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Vertex Operator Realizations of Conformal Superalgebras

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Introduction

One of the first origins of vertex algebras was the explicit constructions of representations of certain Lie algebras by means of so-called vertex operators. The first construction of this kind was done by Lepowsky and Wilson [22], who constructed a vertex operator representation of the affine algebra $A_1^{(1)}$. Their work was later generalized in [17]. Frenkel and Kac [10] and, independently, Segal [30] constructed the basic representations of the simply laced affine Lie algebras using *untwisted vertex operators* (as opposed to the twisted vertex operators of Lepowsky and Wilson). Later vertex operators were used to construct a large family of modules for different types of Lie algebras, including all of the affine Kac–Moody algebras, toroidal algebras, and some other extended affine Lie algebras (see e.g. [3; 11; 12; 13; 25; 32] and references therein). The advantage of vertex operator constructions is that they are very explicit. They have yielded many interesting results for combinatorial identities, modular forms, soliton theory, and so forth.

It seems to be a natural problem to describe all Lie algebras, or at least a large family of Lie algebras, for which the vertex operator constructions of representations work. Our first observation is that, in some of the cases just described, the Lie algebras whose representations are constructed by vertex operators correspond to the *conformal algebras* introduced by Kac [15; 16]; see also [26; 27; 33]. On the other hand, vertex operators give rise to another algebraic structure, called *vertex algebras,* studied extensively in [4; 9; 11; 15], for example. A vertex operator construction of representations of Lie algebras amounts sometimes to an embedding of a conformal algebra into a vertex algebra generated by vertex operators, so that the vertex algebra becomes an *enveloping vertex algebra* of these conformal algebras.

In the present work we make the first step in describing the Lie algebras that are representable by vertex operators. We classify the Lie algebras that can be realized by the untwisted vertex operators of Frenkel–Kac–Segal. The vertex algebra generated by these vertex operators is also called *lattice vertex algebra* because its construction depends on a choice of an integer lattice. In fact there is a construction that, for every lattice Λ with an integer-valued bilinear form $(\cdot|\cdot)$, gives

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a vertex superalgebra $V_\Lambda = \bigoplus_{\lambda\in\Lambda} V_\lambda$ graded by the lattice Λ; see [6; 7; 11; 15; 29] and also Section 1.7 of this paper. If Λ is a simply laced root lattice of a finite-dimensional simple Lie algebra, then V_Λ is a module over the corresponding affine Kac–Moody algebra. Lattice vertex algebras play an important role in different areas of mathematics and physics. In particular, the celebrated Moonshine vertex algebra $V^\natural$ (such that Aut $V^\natural$ is the Monster simple group) is closely related to the lattice vertex algebra of a certain even unimodular lattice of rank 24, called the Leech lattice [4; 11].

The property of a conformal algebra $\mathfrak{L}$ to be generated by the Frenkel–Kac–Segal vertex operators means, in precise terms, that $\mathfrak{L}$ is a subalgebra of the lattice vertex algebra V_Λ corresponding to some integer lattice Λ such that the following statements hold.

- $\mathfrak{L} = \bigoplus_{\lambda\in\Lambda} \mathfrak{L}_\lambda$ is homogeneous with respect to the grading by the lattice Λ. Here $\mathfrak{L}_\lambda = \mathfrak{L} \cap V_\lambda$. Let $\Delta = \{\lambda \in \Lambda \mid \mathfrak{L}_\lambda \neq 0\}$ be the *root system* of $\mathfrak{L}$.
- $|\Delta| < \infty$. In fact, we can somewhat relax this requirement in the case when Λ is semi–positive definite; see Section 4.1.
- $\mathfrak{L}$ is stable under the action of the Heisenberg conformal algebra $\mathfrak{H} \subset V_0$; see Section 1.7.

In this paper we classify the conformal algebras $\mathfrak{L} \subset V_\Lambda$ that satisfy these conditions. Some unexpected results occur already in the case when the rank of Λ is 1 (see Section 4.2). It turns out that, besides well-known realizations of the Clifford and affine $\widehat{\mathrm{sl}}_2$ algebras and the less-known realization of the $N = 2$ simple conformal superalgebra (see e.g. [14]), one also has the realization of the conformal algebra $\mathfrak{K}$ (or rather its central extension, $\hat{\mathfrak{K}}$) obtained by the Tits–Kantor–Koecher construction from a certain Jordan conformal triple system $\mathfrak{J}$; see Section 2. This exhausts all possibilities for rank-1 lattices. We extend this result for the lattices of rank 2 and then generalize the classification for the case of an arbitrary lattice.

We also classify all finite root systems of conformal subalgebras of V_Λ (see Section 4.5). Most of them occur if Λ is positive definite. In this case, all such indecomposable root systems Δ are in fact classical Cartan systems of types other than F_4 and G_2. However, the corresponding Lie algebras are affine Kac–Moody only when Δ is simply laced, and then we are in the situation of [10]. We also explain what happens if the lattice Λ is semi–positive definite, and we outline the relation to the theory of extended affine root systems (EARS; see [1]).

ORGANIZATION OF THE MANUSCRIPT. We start with a review of the theory of conformal superalgebras, following mainly the lecture notes by Kac [15]; see also [27]. In Section 1.4 we construct most of the examples of conformal superalgebras used in the sequel. Then in Sections 1.5 and 1.6 we outline the theory of vertex algebras, again using [15]; in Section 1.7 we describe the construction of lattice vertex superalgebras. In Section 2, we review the Tits–Kantor–Koecher construction and use it to derive the conformal algebra $\mathfrak{K}$. Having done that, we review the so-called boson–fermion correspondence, which is essentially the study of the lattice vertex algebra $V_{\mathbb{Z}}$ corresponding to the lattice $\mathbb{Z}$. Sections 3.1 and 3.2 also mostly follow [15].

In Sections 3.3 and 3.4 we explore the structure of $V_{\mathbb{Z}}$ as a module over the conformal algebra $\mathfrak{W}$ of differential operators on a circle or, equivalently, over the Lie algebra $W_+ = \Bbbk\langle t, p \mid [t, p] = 1\rangle^{(-)}$ of differential operators on a disk. Although the representation theory of the Lie algebra $W = \Bbbk\langle t, t, ^{-1}, p \mid [t, p] = 1\rangle^{(-)}$ of differential operators on a circle, as well as that of the related vertex algebra $\mathcal{W}_{1+\infty}$, has been extensively studied (see e.g. [8; 19]), the representation theory of W_+ seems to be mostly unknown.

In Section 4.1 we give a rigorous formulation of the problem and introduce the necessary definitions. Then, in Section 4.2, we use the foregoing results to study the subalgebras of V_Λ in the case when $\operatorname{rk}\Lambda = 1$; in Section 4.3, we proceed to the case when $\operatorname{rk}\Lambda = 2$. This allows us (in Section 4.4) to classify the root systems for the case when the lattice Λ is positive definite and (in Section 4.5) to describe all finite root systems. Finally, in Section 4.6 we outline the relation between our results and the theory of EARS.

Throughout this paper, all spaces and algebras are over a ground field $\Bbbk$ of characteristic 0.

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1. Conformal and Vertex Algebras

1.1. Formal Series and Conformal Algebras

Let $L = L^{\bar{0}} \oplus L^{\bar{1}}$ be a Lie superalgebra. Consider the space of formal power series $L[[z^{\pm 1}]]$. We will write an element $\alpha \in L[[z^{\pm 1}]]$ in the form

$$\alpha = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1}, \quad \alpha(n) \in L.$$

Denote $L[[z^{\pm 1}]]' = L^{\bar{0}}[[z^{\pm 1}]] \oplus L^{\bar{1}}[[z^{\pm 1}]] \subseteq L[[z^{\pm 1}]]$. The space $L[[z^{\pm 1}]]$ is endowed with a derivation $D = d/dz$ and a family of bilinear products $\boxed{n}$, $n \in \mathbb{Z}_+$, given by

$$(\alpha \boxed{n} \beta)(m) = \sum_{i=0}^{n} \binom{n}{i} [\alpha(n-i), \beta(m+i)]. \tag{1}$$

We say that a pair of formal series $\alpha, \beta \in L[[z^{\pm 1}]]$ are *local* if there is $N = N(\alpha, \beta) \in \mathbb{Z}_+$ such that

$$\sum_{i=0}^{N} (-1)^i \binom{N}{i} [\alpha(n-i), \beta(m+i)] = 0$$

for all $m, n \in \mathbb{Z}$. In particular, we have $\alpha \boxed{n} \beta = 0$ for all $n \geq N$. Dong's lemma [15; 23] states that if $\alpha, \beta, \gamma \in L[[z^{\pm 1}]]$ are three pairwise local formal series, then $\alpha \boxed{n} \beta$ and γ are local for all $n \in \mathbb{Z}_+$.

