\tilde{w}$ the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$ is non-empty. We will also demonstrate a way of calculating the dimension of each non-empty piece in the (v_1, v_2, p_r)-superpiece, again only for SL_3 and Sp_4. (Everything we will do also applies to SL_2, however.) Aggregating all this information over all (v_1, p_r)-pairs will tell us, for each piece of each $X_{\tilde{w}}(\sigma)$, whether it is empty or non-empty and what its dimension is. This gives the emptiness/non-emptiness and dimension of the $X_{\tilde{w}}(\sigma)$ themselves, by the previous results.

We now give some more definitions and propositions that will be needed to develop the ideas of the previous paragraph. Let Γ_{v_1} be the standard minimal gallery from C_M to Q_1, as defined in [14] for SL_3 and Sp_4 (the definition could be generalized to G). Let $\Gamma^f_{(v_1,p_r)}$ be $z\Gamma_{v_1}$, where Q_1 and zQ_1 have relative position p_r, $z \in G(L)$, and z sends A_M to A_M. Let $\Gamma^c_{(v_1,p_r)}$ be some fixed minimal connecting gallery from Q_1 to zQ_1. Define $\bar{\Gamma}_{(v_1,p_r)} = \Gamma_{v_1} \cup \Gamma^c_{(v_1,p_r)} \cup \Gamma^f_{(v_1,p_r)}$.

Let Ω be a gallery in A_M starting at C_M and containing any alcove at most once (so it is non-stuttering, non-backtracking, and does not cross itself). Let $\Omega_1, \Omega_2, \ldots, \Omega_n$ be the alcoves of Ω in order (so $\Omega_1 = C_M$), and let e_i be the edge between Ω_i and Ω_{i+1}.

DEFINITION 2.4. Let j be minimal such that C_M and Ω_j are on opposite sides of the hyperplane h_j in A_M determined by e_j. We say that e_j is the *first choice edge* in Ω.

If j does not exist then there are no choice edges in Ω. If j does exist, then this leads to our next definition.

DEFINITION 2.5. The *hard choice at e_j* is the gallery $\Omega_1, \ldots, \Omega_j, \Omega_{j+1}$, and the *easy choice at e_j* is the gallery $\Omega_1, \ldots, \Omega_j$, $f_{h_j}(\Omega_{j+1}) = \Omega_j$, where f_{h_j} represents the flip of A_M about h_j.

Given the hard choice, we consider $\Omega_1, \ldots, \Omega_j, \Omega_{j+1}, \ldots, \Omega_n$ and find the minimal $k > j$ such that h_k has Ω_k and C_M on opposite sides. This is the next choice edge, given the hard choice at j, and we can make either an easy choice or a hard choice here. Given the easy choice at j we consider $\Omega_1, \ldots, \Omega_j, f_{h_j}(\Omega_{j+1}), \ldots, f_{h_j}(\Omega_n)$, and we find the minimal k such that $k > j$ and such that $f_{h_j}(\Omega_k)$ and C_M are on opposite sides of the hyperplane between $f_{h_j}(\Omega_k)$ and $f_{h_j}(\Omega_{k+1})$. This is the next choice edge, given the easy choice at j, and we can make either a hard or an easy choice here. In this way we construct a binary tree T.

Definition 2.6. T is called the *choice tree* for Ω. Each node in T (except the leaves) corresponds to a choice edge in Ω. At every node except the leaves, T has a branch corresponding to a hard choice and another branch corresponding to an easy choice.

One can show that any non-backtracking path from the root node to a leaf of T corresponds to the retraction (onto A_M centered at C_M) of some gallery (or galleries) starting at C_M and of the same type at Ω. Such a path is equivalent to the choice of a leaf of T, since T is a tree. The gallery Ω itself corresponds to the path obtained by making all hard choices in T. Further, all galleries starting at C_M of the same type as Ω retract in a way specified by some non-backtracking path from the root node of T to a leaf.

Definition 2.7. The *set of comprehensive folding results of* Ω is the set of final alcoves of retractions of galleries starting at C_M that have the same type as Ω.

By *retraction* we always mean the retraction centered at C_M onto A_M. So a comprehensive folding result of Ω can also be thought of as a non-backtracking path F from the root node to a leaf of the choice tree of Ω, or (equivalently) as a leaf of T.

Let $\Omega = \bar{\Gamma}_{(v_1, p_r)}$.

Definition 2.8. The *cf-dimension of* F is $l(\Gamma_{v_1}) + l(\Gamma^c_{(v_1, p_r)}) - n_F - 2$, where n_F is the number of hard choices in F and l represents the length of a gallery (the number of alcoves in it).

3. SL_3

In order to carry out the process outlined in the first half of Section 2 for SL_3, it suffices to consider v_1 in the region pictured in Figure 1. All other v_1 can be obtained from these by rotating by 120° or 240° about the center of C_M. Furthermore, if v_1' is the rotation of v_1 by α (= 120° or 240°) about the center of C_M then it is easy to see that the set $\{\tilde{w} \in \tilde{W}$: the (v_1', v_2', p_r)-piece of $X_{\tilde{w}}(\sigma)$ is non-empty$\}$ is the rotation of the set $\{\tilde{w} \in \tilde{W}$: the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$ is non-empty$\}$ by α about the center of C_M. Further, the correspondence between these two sets given by rotation by α preserves the dimension of the corresponding pieces.

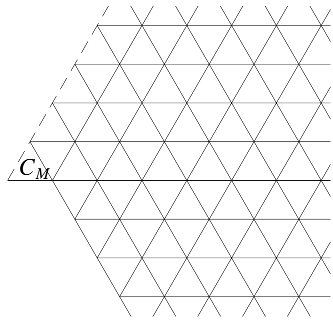

Figure 1 The region containing all vertices v_1 that must be considered for SL_3

Given the restriction (mentioned in Section 2) that p_r be such that $Q_2 \cap \mathcal{B}_1 = \{v_2\}$, we know that Q_2 and $\sigma(Q_2)$ must share exactly one vertex. Thus for SL_3, p_r corresponds to some element of W (the finite Weyl group) of length 2 or 3.