Let $\mathfrak{L} \subset L[[z^{\pm 1}]]'$ be a subspace of pairwise local formal series closed under D and under all products $\boxed{n}$. Then $\mathfrak{L}$ is an example of *Lie conformal superalgebra*. Alternatively, we can define a Lie conformal superalgebra axiomatically as a $\Bbbk[D]$-module $\mathfrak{L} = \mathfrak{L}^{\bar{0}} \oplus \mathfrak{L}^{\bar{1}}$ equipped with a family of products $\boxed{n}$ ($n \in \mathbb{Z}_+$) satisfying the following axioms (see e.g. [15; 27]). For any homogeneous $a, b, c \in \mathfrak{L}$:

C1. (locality) $a \boxed{n} b = 0$ for $n \gg 0$;
C2. $(Da) \boxed{n} b = -na \boxed{n-1} b$;
C3. $D(a \boxed{n} b) = (Da) \boxed{n} b + a \boxed{n} (Db)$;
C4. (quasisymmetry)

$$a \boxed{n} b = -(-1)^{p(a)p(b)} \sum_{i \geq 0} (-1)^{n+i} \frac{1}{i!} D^i (b \boxed{n+i} a); \tag{2}$$

C5. (conformal Jacoby identity)

$$(a \boxed{n} b) \boxed{m} c = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \Big(a \boxed{n-i} (b \boxed{m+i} c) - (-1)^{p(a)p(b)} b \boxed{m+i} (a \boxed{n-i} c) \Big). \tag{3}$$

Here $p(a)$ is the parity of a.

One can prove (see e.g. [15]) that any subspace in $L[[z^{\pm 1}]]'$ of pairwise local series that is closed under the products (1) and $D = d/dz$ satisfies all these axioms.

We will often use the notation $D^{(n)} = (-1)^n \frac{1}{n!} D^n$.

1.2. The Coefficient Algebra

Let $\mathfrak{U}$ be a $\Bbbk[D]$-module. Its *space of coefficients* $U = \operatorname{Coeff} \mathfrak{U}$ is constructed as follows. Consider the space $\mathfrak{U} \otimes \Bbbk[t, t^{-1}]$, where t is an independent variable. We will write $a \otimes t^n = a(n)$ for $a \in \mathfrak{U}$. Let $E = \operatorname{Span}_{\Bbbk}\{(Da)(n) + na(n-1) \mid a \in \mathfrak{U}, n \in \mathbb{Z}\}$. Then let

$$U = \operatorname{Coeff} \mathfrak{U} = \mathfrak{U} \otimes \Bbbk[t, t^{-1}]/E.$$

There is a homomorphism $\mathfrak{U} \to U[[z^{\pm 1}]]$ given by $a \mapsto \sum_n a(n) z^{-n-1}$. This homomorphism is the universal one among all the representations of $\mathfrak{U}$ by formal series: if $\mathfrak{U} \to U'[[z^{\pm 1}]]$ is another $\Bbbk[D]$-homomorphism, then there is a unique homomorphism $U \to U'$ such that the diagram

$$\begin{tikzcd} U[[z^{\pm 1}]] \arrow[rr] & & U'[[z^{\pm 1}]] \\ & \mathfrak{U} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes.

If $\mathfrak{U}$ has a structure of a conformal algebra, then the space $U = \operatorname{Coeff} \mathfrak{U}$ becomes a "usual" algebra. The product on U is defined by

$$[a(m), b(n)] = \sum_{i \geq 0} \binom{m}{i} (a \boxed{i} b)(m + n - i).$$

The sum here makes sense owing to the locality of a and b. In this case U is called the *coefficient algebra* of $\mathfrak{U}$. It still has the universality property mentioned before.

Let $\mathfrak{L}$ be a Lie conformal superalgebra, and let $L = \operatorname{Coeff} \mathfrak{L}$ be its Lie superalgebra of coefficients. Then $L = L_- \oplus L_+$ is a direct sum of subalgebras $L_- = \operatorname{Span}\{a(n) \mid a \in \mathfrak{L}, n < 0\}$ and $L_+ = \operatorname{Span}\{a(n) \mid a \in \mathfrak{L}, n \geq 0\}$. The derivation D is also a derivation of L, acting by $D(a(n)) = -na(n-1)$. We see that L_- and L_+ are closed under the action of D.

1.3. Conformal Modules

As before, let $\mathfrak{L}$ be a Lie conformal superalgebra and let $L = \operatorname{Coeff} \mathfrak{L} = L_- \oplus L_+$ be its Lie superalgebra of coefficients. Denote by $\hat{L}_+ = L_+ \oplus \Bbbk D$ the extension of L_+ by D. A module over $\mathfrak{L}$ is by definition a $\hat{L}_+$-module $\mathfrak{U}$ such that, for any $u \in \mathfrak{U}$ and $a \in \mathfrak{L}$, we have $a(n)u = 0$ for $n \gg 0$. One can view $\mathfrak{U}$ as a $\Bbbk[D]$-module such that, for any $a \in \mathfrak{L}$, $u \in \mathfrak{U}$, and $n \in \mathbb{Z}_+$, there is an action $a \boxed{n} u \in \mathfrak{U}$, so that the semidirect product $\mathfrak{L} \ltimes \mathfrak{U}$ becomes a Lie conformal superalgebra, with $\mathfrak{U}$ as its abelian ideal.

The space of coefficients $U = \operatorname{Coeff} \mathfrak{U}$ becomes a module over L by

$$a(m)u(n) = \sum_{i \geq 0} \binom{m}{i} (a \boxed{i} u)(m+n-i).$$

1.4. Examples

1.4.1. Affine Algebras. Let $\mathfrak{g}$ be an arbitrary Lie superalgebra. Consider the corresponding loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \Bbbk[t, t^{-1}]$. Now, for any $a \in \mathfrak{g}$, define

$$\tilde{a} = \sum_{n \in \mathbb{Z}} at^n z^{-j-1} \in \tilde{\mathfrak{g}}[[z, z^{-1}]].$$

It is easy to see that any two $\tilde{a}, \tilde{b}$ are local with $N(\tilde{a}, \tilde{b}) = 1$ and

$$\tilde{a} \boxed{0} \tilde{b} = \widetilde{[a, b]}.$$

By Dong's lemma, the series $\{\tilde{a} \mid a \in \mathfrak{g}\} \subset \tilde{\mathfrak{g}}[[z, z^{-1}]]$ generate a Lie conformal superalgebra $\mathfrak{G}$. As a $\Bbbk[D]$-module, $\mathfrak{G} \cong \Bbbk[D] \otimes \mathfrak{g}$.