The galleries $\bar{\Gamma}_{(v_1,p_r)}$ for SL_3 can have the general shapes pictured in Figure 2. For clarity, only two of the galleries in this figure have all their parts labeled.

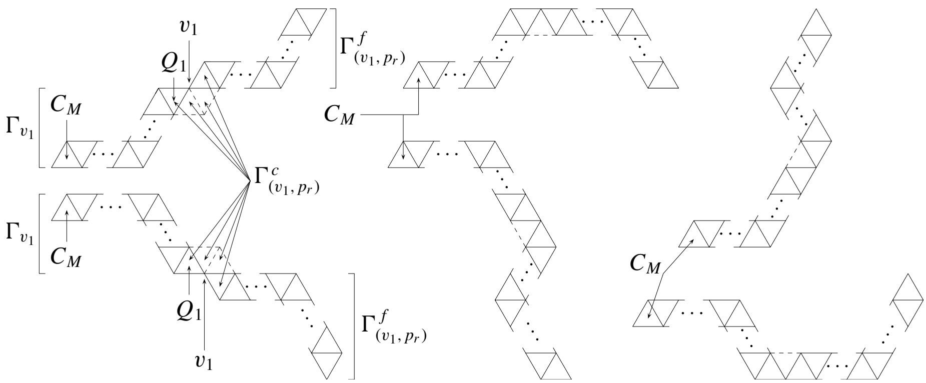

Figure 2 General shapes of the $\bar{\Gamma}_{(v_1,p_r)}$ for SL_3

We now observe that the set of comprehensive folding results of $\Omega = \bar{\Gamma}_{(v_1,p_r)}$ contains the set of possible $\mathrm{inv}(D, \sigma(D))$ for D in the (v_1, v_2, p_r)-superpiece.

PROPOSITION 3.1. *The set of comprehensive folding results of* Ω *coincides with the set of* $\mathrm{inv}(D, \sigma(D))$ *for* D *in the* (v_1, v_2, p_r)*-superpiece. The cf-dimension of* F *is equal to the dimension of the* (v_1, v_2, p_r)*-piece of* $X_{\tilde{w}}(\sigma)$, *where* $\tilde{w}C_M$ *is the comprehensive folding result of* Ω *corresponding to* F.

Proof. We first note that one can make at most one easy choice for each $\Omega = \bar{\Gamma}_{(v_1,p_r)}$ in Figure 2. Once this choice is made, there are no subsequent choice edges. This can be seen simply by analyzing the pictures in Figure 2 on a case-by-case basis. Also, one can see that $\Gamma_{v_1} \cup \Gamma^c_{(v_1,p_r)}$ is minimal. So the first choice edge in Ω occurs between two of the alcoves of $\Gamma^f_{(v_1,p_r)}$. Using these facts, one can show that choice edges in Ω correspond to hyperplanes in A_M that pass between two alcoves of $\Gamma^f_{(v_1,p_r)}$ and that also pass between two alcoves of Γ_{v_1}.

Given a non-backtracking path F in T from the root node to a leaf, we need to produce some gallery Λ such that (a) $y\Gamma_{v_1} = \Lambda$ for some $y \in \mathrm{SL}_3(L)$ with $yQ_1 = Q_2$ for some $Q_2 \in \mathcal{Q}_2$ and (b) $\rho_{C_M}(y^{-1}(\Lambda \cup \Lambda^c \cup \sigma(\Lambda)))$ gives the comprehensive folding result determined by F. Here Λ^c is a minimal gallery from Q_2 to $\sigma(Q_2)$ that has the same type as $\Gamma^c_{(v_1,p_r)}$, and ρ_{C_M} is the retraction onto A_M centered at C_M.

Note first that F determines the relative position of any two alcoves in $\bar{\Lambda} = \Lambda \cup \Lambda^c \cup \sigma(\Lambda)$. In our SL_3 case, F is just an indication of the choice edge at which to make the easy choice, if any (since there is at most one easy choice). We will construct Λ starting from Λ_n, the alcove that contains v_2. We choose $\Lambda_n = Q_2$. The dimension of the set of choices for this construction is $l(\Gamma^c_{(v_1,p_r)}) - 1$, since the structure of (non-affine) Deligne–Lusztig varieties is known [3]. We assume by induction that we have constructed $\Lambda_i, \Lambda_{i+1}, \dots, \Lambda_n$ (and therefore also $\sigma(\Lambda_n), \sigma(\Lambda_{n+1}), \dots, \sigma(\Lambda_i)$) such that the relative position of any two of these $2(n-i+1)$ alcoves is that given by F and such that the dimension of the space of possible such constructions is $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)}$, where $n_{(F,i)}$ is defined as follows. Each choice edge e in Ω has two corresponding integers $1 \le \beta_1, \beta_2 \le n-1$ such that the hyperplane h_e corresponding to e passes between the (β_1)th and (β_1+1)th alcoves of Γ_{v_1} (where the first alcove of Γ_{v_1} is considered to be C_M) and between the (β_2)th and (β_2+1)th alcoves of $\Gamma^f_{(v_1,p_r)}$ (where the nth alcove of $\Gamma^f_{(v_1,p_r)}$ is considered to be the one containing v_1). We define $n_{(F,i)}$ to be the number of choice edges e such that $i \le \beta_1, \beta_2$ and such that F indicates a hard choice at e. Note that

$$l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)} = \begin{cases} l(\Gamma_{v_1}) + l(\Gamma^c_{(v_1,p_r)}) - n_F - 2 & \text{if } i = 1, \\ l(\Gamma^c_{(v_i,p_r)}) - 1 & \text{if } i = n. \end{cases} \qquad (*)$$

We now want to find Λ_{i-1} such that the relative positions of any two of the alcoves $\Lambda_{i-1}, \dots, \Lambda_n, \sigma(\Lambda_n), \dots, \sigma(\Lambda_{i-1})$ is that specified by F. We seek the dimension of the set of such Λ_{i-1}. Let A be some apartment containing Λ_i and $\sigma(\Lambda_i)$, and let $S \subset A$ be the intersection of all apartments that contain Λ_i and $\sigma(\Lambda_i)$. Let d_{i-1} be the edge of Λ_i to which Λ_{i-1} must be attached (this is specified by the requirement that $\bar{\Lambda}$ and Ω be of the same type). Let $\widetilde{\Lambda_{i-1}}$ be the alcove in A obtained by reflecting Λ_i about d_{i-1}, and let $\widetilde{\sigma(\Lambda_{i-1})}$ be the alcove in A obtained by reflecting $\sigma(\Lambda_i)$ about $\sigma(d_{i-1})$. One can see by considering each of the cases pictured in Figure 2 that either exactly one of $\widetilde{\Lambda_{i-1}}$ and $\widetilde{\sigma(\Lambda_{i-1})}$ is in S or neither is in S. Note that the former occurs if and only if $i-1 = \min(\beta_1, \beta_2)$ for β_1, β_2 the two integers corresponding to some choice edge in F.

Let S_{i-1} be the intersection of all apartments containing $S \cup \widetilde{\Lambda_{i-1}}$, and let S^σ_{i-1} be the intersection of all apartments containing $S \cup \widetilde{\sigma(\Lambda_{i-1})}$. One can see by considering the cases in Figure 2 that, if neither $\widetilde{\Lambda_{i-1}}$ nor $\widetilde{\sigma(\Lambda_{i-1})}$ is in S, then $\widetilde{\Lambda_{i-1}}$ is not in S^σ_{i-1} and $\widetilde{\sigma(\Lambda_{i-1})}$ is not in S_{i-1}. Therefore, in this case we can choose any Λ_{i-1} adjacent to Λ_i by d_{i-1}. This in turn determines $\sigma(\Lambda_{i-1})$ adjacent to $\sigma(\Lambda_i)$ by $\sigma(d_{i-1})$, and in such a way that the desired relative positions of all pairs of $\Lambda_{i-1}, \ldots, \Lambda_n, \sigma(\Lambda_n), \ldots, \sigma(\Lambda_{i-1})$ occur. There is one dimension worth of these choices, so the dimension of the construction down to $i-1$ is $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)} + 1$. In this case $n_{(F,i-1)} = n_{(F,i)}$, so $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)} + 1 = l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i-1)}$.

We now consider the case in which exactly one of $\widetilde{\Lambda_{i-1}}$ and $\sigma(\widetilde{\Lambda_{i-1}})$ is in S. We assume that $\widetilde{\Lambda_{i-1}}$ is in S; the other case is similar. This means that $i-1 = \min(\beta_1, \beta_2)$ for β_1, β_2 the two integers corresponding to some choice edge e. If F dictates a hard choice at this point, then to ensure the proper relative position of $\Lambda_{i-1}, \ldots, \Lambda_n, \sigma(\Lambda_n), \ldots, \sigma(\Lambda_{i-1})$ we must choose $\Lambda_{i-1} \subset S$. There is only one such choice, causing no increase in the dimension of the construction. If F dictates an easy choice, we may choose any Λ_{i-1} not in A but attached to Λ_i via d_{i-1}. There is one dimension worth of such choices, increasing dimension by one. In the former case, $n_{(F,i-1)} = n_{(F,i)} + 1$; in the latter, $n_{(F,i+1)} = n_{(F,i)}$. In both cases, the dimension of the new structure is $l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i-1)}$. This finishes the proof of the proposition. □

The result of all this is that we can calculate the values of $\mathrm{inv}(D, \sigma(D))$ for D in the (v_1, v_2, p_r)-superpiece, and for each $\tilde{w}$ in this set we can calculate the dimension of the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$. This can all be done through straightforward computation of comprehensive folding results and cf-dimensions. For instance, using v_1 and p_r leading to the $\bar{\Gamma}_{(v_1,p_r)}$ pictured in Figure 3, we derive the results pictured in Figure 4. The numbers in Figure 4 are the dimensions of the (v_1, v_2, p_r)-pieces of the $X_{\tilde{w}}(\sigma)$, with $\tilde{w}$ corresponding to the alcoves on which

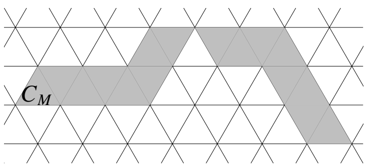

Figure 3 An example of $\bar{\Gamma}_{(v_1,p_r)}$

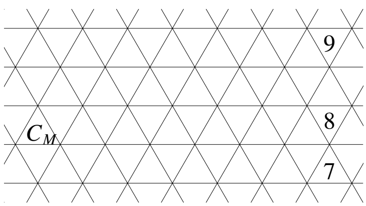

Figure 4 Comprehensive folding results and cf-dimensions from the example in Figure 3