In practice we are often interested in central extensions of loop algebras. Assume that $\mathfrak{g}$ has trivial odd part and that $\mathfrak{g}$ is equipped with an invariant symmetric bilinear form $(\cdot|\cdot)$. Consider then the Lie algebra $\operatorname{Aff}(\mathfrak{g}) = (\mathfrak{g} \otimes \Bbbk[t, t^{-1}]) \oplus \Bbbk \mathsf{c}$ with the brackets given by

$$[a(m), b(n)] = [a, b](m+n) + \delta_{m,-n} m (a|b) \mathsf{c}.$$

The algebra $\operatorname{Aff}(\mathfrak{g})$ is called the *affinization* of $\mathfrak{g}$. It is the coefficient algebra of a conformal algebra $\mathfrak{Aff}(\mathfrak{g}) \subset G[[z, z^{-1}]]$, which is generated by the series $\tilde{a} = \sum_n a(n) z^{-n-1}$ for $a \in \mathfrak{g}$ and $\mathsf{c} = \mathsf{c}(-1)$ so that $D\mathsf{c} = 0$ and

$$\tilde{a} \boxed{0} \tilde{b} = \widetilde{[a, b]}, \qquad \tilde{a} \boxed{1} \tilde{b} = (a|b)\mathsf{c}.$$

If $\mathfrak{g}$ is an abelian Lie algebra, then we know that the corresponding Affine algebra $\mathrm{Aff}(\mathfrak{g})$ is a Heisenberg algebra and that $\mathfrak{Aff}(\mathfrak{g})$ is a Heisenberg conformal algebra. In the physics literature, the series $\tilde{a}$ are sometimes referred to as *bosons* in this case.

1.4.2. THE CLIFFORD ALGEBRA. As another example, take $\mathfrak{g}$ to be a two-dimensional odd linear space spanned over $\Bbbk$ by g_1 and g_{-1}. Consider the central extension $\mathrm{Cl} = \mathfrak{g} \otimes \Bbbk[t, t^{-1}] \oplus \Bbbk\mathsf{c}$ of the corresponding loop algebra with the brackets given by

$$[g_\varepsilon(m), g_{-\varepsilon}(n)] = \delta_{m+n,-1}\mathsf{c}, \quad \varepsilon = \pm 1;$$

the rest of the brackets are 0. We let c to be even. The algebra Cl is called the Clifford Lie superalgebra. It is the coefficient algebra of the conformal Lie superalgebra $\mathfrak{Cl} \subset \mathrm{Cl}[[z, z^{-1}]]$ spanned over $\Bbbk[D]$ by $\gamma_\varepsilon = \tilde{g}_\varepsilon$ ($\varepsilon = \pm 1$) and $\mathsf{c} = \mathsf{c}(-1)$, with the products given by $\gamma_\varepsilon \boxed{0}\, \gamma_{-\varepsilon} = \mathsf{c}$ (the rest of the products are 0). The series $\gamma_{\pm\varepsilon}$ are sometimes called *fermions* by physicists.

The Clifford algebra Cl is doubly graded: set $p(\gamma_\varepsilon(n)) = \varepsilon$ and $d(\gamma_\varepsilon(n)) = -n - \frac{1}{2}$ for $\varepsilon = \pm 1$ and $n \in \mathbb{Z}$, so we get

$$\mathrm{Cl} = \bigoplus_{p \in \mathbb{Z}} \mathrm{Cl}_p, \qquad \mathrm{Cl}_p = \bigoplus_{d \in \mathbb{Z}/2} \mathrm{Cl}_{p,d}.$$

The conformal Clifford algebra $\mathfrak{Cl}$ is also doubly graded such that $d(\gamma_\varepsilon) = \frac{1}{2}$, $p(\gamma_\varepsilon) = \varepsilon$, $p(\mathsf{c}) = d(\mathsf{c}) = 0$, and $\mathfrak{Cl}_{p,d} \boxed{n}\, \mathfrak{Cl}_{p',d'} \subset \mathfrak{Cl}_{p+p',d+d'-n-1}$, $D\mathfrak{Cl}_{p,d} \subset \mathfrak{Cl}_{p,d+1}$.

1.4.3. THE LIE ALGEBRA OF DIFFERENTIAL OPERATORS. Another example of conformal algebras is obtained from the Lie algebra of differential operators.

Let $A = \Bbbk\langle p, t^{\pm 1} \mid [t, p] = 1\rangle$ be the (localization in t of) the associative Weyl algebra. Let $W = A^{(-)}$ be the corresponding Lie algebra. It is the coefficient algebra of a Lie conformal algebra $\mathfrak{W} \subset W[[z, z^{-1}]]$ that is spanned over $\Bbbk[D]$ by elements

$$p_m = \sum_{n \in \mathbb{Z}} \frac{1}{m!} p^m t^n z^{-n-1} \subset W[[z, z^{-1}]]$$

with the multiplication table

$$\begin{aligned} p_m \boxed{k}\, p_n = &\binom{m+n-k}{m} p_{m+n-k} \\ &- (-1)^k \sum_{s=0}^{m-k} \binom{m+n-k-s}{n} D^{(s)} p_{m+n-k-s}. \end{aligned} \tag{4}$$

The algebra W has a unique central extension $\hat{W} = W \oplus \Bbbk\mathsf{c}$ that is defined by the 2-cocycle $\phi\colon W \times W \to \Bbbk$ given by

$$\phi(p^m t^k, p^n t^l) = \delta_{m+n,k+l} (-1)^m m!\, n! \binom{k}{m+n+1}. \tag{5}$$

The algebra $\hat{W}$ is usually referred to as $\mathcal{W}_{1+\infty}$ (see e.g. [8]). It is the coefficient algebra of a central extension $\hat{\mathfrak{W}} = \mathfrak{W} \oplus \Bbbk\mathsf{c}$ of the conformal Lie algebra $\mathfrak{W}$ that is defined by the conformal 2-cocycle [2] $\phi_k : \mathfrak{W} \times \mathfrak{W} \to \Bbbk\mathsf{c}$ ($k \in \mathbb{Z}_+$) given by

$$\phi_k(p_m, p_n) = \delta_{k,m+n+1}(-1)^m \mathsf{c}. \tag{6}$$

The algebras W and $\hat{W}$ are graded by setting $\deg p = \deg t^{-1} = 1$, $\deg t = -1$, and $\deg \mathsf{c} = 0$. The conformal algebras $\mathfrak{W}$ and $\hat{\mathfrak{W}}$ inherit the gradation from W and $\hat{W}$ respectively, so that we have $\deg(p_m) = m + 1$, $\deg D = 1$, and $\deg \boxed{k} = -k - 1$.

1.4.4. The Virasoro Conformal Algebra. Here are all nonzero products in $\hat{\mathfrak{W}}$ between p_0 and p_1:

$$p_0 \boxed{1}\, p_0 = \mathsf{c}, \quad p_0 \boxed{1}\, p_1 = p_0, \quad p_0 \boxed{2}\, p_1 = \mathsf{c},$$
$$p_1 \boxed{0}\, p_1 = Dp_1, \quad p_1 \boxed{1}\, p_1 = 2p_1, \quad p_1 \boxed{3}\, p_1 = -\mathsf{c},$$
$$p_1 \boxed{0}\, p_0 = Dp_0, \quad p_1 \boxed{1}\, p_0 = p_0, \quad p_1 \boxed{2}\, p_0 = -\mathsf{c}.$$

The element $p_0 \in \hat{\mathfrak{W}}$ generates a copy of the Heisenberg conformal algebra $\mathfrak{H} = \mathfrak{Aff}(\Bbbk) \subset \hat{\mathfrak{W}}$, introduced in §1.4.1. Its coefficient algebra $H = \operatorname{Coeff} \mathfrak{H}$ is the affinization of the one-dimensional trivial Lie algebra, so we have

$$[p_0(m), p_0(n)] = \delta_{m,-n} m \mathsf{c}.$$

The element $p_1 \in \hat{\mathfrak{W}}$ generates the Virasoro conformal algebra $\mathfrak{Vir} \subset \hat{\mathfrak{W}}$. Its coefficient algebra $\mathrm{Vir} = \operatorname{Coeff} \mathfrak{Vir}$ is spanned by c and $p_1(m)$ for $m \in \mathbb{Z}$ with the brackets given by

$$[p_1(m), p_1(n)] = (m - n)p_1(m + n - 1) - \delta_{m+n,2}\binom{m}{3}\mathsf{c}.$$

Together p_0 and p_1 span a semidirect product $\mathfrak{Vir} \ltimes \mathfrak{H} \subset \hat{\mathfrak{W}}$.

Note that $\deg p_0 = 1$ and $\deg p_1 = 2$.