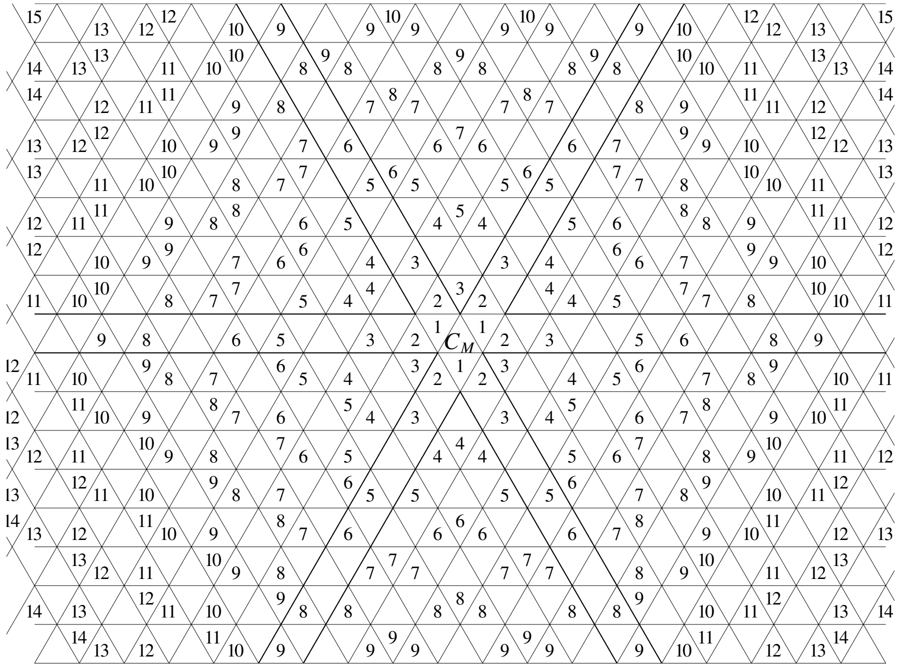

Figure 5 Main result in diagram form for SL_3

the numbers are written. Alcoves with no numbers have empty (v_1, v_2, p_r)-pieces. We did an analogous computation for every v_1 in the region shown in Figure 1 and for every p_r for which the corresponding $w \in W$ has $l(w) \geq 2$. We rotated all results about the center of C_M by 120° and 240°, combining these with the unrotated results. For any alcove that contained more than one number at that point, we took the maximum (although in all cases for which two numbers occurred in the same alcove, these numbers turned out to be equal). The outcome of this process is Figure 5, which shows the $X_{\tilde{w}}(\sigma)$ that are non-empty (those corresponding to alcoves that have numbers in them) as well as the dimensions of those non-empty $X_{\tilde{w}}(\sigma)$. The bold lines in that figure correspond to the shrunken Weyl chambers (to be discussed in Section 6).

Something observed in the course of the computation is that two different numbers never occurred in the same alcove. This means that, for any fixed $\tilde{w}$, the non-empty pieces of $X_{\tilde{w}}(\sigma)$ all have the same dimension. As we will see in the next section, this may be related to the fact that all vertices in the building for SL_3 are special.

4. Sp_4

For Sp_4, it suffices to consider v_1 in the region pictured in Figure 6. All other v_1 can be obtained from these by reflecting about the line of symmetry of C_M. Once

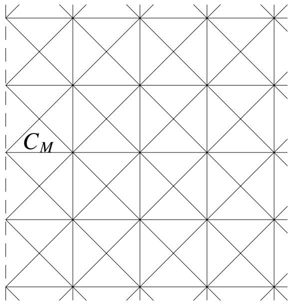

Figure 6 The region containing all vertices v_1 that must be considered for Sp_4

results are obtained for v_1 in the region specified, we will have to reflect the results across the line of symmetry of C_M as well. Note that v_1 can be special or non-special for Sp_4, whereas only the special case was possible for SL_3.

Given the restriction that p_r be such that $Q_2 \cap \mathcal{B}_1 = \{v_2\}$, it follows that Q_2 and $\sigma(Q_2)$ must share exactly one vertex. Therefore, p_r corresponds to some element of W of length 2, 3, or 4 for v_1 special and to some element of length 2 for v_1 non-special.

The galleries $\bar{\Gamma}_{(v_1,p_r)}$ have the general shapes pictured in Figure 7 for the case in which v_1 is non-special. For clarity, only two of the galleries appearing in Figure 7 have all of their parts labeled. Figure 8 contains general shapes of the Γ_{v_1} for v_1 special. The twenty different general shapes of the $\bar{\Gamma}_{(v_1,p_r)}$ can be deduced from these four possible Γ_{v_1} by determining $\Gamma^f_{(v_1,p_r)}$ and $\Gamma^c_{(v_1,p_r)}$ from each Γ_{v_1} using each of the five possible p_r.

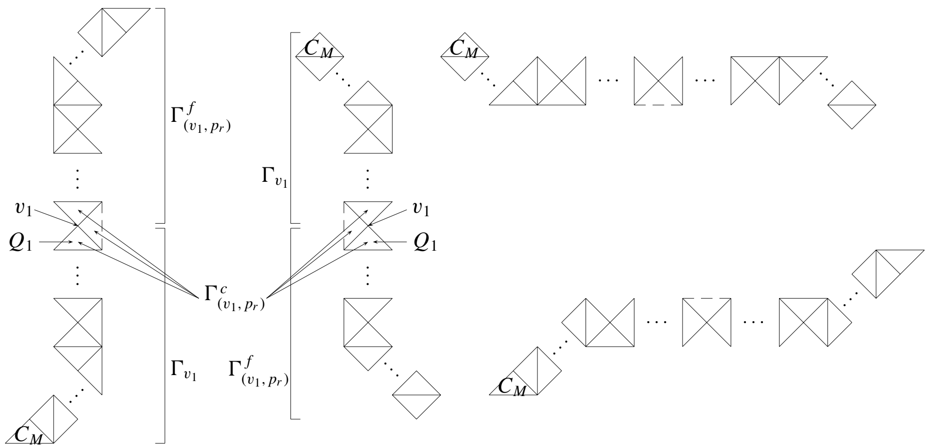

Figure 7 General shapes of the $\bar{\Gamma}_{(v_1,p_r)}$ for Sp_4, non-special v_1

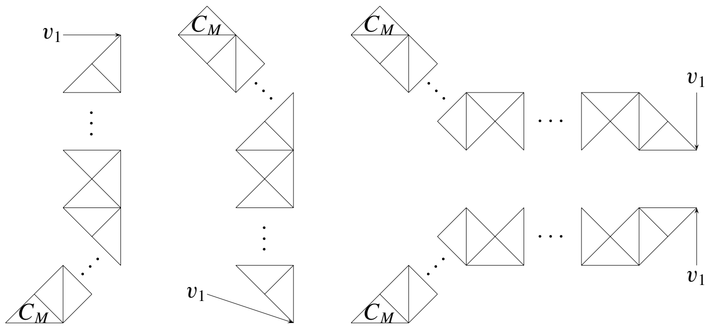

Figure 8 General shapes of the Γ_{v_1} for Sp_4, special v_1

The set of comprehensive folding results of $\Omega = \bar{\Gamma}_{(v_1,p_r)}$ contains the set of possible $\mathrm{inv}(D,\sigma(D))$ for D in the (v_1,v_2,p_r)-superpiece.

PROPOSITION 4.1. *The set of comprehensive folding results of Ω coincides with the set of* $\mathrm{inv}(D,\sigma(D))$ *for D in the (v_1,v_2,p_r)-superpiece. The cf-dimension of a leaf, F, of the tree T of Ω is equal to the dimension of the (v_1,v_2,p_r)-piece of $X_{\tilde{w}}(\sigma)$, where $\tilde{w}C_M$ is the comprehensive folding result of Ω corresponding to F.*

Proof. We first note that F can contain at most two easy choices. In fact, the maximum number of easy choices that F can contain is $-m+4$, where m is the length of p_r in W. This result is obtained by considering cases. For SL_3, the maximum number of easy choices is $-m+3$. As in the SL_3 case, for Sp_4 we have that $\Gamma_{v_1} \cup \Gamma^c_{(v_1,p_r)}$ is minimal.

DEFINITION 4.1. A *non-primal choice edge* is a non-leaf node in T that occurs below some easy choice in T (i.e., the non-backtracking path from the root node to the node in question passes through an edge in T corresponding to an easy choice).

DEFINITION 4.2. A *primal choice edge* is any choice edge that is not non-primal.

All choice edges for SL_3 are primal. For SL_3 and Sp_4, all primal choice edges in $\Omega = \bar{\Gamma}_{(v_1,p_r)}$ correspond to hyperplanes in A_M that pass between two alcoves of $\Gamma^f_{(v_1,p_r)}$ and that also pass between two alcoves of Γ_{v_1}.

Given a primal choice edge in F, we define the two corresponding integers $1 \leq \beta_1,\beta_2 \leq n-1$ as in the SL_3 case. We will also define $1 \leq \beta_1,\beta_2 \leq n-1$ for a non-primal choice edge e, but in a slightly different way. Since e is non-primal, there is some choice edge d above e in F at which F makes the easy choice. Let h_d be the hyperplane in A_M determined by the edge d in Ω (so h_d is a hyperplane separating two alcoves of $\Gamma^f_{(v_1,p_r)}$ and also two alcoves of Γ_{v_1}). Let f_{h_d} be the flip in A_M about h_d. Consider the gallery in A_M obtained by applying f_{h_d} to the alcoves in Ω that occur after d (here C_M is considered to be the first alcove of Ω). Let $\tilde{e} = f_{h_d}(e)$, and let $h_{\tilde{e}}$ be the hyperplane in A_M determined by $\tilde{e}$. Let β_2 be such that h_e passes between the (β_2)th and (β_2+1)th alcoves of $\Gamma^f_{(v_1,p_r)}$ (here the nth alcove of $\Gamma^f_{(v_1,p_r)}$ is considered to be the one containing v_1). If $h_{\tilde{e}}$ passes between two alcoves of Γ_{v_1}, then let β_1 be such that $h_{\tilde{e}}$ passes between the (β_1)th and (β_1+1)th alcoves of Γ_{v_1} (here C_M is considered to be the first alcove of Γ_{v_1}). Otherwise let $\beta_1 = n$.

Now, given a non-backtracking path F from the root node of T to a leaf, we want to produce a gallery Λ such that (a) $y\Gamma_{v_1} = \Lambda$ for some $y \in \mathrm{Sp}_4(L)$ with $yQ_1 = Q_2$ for some $Q_2 \in \mathcal{Q}_2$ and (b) $\rho_{C_M}(y^{-1}(\Lambda \cup \Lambda^c \cup \sigma(\Lambda)))$ gives the comprehensive folding result determined by F. Here, as before, Λ^c is a minimal gallery from Q_2 to $\sigma(Q_2)$ that has the same type as $\Gamma^c_{(v_1,p_r)}$.

We choose $\Lambda_n = Q_2$. The dimension of the set of such choices is $l(\Gamma^c_{(v_1,p_r)}) - 1$ [3]. We assume by induction that we have constructed $\Lambda_i, \dots, \Lambda_n$ and $\sigma(\Lambda_n), \dots, \sigma(\Lambda_i)$ such that the relative position of any two of these alcoves is that given by F and such that the dimension of the space of choices for this construction is $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)}$, where $n_{(F,i)}$ is defined to be the number of choice edges e in F such that $i \le \beta_1, \beta_2$ and such that F indicates a hard choice at e. Here β_1, β_2 are the integers corresponding to e, defined in the new way. Equation $(*)$ of Section 3 holds under the new definitions as well.