1.4.5. $N = 2$ Simple Lie Conformal Superalgebra. A conformal algebra $\mathfrak{A}$ is said to be of a *finite type* if it is a finitely generated module over $\Bbbk[D]$. The algebras $\mathfrak{Vir}$, $\mathfrak{Cl}$ and $\mathfrak{Aff}(\mathfrak{g})$ for finite-dimensional $\mathfrak{g}$ defined previously are of a finite type. All simple and semisimple Lie conformal superalgebras of finite type are classified by Kac in [14]; see also [5] for the non-super case.

Besides the algebras mentioned already, in the sequel we will need the following simple finite-type Lie conformal superalgebra, called the $N = 2$ Lie conformal superalgebra. It is spanned over $\Bbbk[D]$ by two odd elements γ_{-1}, γ_{+1} and two even elements v, h. Elements v and h generate (respectively) the Virasoro and Heisenberg Lie conformal algebra, so the even part of the $N = 2$ superalgebra is equal to $\mathfrak{Vir} \ltimes \mathfrak{H}$. The remaining nonzero products between the generators are

$$\gamma_{-1} \boxed{0}\, \gamma_1 = v + \tfrac{1}{2}Dh, \quad \gamma_{-1} \boxed{1}\, \gamma_1 = h, \quad v \boxed{0}\, \gamma_{\pm 1} = D\gamma_{\pm 1},$$
$$v \boxed{1}\, \gamma_{\pm 1} = \tfrac{3}{2}\gamma_{\pm 1}, \quad h \boxed{0}\, \gamma_{\pm 1} = \pm\gamma_{\pm 1}.$$

Use (2) to derive products in the other order.

1.5. Vertex Algebras

In order to define vertex algebras, we need to consider the so-called *vertex operators* instead of formal power series. Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a vector superspace. Denote by $\mathrm{gl}(V)$ the Lie superalgebra of all $\Bbbk$-linear operators on V. Consider the space $\mathrm{vo}(V) \subset \mathrm{gl}(V)[[z, z^{-1}]]'$ of *vertex operators* on V given by

$$\mathrm{vo}(V) = \left\{ \sum_{n\in\mathbb{Z}} a(n) z^{-n-1} \;\middle|\; \forall v \in V,\, a(n)v = 0 \text{ for } n \gg 0 \right\}.$$

For $\alpha(z) \in \mathrm{vo}(V)$, denote

$$\alpha_-(z) = \sum_{n<0} \alpha(n) z^{-n-1} \quad \text{and} \quad \alpha_+(z) = \sum_{n\geq 0} \alpha(n) z^{-n-1}.$$

Denote also by $\mathbb{1} = \mathbb{1}_{\mathrm{vo}(V)} \in \mathrm{vo}(V)$ the identity operator such that $\mathbb{1}(-1) = \mathrm{Id}_V$; all other coefficients are 0.

In addition to the products $\boxed{n}$ ($n \in \mathbb{Z}_+$) defined by (1), the space of vertex operators $\mathrm{vo}(V)$ has products $\boxed{n}$ for $n < 0$. Define first $\boxed{-1}$ by

$$\alpha(z) \boxed{-1} \beta(z) = \alpha_-(z)\beta(z) + \beta(z)\alpha_+(z).$$

Note that the products of vertex operators here make sense in that, for any $v \in V$, we have $\alpha(n)v = \beta(n)v = 0$ for $n \gg 0$.

Next, for any $n < 0$ set

$$\alpha(z) \boxed{n} \beta(z) = \frac{1}{(-n-1)!} (D^{-n-1}\alpha(z)) \boxed{-1} \beta(z),$$

where $D = \frac{d}{dz}$. It is easy to see that

$$\alpha \boxed{-1} \mathbb{1} = \alpha, \quad \alpha \boxed{-2} \mathbb{1} = D\alpha, \quad \mathbb{1} \boxed{n} \alpha = \delta_{-1,n}\alpha.$$

It also follows easily from definitions that D is a derivation of all these products:

$$D(\alpha \boxed{n} \beta) = D\alpha \boxed{n} \beta + \alpha \boxed{n} D\beta.$$

We have the following explicit formula for the products: If

$$(\alpha \boxed{n} \beta)(z) = \sum_m (\alpha \boxed{n} \beta)(m) z^{-m-1},$$

then

$$\begin{aligned}(\alpha \boxed{n} \beta)(m) = &\sum_{s\leq n} (-1)^{s+n} \binom{n}{n-s} \alpha(s)\beta(m+n-s) \\ &- (-1)^{p(a)p(b)} \sum_{s\geq 0} (-1)^{s+n} \binom{n}{s} \beta(m+n-s)\alpha(s).\end{aligned}$$

The notion of locality introduced in Section 1.1 applies also to vertex operators without any changes. Dong's lemma holds for vertex operators instead of formal power series as well.

A *vertex superalgebra* is a subspace of vertex operators $\mathfrak{V} \subset \mathrm{vo}(V)$ such that $\mathbb{1} \in \mathfrak{V}$, $\alpha \boxed{n} \beta \in \mathfrak{V}$ for every $\alpha, \beta \in \mathfrak{V}$ and $n \in \mathbb{Z}$, and every $\alpha, \beta \in \mathfrak{V}$ are local to each other. This definition is due to Li [23]. For an axiomatic definition of vertex superalgebras that is equivalent to this description, we refer the reader to [4; 7; 9; 11; 15].

For a vertex superalgebra $\mathfrak{V}$ one can consider the *left adjoint action map* $Y\colon \mathfrak{V} \to \mathrm{vo}(\mathfrak{V})$ given by $Y(a)(z) = \sum_{n\in\mathbb{Z}} (a \boxed{n} \cdot) z^{-n-1}$. One of the main properties of vertex superalgebras is that Y is a vertex superalgebra homomorphism; in particular, $Y(a \mathbin{\textcircled{n}} b) = Y(a) \mathbin{\textcircled{n}} Y(b)$, which is equivalent to the following generalization of the conformal Jacoby identity (3):

$$
\begin{aligned}
(a \boxed{n} b) \boxed{m} c = \sum_{s\le n} (-1)^{s+n} \binom{n}{n-s} a \boxed{s} (b \boxed{m+n-s} c) \\
- (-1)^{p(a)p(b)} \sum_{s\ge 0} (-1)^{s+n} \binom{n}{s} b \boxed{m+n-s} (a \boxed{s} c)
\end{aligned}
\tag{7}
$$

for all $m, n \in \mathbb{Z}$.

We note that vertex superalgebras are in particular conformal superalgebras. Moreover, it can be shown that the quasisymmetry identity (2) holds in vertex superalgebras for all integer n.

1.6. Enveloping Vertex Algebras of a Conformal Algebra

Let again $\mathfrak{C}$ be a conformal superalgebra and $L = \operatorname{Coeff}\mathfrak{C} = L_- \oplus L_+$ its coefficient Lie superalgebra. Denote by $\hat{L} = L \oplus \Bbbk D$ the extension of L by the derivation D (see Section 1.2). Consider a homomorphism of conformal superalgebras $\rho\colon \mathfrak{C} \to \mathfrak{V}$ of $\mathfrak{C}$ into a vertex superalgebra $\mathfrak{V}$. If $\mathfrak{V}$ is generated as a vertex superalgebra by $\rho(\mathfrak{C})$ then we call it an *enveloping vertex superalgebra* of $\mathfrak{C}$.