We now want to construct Λ_{i-1}. As for SL_3, we let A be some apartment containing Λ_i and $\sigma(\Lambda_i)$. Let $S \subset A$ be the intersection of all apartments that contain both Λ_i and $\sigma(\Lambda_i)$. Let d_{i-1} be the edge of Λ_i to which Λ_{i-1} must be attached. Let $\widetilde{\Lambda_{i-1}}$ be the alcove in A obtained by reflecting Λ_i about d_{i-1}, and let $\widetilde{\sigma(\Lambda_{i-1})}$ be the alcove in A obtained by reflecting $\sigma(\Lambda_i)$ about $\sigma(d_{i-1})$. Let S_{i-1} be the intersection of all apartments containing $S \cup \widetilde{\Lambda_{i-1}}$ and let S^σ_{i-1} be the intersection of all apartments containing $S \cup \widetilde{\sigma(\Lambda_{i-1})}$.

One can see by considering cases that either

(1) $\widetilde{\Lambda_{i-1}}, \widetilde{\sigma(\Lambda_{i-1})} \not\subset S$, $\widetilde{\Lambda_{i-1}} \not\subset S^\sigma_{i-1}$, and $\widetilde{\sigma(\Lambda_{i-1})} \not\subset S_{i-1}$, or
(2) $\widetilde{\Lambda_{i-1}}, \widetilde{\sigma(\Lambda_{i-1})} \not\subset S$, $\widetilde{\Lambda_{i-1}} \subset S^\sigma_{i-1}$, and $\widetilde{\sigma(\Lambda_{i-1})} \subset S_{i-1}$, or
(3) $\widetilde{\Lambda_{i-1}} \subset S$ and $\widetilde{\sigma(\Lambda_{i-1})} \not\subset S$, or
(4) $\widetilde{\Lambda_{i-1}} \not\subset S$ and $\widetilde{\sigma(\Lambda_{i-1})} \subset S$.

One can also see by considering cases that $i-1 = \min(\beta_1, \beta_2)$ (for β_1, β_2 the two integers associated to some choice edge e) if and only if we are in case (2), (3), or (4). In contrast to the SL_3 case, it is possible for neither $\widetilde{\Lambda_{i-1}}$ nor $\widetilde{\sigma(\Lambda_{i-1})}$ to be in S while still $\widetilde{\Lambda_{i-1}} \subset S^\sigma_{i-1}$ and $\widetilde{\sigma(\Lambda_{i-1})} \subset S_{i-1}$. To see this, consider the case in which p_r corresponds to an element of W of length 3 (pictured in Figure 9). This is the only situation in which case (2) arises.

In case (1) we can choose Λ_{i-1} to be any alcove adjacent to Λ_i by d_{i-1}. In this case, the dimension of the space of choices for the construction increases by one and is therefore equal to $l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i)}$. We also have $n_{(F,i)} = n_{(F,i-1)}$.

Cases (3) and (4) occur only when p_r corresponds to an element of W of length 2. We address case (3); the other case is similar. We know $i-1 = \min(\beta_1, \beta_2)$ for β_1 and β_2 the two integers corresponding to some choice edge e. If F dictates a

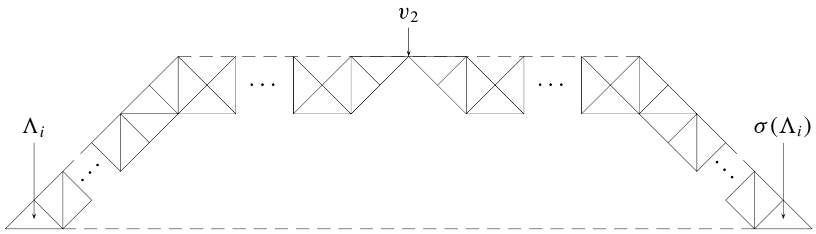

Figure 9 $\Lambda_i, \dots, \Lambda_n, \sigma(\Lambda_n), \dots, \sigma(\Lambda_i)$

hard choice at e, we choose Λ_{i-1} in S. In this case there is no increase in the dimension of the space of choices of the construction and, since $n_{(F,i-1)} = n_{(F,i)} + 1$, the dimension of the new space of choices is $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)} = l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i-1)}$. If F dictates an easy choice at e, we choose Λ_{i-1} to be any alcove attached to Λ_i at d_{i-1} but not in S. There is one dimension worth of such choices, so the dimension of the space of choices of the construction increases by one. We have $n_{(F,i-1)} = n_{(F,i)}$, so the new dimension is $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)} + 1 = l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i-1)}$.

We now consider case (2), which occurs only when p_r corresponds to an element of W of length 3. The construction $\Lambda_i, \dots, \Lambda_n, \sigma(\Lambda_n), \dots, \sigma(\Lambda_i)$ is contained in an apartment and has the general shape pictured in Figure 9. The dashed lines in this figure represent the boundary of S. Any choice of Λ_{i-1} determines a $g(\Lambda_{i-1})$ attached to $\sigma(\Lambda_i)$ via $\sigma(d_{i-1})$, just by taking the alcove adjacent to $\sigma(\Lambda_i)$ via $\sigma(d_{i-1})$ in the intersection of all apartments containing Λ_{i-1} and S. By Lemma 4.2, the number of choices of Λ_{i-1} with $g(\Lambda_{i-1}) = \sigma(\Lambda_{i-1})$ is non-zero and finite. So if F requires a hard choice, take Λ_{i-1} with $g(\Lambda_{i-1}) = \sigma(\Lambda_{i-1})$. Then dimension does not increase and is therefore equal to $l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)}$, which is $l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i-1)}$ because $n_{(F,i-1)} = n_{(F,i)} + 1$. If F requires an easy choice, take Λ_{i-1} with $g(\Lambda_{i-1}) \neq \sigma(\Lambda_{i-1})$. Then dimension increases by one and is thus

$$l(\Gamma^c_{(v_1,p_r)}) + (n-i) - 1 - n_{(F,i)} + 1 = l(\Gamma^c_{(v_1,p_r)}) + (n-(i-1)) - 1 - n_{(F,i-1)},$$

since $n_{(F,i-1)} = n_{(F,i)}$. This concludes the proof of Proposition 4.1. □

LEMMA 4.2. *The number of choices of* Λ_{i-1} (*in the preceding paragraph*) *with* $g(\Lambda_{i-1}) = \sigma(\Lambda_{i-1})$ *is non-zero and finite.*