An L-module (or $\hat{L}$-module) U is called a *highest weight* module if it is generated as a module over L by a single element $u \in U$ such that $L_+ u = Du = 0$. In this case, u is called a *highest weight vector.*

It is well known [15; 26; 27] that the enveloping vertex superalgebra $\mathfrak{V}$ has the structure of a highest weight module over $\hat{L} = \operatorname{Coeff}\mathfrak{C} \oplus \Bbbk D$ with the highest weight vector $\mathbb{1}$ defined by $a(n)v = \rho(a) \boxed{n} v$. Ideals of $\mathfrak{V}$ are $\hat{L}$-submodules. If $\rho_1\colon \mathfrak{C} \to \mathfrak{V}_1$ and $\rho_2\colon \mathfrak{C} \to \mathfrak{V}_2$ are two enveloping vertex algebras of $\mathfrak{C}$ and if $\psi\colon \mathfrak{V}_1 \to \mathfrak{V}_2$ is a vertex algebra homomorphism such that $\rho_1\psi = \rho_2$, then ψ is an $\hat{L}$-module homomorphism. Conversely, any highest weight module V over $\hat{L}$ with the highest weight vector $\mathbb{1}$ has a structure of enveloping vertex algebra of $\mathfrak{C}$ with the map $\rho\colon \mathfrak{C} \to V$ given by $\rho(a) = a(-1)\mathbb{1}$. In this case: $a(n)v = \rho(a) \boxed{n} v$ for $a \in \mathfrak{C}$; submodules of V are vertex ideals; and, if $\rho\colon V_1 \to V_2$ is a homomorphism of two highest weight L-modules such that $\rho(\mathbb{1}) = \mathbb{1}$, then ρ is a vertex algebra homomorphism.

We also mention the notion of universal (or Verma) highest weight module over L. It is defined by $V(L) = \operatorname{Ind}_{L_+}^{L} \Bbbk\mathbb{1} = U(L) \otimes_{U(L_+)} \Bbbk\mathbb{1}$. The action of the derivation D on L can be naturally extended to the action on $V(L)$, so $V(L)$ becomes an

$\hat{L}$-module. The Verma module is universal in the sense that, for any other highest weight module U with highest weight vector u, there is unique homomorphism $V(L) \to U$ such that $\mathbb{1} \mapsto u$. Hence the theorem implies that the enveloping vertex algebra corresponding to the Verma module $V(L)$ is universal in the obvious sense. It is called the *universal enveloping vertex algebra* of $\mathfrak{L}$.

1.7. Lattice Vertex Algebras

In this section we construct an important example of vertex superalgebras, called lattice vertex superalgebras. We mostly follow [15], but see also [6; 10; 11; 29].

We start from an integer lattice Λ of rank ℓ. It comes with an integer-valued bilinear form $(\cdot|\cdot)$. Let $\mathfrak{h} = \Lambda \otimes_{\mathbb{Z}} \Bbbk$, and we extend the form to $\mathfrak{h}$. Let $\nu\colon \mathfrak{h} \to \mathfrak{h}^*$ be the usual identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of this form.

Let $\mathfrak{H} = \mathfrak{Aff}(\mathfrak{h})$ be the Heisenberg conformal algebra corresponding to the space $\mathfrak{h}$ (see §1.4.1), and let $H = \mathrm{Aff}(\mathfrak{h}) = \mathrm{Coeff}\,\mathfrak{H}$ be its coefficient Heisenberg algebra. Take some $\beta \in \Lambda$ and let V_β be the canonical relation representation of H, that is, a highest weight irreducible H-module generated by the highest weight vector v_β such that $h(0) = (h|\beta)\,\mathrm{Id}$, $\mathsf{c} = \mathrm{Id}$.

It follows from Section 1.6 that V_0 is an enveloping vertex algebra of $\mathfrak{H}$, $v_0 = \mathbb{1}$. It can be shown that V_β is a module over the vertex algebra V_0. We define $V_\Lambda = \bigoplus_{\beta\in\Lambda} V_\beta = V_0 \otimes \Bbbk[\Lambda]$.

Let $\varepsilon\colon \Lambda \times \Lambda \to \{\pm 1\}$ be a bimultiplicative map such that

$$\varepsilon(\alpha,\beta) = (-1)^{(\alpha|\alpha)(\beta|\beta)}(-1)^{(\alpha|\beta)}\varepsilon(\beta,\alpha) \tag{8}$$

for any $\alpha, \beta \in \Lambda$. We remark that it is enough to check the identity (8) only when α and β belong to some $\mathbb{Z}$-basis of Λ; then (8) will follow for general α, β by bimultiplicativity.

Let

$$1 \longrightarrow \{\pm 1\} \longrightarrow \hat{\Lambda} \longrightarrow \Lambda \longrightarrow 1$$

be the extension of Λ corresponding to the cocycle ε. Let $e\colon \Lambda \to \hat{\Lambda}$ be a section of this extension. The extended lattice $\hat{\Lambda}$ acts on the group algebra $\Bbbk[\Lambda]$ of Λ by $e(\alpha)e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta}$.

The main result [6; 7; 11; 15] is that there is a vertex superalgebra structure on $V = V_\Lambda$ such that $a \boxed{n}\, v = a(n)v$ for any $a \in \mathfrak{H} \subset V_0$ and $v \in V$, and the products between the generators v_α are defined by

$$\begin{aligned} v_\alpha \boxed{n}\, v_\beta &= \varepsilon(\alpha,\beta)\frac{1}{m!}(D - \beta(-1))^m v_{\alpha+\beta} \\ &= \varepsilon(\alpha,\beta) \sum_{k\in\mathcal{P}(m)} \prod_{j\geq 1}\left(\frac{\alpha(-j)}{j!}\right)^{k_j} v_{\alpha+\beta}, \end{aligned} \tag{9}$$

where $m = -(\alpha|\beta) - n - 1$ and $\mathcal{P}(m) = \{k = (k_1, k_2, \dots) \mid k_i \geq 0, \sum_{i\geq 1} ik_i = m\}$ is the set of partitions of m.

In particular, $v_\alpha \boxed{n}\, v_\beta = 0$ if $n \geq -(\alpha|\beta)$; we also have $v_\alpha \boxed{-(\alpha|\beta)-1}\, v_\beta = \varepsilon(\alpha,\beta)v_{\alpha+\beta}$ and $v_\alpha \boxed{-(\alpha|\beta)-2}\, v_\beta = \varepsilon(\alpha,\beta)\alpha(-1)v_{\alpha+\beta}$. Note that $V_\alpha \boxed{n}\, V_\beta \subset V_{\alpha+\beta}$.

The even and odd parts of V are

$$V^{\bar{0}} = \bigoplus_{\substack{\alpha\in\Lambda:\\(\alpha|\alpha)\in 2\mathbb{Z}}} V_\alpha, \qquad V^{\bar{1}} = \bigoplus_{\substack{\alpha\in\Lambda:\\(\alpha|\alpha)\in 2\mathbb{Z}+1}} V_\alpha$$

(respectively). The vertex algebra V is simple if and only if the form $(\cdot|\cdot)$ is non-degenerate.

Under the left adjoint action map $Y\colon V \to \mathrm{vo}(V)$, the elements v_α are mapped to the so-called vertex operators

$$Y(v_\alpha) = \Gamma_\alpha(z) = e(\alpha) z^{\alpha(0)} E_-(\alpha,z) E_+(\alpha,z),$$

where

$$E_\pm(\alpha,z) = \exp \sum_{n \gtrless 0} -\frac{\alpha(n)}{n} z^{-n} \in \mathrm{vo}(V)$$

and where the vertex operator $z^{\alpha(0)} \in \mathrm{vo}(V_{\mathbb{Z}})$ is defined by

$$z^{\alpha(0)}|_{V_\mu} = \sum_{n\in\mathbb{Z}} \delta_{n,(\alpha|\mu)} z^n.$$

We also have

$$[h(n), e(\alpha)] = \delta_{n,0}(\alpha|h)e(\alpha).$$

Besides the grading by the lattice Λ, the vertex superalgebra V has another grading by the group $\frac{1}{2}\mathbb{Z}$, so that $\deg v_\alpha = \frac{1}{2}(\alpha|\alpha)$, $\deg a = 1$ for every $a \in \mathfrak{H}$, $\deg \boxed{n} = -n-1$, and $\deg D = 1$. We have the decomposition

$$V_\beta = \bigoplus_{d\in(\beta|\beta)/2+\mathbb{Z}_+} V_{\beta,d}.$$

Let $(\alpha_1,\dots,\alpha_\ell)$ and $(\beta_1,\dots,\beta_\ell)$ be dual bases of $\mathfrak{h}$ (i.e., such that $(\alpha_i|\beta_j) = \delta_{ij}$). Then the element $\omega = \frac{1}{2}\sum_{i=1}^{\ell} \alpha_i \boxed{-1} \beta_i \in V_0$ generates a copy of the Virasoro Lie conformal algebra $\mathfrak{Vir}$ (defined in §1.4.4) such that $\omega \boxed{0} v = DV$ for all $v \in V$, $\omega \boxed{1} v = (\deg v)v$ for all homogeneous $v \in V$, $\omega \boxed{2} \omega = 0$, and $\omega \boxed{3} \omega = \frac{1}{2}\mathbb{1}$.

We will identify the Heisenberg conformal algebra $\mathfrak{H}$ with its image in V_0 under the map $\tilde{h} \mapsto h(-1)\mathbb{1}$ for $h \in \mathfrak{h}$, $\mathsf{c} \mapsto \mathbb{1}$.

We remark that the vertex superalgebra structure of V_Λ is quite explicit. A basis of V_Λ is given by all expressions of the form

$$\alpha_1(n_1)\alpha_2(n_2)\dots\alpha_l(n_l)v_\beta, \quad \alpha_i,\beta\in\Lambda,\ 0 > n_i \in \mathbb{Z},$$

and the products of these elements are easily calculated using the formula (9), the identities

$$[\alpha(m), v_\beta(n)] = (\alpha|\beta)v_\beta(m+n), \qquad [\alpha(m),\beta(n)] = (\alpha|\beta)m\delta_{m,-n}$$

in Coeff V_Λ for $\alpha,\beta \in \Lambda$ and $m,n \in \mathbb{Z}$, and the identities (2) and (7) of vertex superalgebras.

2. Jordan Triple Systems and Tits–Kantor–Koecher Construction

2.1. The Tits–Kantor–Koecher Construction

Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a three-graded Lie algebra such that $[L_i, L_j] \subset L_{i+j}$ whenever $i, j, i+j \in \{-1, 0, 1\}$ and $[L_1, L_1] = [L_{-1}, L_{-1}] = 0$. Assume that $L_0 = [L_{-1}, L_1]$ and $L_0 \cap Z(L) = 0$, where $Z(L)$ is the center of L. Consider a pair of trilinear maps

$$\varphi_+ \colon L_1 \times L_{-1} \times L_1 \to L_1 \quad \text{and} \quad \varphi_- \colon L_{-1} \times L_1 \times L_{-1} \to L_{-1}$$

given by $\varphi_\pm(a,b,c) = \frac{1}{2}[[a,b],c]$. The tuple $J = \{(L_{-1}, L_1), \varphi_\pm\}$ is known as a *Jordan pair.* All Jordan pairs can be obtained in this way. In fact, one can define Jordan pairs formally by imposing certain axioms on the maps $\varphi_{\pm 1}$. From an abstractly defined Jordan pair $J = \{(L_{-1}, L_1), \varphi_\pm\}$ one can construct a three-graded Lie algebra $L(J) = L_{-1} \oplus L_0 \oplus L_1$, where $L_0 = \mathcal{D}(J)$ is the Lie algebra of inner derivations of J. This is known as the Tits–Kantor–Koecher (TKK) construction (see [20; 21; 31]). A Jordan pair J is simple if and only if the TKK Lie algebra $L(J)$ is simple.

A derivation $d = (d_-, d_+)$ of a Jordan pair $J = \{(L_{-1}, L_1), \varphi_{\pm 1}\}$ is a pair of linear maps $d_- \colon L_- \to L_-$ and $d_+ \colon L_+ \to L_+$ such that

$$d_\pm(\varphi_\pm(a,b,c)) = \phi_\pm(d_\pm(a),b,c) + \varphi_\pm(a, d_\mp(b), c) + \varphi_\pm(a,b,d_\pm(c)).$$

As is the case for other algebraic structures, the set of all derivations of J is a Lie algebra under the usual commutator operation. Let $a \in L_{-1}$ and $b \in L_1$. It turns out that $d_{ab} = (\varphi_-(a,b,\cdot), \varphi_+(b,a,\cdot))$ is a derivation of J that is known as an *inner derivation*. The Lie algebra L_0 can be identified with the set $\mathcal{D}(J) = \{d_{ab} \mid a \in L_{-1}, b \in L_1\}$ of all inner derivations of J by $d_{ab} = \frac{1}{2}[a,b]$.

There is an important case when $L_{-1} = L_1$ and $\varphi_+ = \varphi_-$; then J is called a *Jordan triple system*. In terms of the three-graded Lie algebra $L = L(J)$, this means that there is an involution $\sigma : L \to L$ such that $\sigma(L_\varepsilon) = L_{-\varepsilon}$ ($\varepsilon = \pm 1$) is the identification of L_{-1} and L_1. All Jordan pairs we deal with in this paper are actually Jordan triple systems.

2.2. Example: Associative Algebras

Here we consider some important examples of Jordan triple systems. Let A be an associative algebra. We define a triple operation $\varphi \colon A \times A \times A \to A$ by $\varphi(a,b,c) = \frac{1}{2}(abc + cba)$. The TKK Lie algebra $L(A)$ can be identified with a subalgebra of the Lie algebra $\mathrm{gl}_2(A)$ of 2×2 matrices over A modulo the center:

$$L(A) = \mathrm{Span}_{\Bbbk}\left\{ \begin{pmatrix} -dc & b \\ a & cd \end{pmatrix} \in \frac{\mathrm{gl}_2(A)}{Z(\mathrm{gl}_2(A))} \;\middle|\; a,b,c,d \in A \right\}.$$

Here $L(A)_{-1}$ consists of lower triangular matrices, $L(A)_1$ consists of upper triangular matrices, and $L(A)_0$ is the space of all diagonal matrices in $L(A)$. Quite often it happens that $L(A) = \mathrm{gl}_2(A)/Z(\mathrm{gl}_2(A))$.

Now assume that there is an involution or an anti-involution $\tau : A \to A$ on A. Then both the set A^τ of τ-stable elements and the set $A^{-\tau}$ of those elements that change sign under the action of τ are closed under the triple operation. The corresponding TKK Lie algebra $L(A^{\pm\tau})$ can be still represented by 2×2 matrices over A. Consider the case when $\tau : A \to A$ is an anti-involution. Then

$$L(A^{\pm\tau}) = \left\{ \begin{pmatrix} -\tau(x) & b \\ a & x \end{pmatrix} \,\middle|\, a, b \in A^{\pm\tau},\ x \in \mathcal{D} \subset A^{(-)} \right\},$$

where $\mathcal{D} \subset A^{(-)}$ is a Lie subalgebra of $A^{(-)}$ generated by all elements of the form ab for $a, b \in A^{\pm\tau}$. We see that $\mathcal{D} = L(A^{\pm\tau})_0$ is precisely the Lie algebra of inner derivations of the Jordan pair $(A^{\pm\tau}, A^{\pm\tau})$. It acts on $A^{\pm\tau}$ by $x.a = xa + a\tau(x)$, where $x \in \mathcal{D}$ and $a \in A^{\pm\tau}$. The involution $\sigma : L(A^{\pm\tau}) \to L(A^{\pm\tau})$ acts by $\left(\begin{smallmatrix} -\tau(x) & b \\ a & x \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x & a \\ b & -\tau(x) \end{smallmatrix}\right)$ and so $\sigma|_{\mathcal{D}} = -\tau$.

2.3. *The Conformal Algebra* $\hat{\mathfrak{K}}$

Now let $A = \Bbbk\langle p, t, t^{-1} \mid [t, p] = 1\rangle$ be the localized Weyl algebra, the one we have dealt with in §1.4.3. It has an anti-involution $\tau : A \to A$ defined by $\tau(t) = t$ and $\tau(p) = -p$. The space $J = A^{-\tau}$ is a Jordan triple subsystem of A. It is easy to see that J is simple. Let $K = L(J) = J \oplus \mathcal{D}(J) \oplus J$ be the TKK Lie algebra corresponding to J.