Proof. We can identify the set $\{\Lambda_{i-1}\}$ with $\mathbb{A}^1$ over $\bar{\mathbb{F}}_q$, where $\mathbb{F}_q$ is the residue field of F. We can identify the set $\{\sigma(\Lambda_{i-1})\}$ with the set $\{\Lambda_{i-1}\}$ (and therefore with $\mathbb{A}^1$) using g. Hence the map $\sigma : \{\Lambda_{i-1}\} \to \{\sigma(\Lambda_{i-1})\}$ given by the action of σ on $\mathcal{B}_\infty$ gives a map $\psi : \mathbb{A}^1 \to \mathbb{A}^1$. But σ also acts on $\mathbb{A}^1(\bar{\mathbb{F}}_q)$ as the (algebraic) Frobenius, and one can show that if $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is defined by $\psi = \varphi \circ \sigma$ then φ is an algebraic isomorphism of $\mathbb{A}^1$. Thus, $\varphi(x) = ax + b$ with $a \neq 0$. The fixed points of ψ correspond to $x \in \mathbb{A}^1$ such that $a\sigma(x) + b = x$, which has exactly q solutions since $\sigma(x) = x^q$. □

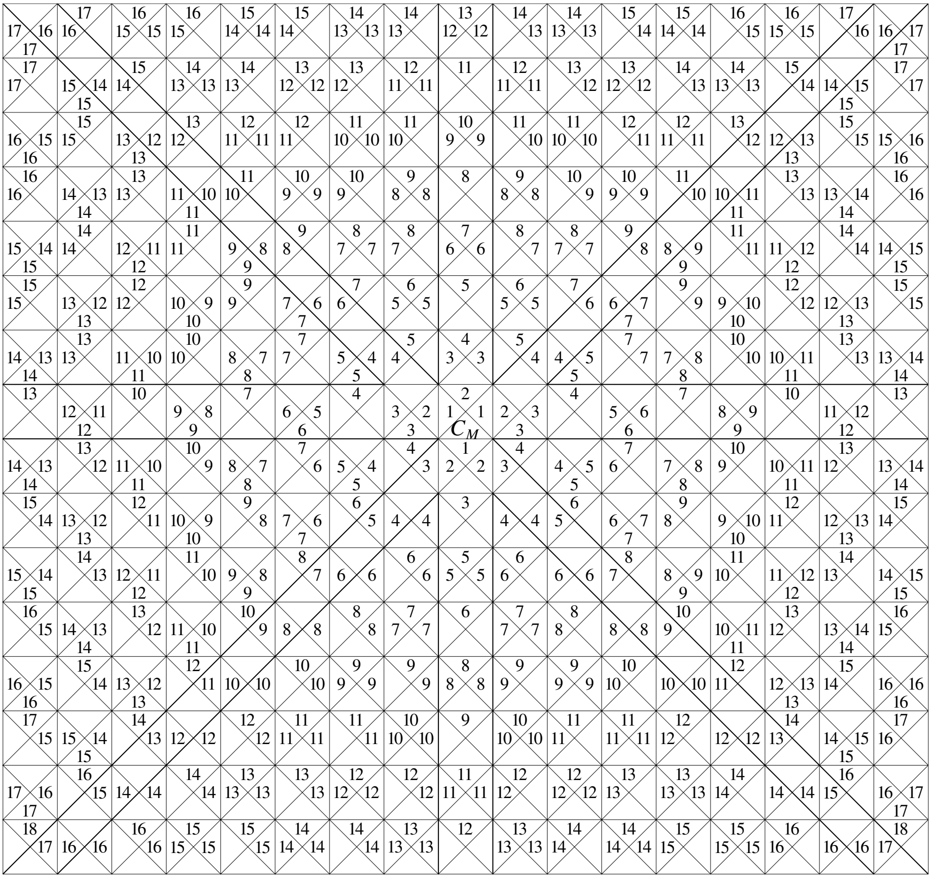

Figure 10 Main result in diagram form for Sp_4

So now we can compute the emptiness/non-emptiness as well as the dimension of the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$ for each $\tilde{w}$ by doing straightforward computations with cf-dimension. We did this for all v_1 and reflected the results across the line of symmetry of C_M, taking maxima whenever two numbers appeared in the same alcove. The results of this process can be seen in Figure 10.

In the course of the computation we observed that, if the (v_1, v_2, p_r)-piece of $X_{\tilde{w}}(\sigma)$ and the (v_1', v_2', p_r')-piece of $X_{\tilde{w}}(\sigma)$ had different dimensions, then exactly one of v_1, v_1' was non-special and the corresponding piece had the smaller dimension.

5. Application to $\dim(X^K_{\tilde{w}}(\sigma))$

Let $\tilde{w} \in \tilde{W}$ and let μ be a dominant cocharacter. Let π be the uniformizer in F. The map $p\colon G(L)/I \to G(L)/K$ gives a map $X^I_{\tilde{w}}(b\sigma) \to X^K_{\mu(\pi)}(b\sigma)$ whenever $I\tilde{w}I \subset K\mu(\pi)K$. The non-empty fibers of the map p are always K/I, which has dimension equal to the length δ of the longest element of the finite Weyl group W. Moreover, any point in $X^K_{\mu(\pi)}(b\sigma)$ is hit by a point in $X^I_{\tilde{w}}(b\sigma)$ for some $\tilde{w}$ with $I\tilde{w}I \subset K\mu(\pi)K$. If $\mathcal{S}_{\mu(\pi)} \subset \tilde{W}$ is defined so that $\coprod_{\tilde{w}\in\mathcal{S}_{\mu(\pi)}} I\tilde{w}I = K\mu(\pi)K$, then $\dim(X^K_{\mu(\pi)}(b\sigma)) = \max_{\tilde{w}\in\mathcal{S}_{\mu(\pi)}}(\dim(X^I_{\tilde{w}}(b\sigma))) - \delta$ because $p^{-1}(X^K_{\mu(\pi)}(b\sigma)) = \bigcup_{\tilde{w}\in S_{\mu(\pi)}} X^I_{\tilde{w}}(b\sigma)$. We applied this formula to $G = \mathrm{SL}_2$, SL_3, and Sp_4 (all with $b = 1$) and found that $\dim(X^K_{\mu(\pi)}(\sigma)) = \langle \mu, \rho \rangle$, where ρ is half the sum of the positive roots of G. This result supports Rapoport's Conjecture 5.10 in [13].