LEMMA 1. *The Lie algebra K is the coefficient algebra of a simple conformal algebra $\mathfrak{K} \subset K[[z, z^{-1}]]$.*

Proof. First we construct the $\Bbbk[D]$-module $\mathfrak{J}$ such that $J = \operatorname{Coeff} \mathfrak{J}$. The anti-involution τ acts also on the $\Bbbk[D]$-module $\mathfrak{A} \subset A[[z, z^{-1}]]$ that is generated by the series $p_m = \sum_n p_m(n) z^{-n-1} \in A[[z, z^{-1}]]$; see §1.4.3. (In fact, $\mathfrak{A}$ has a structure of an *associative* conformal algebra; see [16; 27].) We let $\mathfrak{J} = \mathfrak{A}^{-\tau}$ be the $\Bbbk[D]$-submodule of $\mathfrak{A}$ consisting of those elements of $\mathfrak{A}$ that change sign under the action of τ. We have $J = \operatorname{Coeff} \mathfrak{J}$.

The Weyl algebra A is spanned by the coefficients $p_m(n) = \frac{1}{m!} p^m t^n$ for $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$ (see §1.4.3). Therefore, the space J is spanned by the elements

$$u_m(n) = \tau(p_m(n)) - p_m(n) = -\frac{1}{m!} p^m t^n + (-1)^m \sum_{i \geq 0} \binom{n}{i} \frac{1}{(m-i)!} p^{m-i} t^{n-i}$$

for all $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$. Then the series

$$u_m = \sum_{n \in \mathbb{Z}} u_m(n) z^{-n-1} = -p_m + (-1)^m \sum_{i=0}^{m} D^{(i)} p_{m-i} \in \mathfrak{J}$$

span $\mathfrak{J}$ over $\Bbbk[D]$. Since $u_m(n) + \tau(u_m(n)) = 0$ we have

$$u_m + (-1)^m \sum_{i \geq 0} D^{(i)} u_{m-i} = 0$$

and so, if m is even, it follows that

$$u_m = -\frac{1}{2}\sum_{i\geq 1} D^{(i)} u_{m-i}. \tag{10}$$

The remaining series $\{u_m \mid m \in 2\mathbb{Z}_+ + 1\}$ form a basis of $\mathfrak{J}$ over $\Bbbk[D]$.

Some simple calculations show that the algebra $\mathcal{D}(J)$ of inner derivations of $J = A^{-\tau}$ is equal to the whole $W = A^{(-)}$. Hence J is a module over W, where the action is given by $x.a = xa + a\tau(x)$ for $x \in W$ and $a \in J$. This action induces the following action of $\mathfrak{W}$ on $\mathfrak{J}$:

$$p_m(k)u_n = \binom{m+n-k}{m} u_{m+n-k} - \delta_{k,0}(-1)^n \sum_{i\geq 0}\binom{m+n-i}{m} D^{(i)} u_{m+n-i} \tag{11}$$

for all $m, n \in \mathbb{Z}_+$.

Thus we obtain the TKK conformal algebra $\mathfrak{K} = \mathfrak{K}_{-1} \oplus \mathfrak{K}_0 \oplus \mathfrak{K}_1 \subset K[[z,z^{-1}]]$, where a $\Bbbk[D]$-basis of $\mathfrak{K}_0 = \mathfrak{W}$ is given by $p_m \in W[[z,z^{-1}]]$ for $m \in \mathbb{Z}_+$, a $\Bbbk[D]$-basis of $\mathfrak{K}_{-1} = \mathfrak{J}$ is given by $u_m \in J[[z,z^{-1}]] = K_{-1}[[z,z^{-1}]]$ for $m \in 2\mathbb{Z}_+ + 1$, and $\mathfrak{K}_1 = \mathfrak{J}$ is spanned over $\Bbbk[D]$ by the basis $\sigma(u_m) \in K_1[[z,z^{-1}]]$ for $m \in 2\mathbb{Z}_+ + 1$. Here $\sigma: \mathfrak{K} \to \mathfrak{K}$ is the involution identifying $\mathfrak{K}_1$ with $\mathfrak{K}_{-1}$. Since the coefficients of these series span K and are linearly independent, we have $K = \operatorname{Coeff}\mathfrak{K}$.

The formula (11) shows that p_m and u_n in $K[[z,z^{-1}]]$ are local, hence so are p_m and $\sigma(u_n)$. The series u_m and $\sigma(u_n)$ are local as well because the product $K_{-1} \times K_1 \to K_0$ is just the associative product in A if we identify $K_{-1} = K_1 = A^{-\tau} \subset A$ and $K_0 = A$ as linear spaces. □

Here is the multiplication table in $\mathfrak{K}$. The products of p_m and p_n are given by (4), the products $p_m \boxed{k}\, u_n = p_m(k)u_n$ are given by (11);

$$\begin{aligned} p_m \boxed{k}\, \sigma(u_n) &= (-1)^{m+1}\binom{m+n}{m}\sigma(u_{m+n-k}) \\ &\quad + (-1)^{m+n}\sum_{i=0}^{n}\binom{m+n-k-i}{m-k} D^{(i)}\sigma(u_{m+n-k-i}); \end{aligned} \tag{12}$$

$$\begin{aligned} &u_m \boxed{k}\, \sigma(u_n) \\ &\quad = \left(\binom{m+n-k}{m} - (-1)^m\binom{m+n}{m}\right) p_{m+n-k} \\ &\quad\quad - (-1)^n \sum_{j,l}(-1)^l\binom{k}{l}\left(\binom{j-k}{m} - (-1)^m\binom{j-l}{m}\right) D^{(m+n-j)} p_{j-k}. \end{aligned} \tag{13}$$

REMARK. We see that $\mathfrak{J}$ has every right to be called a conformal Jordan triple system. We could have defined the conformal triple operation on $\mathfrak{J}$; this would be a family of trilinear maps depending on two integer parameters.

The algebras K and $\mathfrak{K}$ have central extensions $\hat{K} = K \oplus \Bbbk\mathrm{c}$ and $\hat{\mathfrak{K}} = \mathfrak{K} \oplus \Bbbk\mathrm{c}$ (respectively) defined by the 2-cocycles $\phi: K \times K \to \Bbbk\mathrm{c}$ and $\phi_k: \mathfrak{K} \times \mathfrak{K} \to \Bbbk\mathrm{c}$ ($k \in \mathbb{Z}_+$) that are given by

$$\phi(u_m(k),\sigma(u_n(l))) = \delta_{k+l,m+n}(-1)^m\binom{k}{m+n+1}\left(\binom{m+n}{m}-1\right),$$
$$\phi_k(u_m,\sigma(u_n)) = \delta_{k,m+n+1}(-1)^m\left(\binom{m+n}{m}-1\right). \tag{14}$$

REMARK. It is possible to show that conformal algebra $\mathfrak{K}$ (and even its subalgebra $\mathfrak{W} \ltimes \mathfrak{J}$) is not embeddable into an associative conformal algebra, though it is finitely generated and has linear locality function (see [28]). It is proved in [28] that the linearity of locality function is a *necessary* condition for embedding of a finitely generated conformal algebra into an associative conformal algebra.

3. Boson–Fermion Correspondence

3.1. Lie Algebras of Matrices

Let gl_∞ be the Lie algebra of infinite matrices that have only finitely many nonzero entries. Denote by E_{ij} the elementary matrix that has the only nonzero entry at the ith row and jth column. Then we have

$$[E_{i,j}, E_{k,l}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}.$$

Let M be the Lie algebra of infinite matrices that have only finitely many nonzero diagonals. The algebras gl_∞ and M are both graded by setting the degree of E_{ij} equal to $j-i$.

It is well known [18] that gl_∞ and M have unique central extensions $\widehat{\mathrm{gl}}_\infty = \mathrm{gl}_\infty \oplus \Bbbk\mathsf{c} \subset \hat{M} = M \oplus \Bbbk\mathsf{c}$ defined by the 2-cocycle $\phi(A,B) = \mathrm{tr}([A,J]B)$, where $J = \sum_{i<0} E_{ii} \in M$. We set $\deg\mathsf{c} = 0$. The values of ϕ on the elementary matrices are given by

$$\phi(E_{i,j},E_{k,l}) = \begin{cases} \delta_{il}\delta_{jk} & \text{if } j<0 \text{ and } i\ge 0,\\ -\delta_{il}\delta_{jk} & \text{if } i<0 \text{ and } j\ge 0,\\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

The associative Weyl algebra $A = \Bbbk\langle t,t^{-1},p \mid [t,p]=1\rangle$ is embedded into the associative algebra of infinite matrices by

$$t \mapsto \sum_{i\in\mathbb{Z}} E_{i,i-1}, \quad t^{-1} \mapsto \sum_{i\in\mathbb{Z}} E_{i,i+1}, \quad p \mapsto -\sum_{i\in\mathbb{Z}}(i+1)E_{i,i+1}.$$

Therefore, the Lie algebra $W = A^{(-)}$ (defined in §1.4.3) is embedded into the Lie algebra M.