6. A Partial Formula for $\dim(X^I_{\tilde{w}}(\sigma))$ for SL_2, SL_3, and Sp_4

Suppose that G is a simply connected group and that $\tilde{w} \in \tilde{W}$. Let $\tilde{w} = tw$, where $w \in W$ and t acts on A_M by translation. Let $\eta_2(\tilde{w}) = \alpha \in W$, where $\tilde{w}C_M$ is in the same Weyl chamber as αC_M. Let $\eta_1\colon \tilde{W} \to W$ be the quotient map by the subgroup of translations. Let S be the set of simple reflections in W, and let W_T be the subgroup of W generated by $T \subset S$.

Let $h_1, \dots, h_{n+1}$ be the hyperplanes in A_M that contain one of the codimension-1 sub-simplices of C_M. Here n is the rank of G. Let $h_i^{(j)}$ be the hyperplanes in A_M parallel to h_i, with $h_i^{(0)} = h_i$. Choose $h_i^{(1)}$ to be as close as possible to h_i but on the other side of C_M. We define the *union of shrunken Weyl chambers* to be the set of all alcoves that are not between $h_i^{(0)}$ and $h_i^{(1)}$ for any i.

If $\tilde{w}C_M$ is in the union of shrunken Weyl chambers and if $G = \mathrm{SL}_2, \mathrm{SL}_3$, or Sp_4, then $X_{\tilde{w}}(\sigma)$ is non-empty if and only if $\eta_2(\tilde{w})^{-1}\eta_1(\tilde{w})\eta_2(\tilde{w}) \in W \setminus \bigcup_{T\subset S} W_T$, and in this case

$$\dim(X_{\tilde{w}}(\sigma)) = \frac{l_{\tilde{W}}(\tilde{w}) + l_W(\eta_2(\tilde{w})^{-1}\eta_1(\tilde{w})\eta_2(\tilde{w}))}{2}.$$

Here l_W is length in W and $l_{\tilde{W}}$ is length in $\tilde{W}$, as Coxeter groups.

One can examine Figures 5 and 10 to see that the stated equality holds for SL_3 and Sp_4. It also holds for SL_2. Note, though, that the new formula says nothing about the dimension or emptiness/non-emptiness of $X_{\tilde{w}}(\sigma)$ for $\tilde{w}$ not in the union of the shrunken Weyl chambers. Figures 5 and 10 give this information for SL_3 and Sp_4. The complement of the union of the shrunken Weyl chambers for SL_2 is just C_M, an easy special case.

The displayed equality might not hold for SL_4. One problem is that, for SL_2, SL_3, and Sp_4, there are other ways to specify the set $W \setminus \bigcup_{T\subset S} W_T$. In particular, $W \setminus \bigcup_{T\subset S} W_T = \{w \in W : l_W(w) \geq \mathrm{rank}(G)\}$ for these groups, and the SL_4 analogues of these two sets are not the same. We think the first formulation is more likely to be appropriate for a general statement.

The referee pointed out that, for SL_3 (although not for Sp_4), $\dim(X_{\tilde{w}}(\sigma)) = \lceil l_{\tilde{W}}(\tilde{w})/2 \rceil + 1$ for any alcove $\tilde{w}C_M$ for which $X_{\tilde{w}}(\sigma)$ is non-empty and whose intersection with C_M has codimension > 1. Hence, for these alcoves, dimension depends only on the length of $\tilde{w}$. The same formula does not hold for SL_2, but dimension depends only on length of $\tilde{w}$ in that case as well.

7. Related Results

Some of the emptiness/non-emptiness results of this paper were also obtained using other methods in the author's Ph.D. thesis [14]. These other methods are more computationally intensive and do not provide dimension information, but they do extend (to some extent) to $b \neq 1$. Some of the results from [14] can be combined to suggest Conjecture 7.1.

We restrict G to be one of the groups SL_2, SL_3, or Sp_4. Let D be a Weyl chamber in A_M, and let D' be the intersection of D with the union of shrunken Weyl chambers. Then we call D' a shrunken Weyl chamber. Let b be a representative of a σ-conjugacy class that meets the main torus of G. We can choose b so that it acts on A_M by translation and such that bC_M is in the main Weyl chamber. We define the *b-shifted shrunken Weyl chamber* associated to D to be $wbw^{-1}D'$, where $w \in W$ is the Weyl group element corresponding to the Weyl chamber D.

CONJECTURE 7.1. *Let b and G be restricted as just described, and let $\tilde{w}C_M$ be in the union of the b-shifted shrunken Weyl chambers. Then $X_{\tilde{w}}(b\sigma)$ is non-empty if and only if $\eta_2(\tilde{w})^{-1}\eta_1(\tilde{w})\eta_2(\tilde{w}) \in W \setminus \bigcup_{T \subset S} W_T$.*

This conjecture is shown to hold true in [14] for several values of b. Information about $\tilde{w}$ that is not in the b-shifted shrunken Weyl chambers is also given in [14] for the same b-values, but we have been unable to describe these results with a formula.

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