LEMMA 2. *The restriction of the* 2*-cocycle* ϕ *on* W *precisely coincides with the* 2*-cocycle on* W *given by* (5).

Proof. Express the linear generators of W in terms of elementary matrices:

$$p_m(n) = \frac{1}{m!}p^m t^n = (-1)^m\sum_{i\in\mathbb{Z}}\binom{i+m}{m}E_{i,i+m-n}. \tag{16}$$

Then, using (15), we obtain by some calculations that

$$\phi(p_m(k), p_n(l)) = \delta_{m+n,k+l}(-1)^m \binom{k}{m+n+1}. \qquad \square$$

As a consequence, the central extension $\hat{W}$ of W is embedded into the Lie algebra $\hat{M}$.

3.2. The Clifford Vertex Superalgebra

Let $\mathfrak{Cl}$ be the Clifford conformal superalgebra (defined in §1.4.2) and let $\mathrm{Cl} = \mathrm{Coeff}\,\mathfrak{Cl}$ be its coefficient Lie algebra. Let $U = U(\mathrm{Cl})/(\mathsf{c} = 1)$ be the quotient of the universal enveloping algebra of Cl over the ideal generated by the relation $\mathsf{c} = 1$.

The following lemma is proved by a straightforward computation (see e.g. [13]).

LEMMA 3. *Let $e_{ij} = \gamma_{-1}(i)\gamma_1(-j-1) \in U$ for $i, j \in \mathbb{Z}$. Let*

$$\hat{e}_{ij} = \begin{cases} -\gamma_1(-j-1)\gamma_{-1}(i) & \text{if } i = j \geq 0, \\ e_{ij} & \text{otherwise.} \end{cases}$$

Then the mapping $E_{ij} \mapsto e_{ij}$ defines an embedding of the Lie algebra gl_∞ of infinite matrices (see Section 3.1) into U, and the map $E_{ij} \mapsto \hat{e}_{ij}$, $c \mapsto 1$, defines an embedding of the Lie algebra $\widehat{\mathrm{gl}}_\infty$ into U.

Note that $d(\hat{e}_{ij}) = j - i = \deg E_{ij}$ and $p(\hat{e}_{ij}) = 0$ (see §1.4.2 for notation).

Now consider the universal highest weight module V over Cl (see Section 1.6). By definition, V is generated over Cl by a single element $\mathbb{1}$ such that $\mathsf{c}\mathbb{1} = \mathbb{1}$, $\mathrm{Cl}_+\mathbb{1} = 0$, and $V = U(\mathrm{Cl}) \otimes_{U(\mathrm{Cl}_+)\oplus \Bbbk\mathsf{c}} \Bbbk\mathbb{1}$. As a linear space, V can be identified with the Grassman algebra $\Bbbk[\gamma_\varepsilon(n) \mid \varepsilon = \pm 1, n < 0]$, on which c acts as identity, $\gamma_\varepsilon(n)$ for $n < 0$ acts by multiplication on the corresponding variable, and if $n \geq 0$ then $\gamma_\varepsilon(n)$ acts as an odd derivation. It follows that V is an irreducible Cl-module.

The module V inherits the double grading from Cl, so we have

$$V = \bigoplus_{p\in\mathbb{Z}} V_p, \qquad V_p = \bigoplus_{d\in\mathbb{Z}/2} V_{p,d}.$$

It is easy to see that if $V_{p,d} \neq 0$ then $d - p/2 \in \mathbb{Z}$ and $d \geq p^2/2$. Indeed, let

$$w = \gamma_{-1}(n_1) \wedge \gamma_{-1}(n_2) \wedge \cdots \wedge \gamma_{-1}(n_k) \wedge \gamma_1(m_1) \wedge \gamma_1(m_2) \wedge \cdots \wedge \gamma_1(m_l) \in V,$$
$$n_1 < n_2 < \cdots < n_k < 0, \qquad m_1 < m_2 < \cdots < m_l < 0,$$

be such that $p(w) = p > 0$. Then $l \geq p$ and $d(w) \geq -m_1 - m_2 - \cdots - m_l - l/2 \geq p^2/2$.

REMARK. There is an alternative construction of V using semi-infinite wedge products (see e.g. [13, Chap. 14]). It implies that in fact $\dim_\Bbbk V_{p,d} = P(d - p^2/2)$, where $P(n)$ is the classical partition function.

The module V has the structure of an enveloping vertex superalgebra of $\mathfrak{Cl}$ (see Section 1.6) such that the embedding $\mathfrak{Cl} \to V$ is given by $a \mapsto a(-1)\mathbb{1}$. We will identify $\mathfrak{Cl}$ with its image in V. We have

$$V^{\bar{0}} = \bigoplus_{p \in 2\mathbb{Z}} V_p, \qquad V^{\bar{1}} = \bigoplus_{p \in 2\mathbb{Z}+1} V_p.$$

A certain completion $\bar{U}$ of the algebra U acts on the vertex algebra V—that is, some infinite sums of the elements of U make sense as operators on V. In particular, the closure of the algebra $\widehat{\mathrm{gl}}_\infty \subset U$ spanned by $\hat{e}_{ij}$ (see Lemma 3) is the algebra $\hat{M} \subset \bar{U}$. It follows also that the algebra $\hat{W} \subset \hat{M}$ acts on V and there is a map $\hat{\mathfrak{W}} \to V$ given by $a \mapsto a(-1)\mathbb{1}$. The following lemma describes the image of $\hat{\mathfrak{W}} \subset V$ (see e.g. [15]).

LEMMA 4. *The mapping $p_m \mapsto \gamma_{-1} \boxed{-m-1}\, \gamma_1$ defines an embedding of the conformal algebra $\hat{\mathfrak{W}}$ into $V_0 \subset V$.*

Proof. Using (16) and Lemma 3, we obtain

$$\begin{aligned}(\gamma_{-1} \boxed{-m-1}\, \gamma_1)(n) &= \sum_{i \le -m-1} (-1)^m \binom{m+i}{m} \gamma_{-1}(i)\gamma_1(n-m-1-i) \\ &\quad - \sum_{i \ge 0} (-1)^m \binom{m+i}{m} \gamma_1(n-m-1-i)\gamma_{-1}(i) \\ &= (-1)^m \sum_{i \in \mathbb{Z}} \binom{m+i}{m} \hat{e}_{i,i-m+n} = p_m(n). \qquad \square\end{aligned}$$

Now let us apply the construction of Section 1.7 to the lattice $\Lambda = \mathbb{Z}$ that is generated by a single vector α such that $(\alpha|\alpha) = 1$. Then formula (9) implies that the elements $v_{\pm\alpha} \in V_{\mathbb{Z}}$ generate a conformal superalgebra isomorphic to the Clifford algebra $\mathfrak{Cl}$. As a result we get that the vertex algebra V is canonically isomorphic to the vertex algebra $V_{\mathbb{Z}}$ corresponding to the lattice $\mathbb{Z}$ constructed in Section 1.7. This statement is known as "boson–fermion correspondence".

Let us describe the image of the algebra $\hat{\mathfrak{W}}$ in V_0 in terms of the lattice construction. It could be easily proved that

$$p_0 = -\tilde{\alpha} \in \mathfrak{H} \subset V_0 \quad \text{and} \quad p_1 = \tfrac{1}{2}\tilde{\alpha}\, \boxed{-1}\, \tilde{\alpha} - \tfrac{1}{2} D\tilde{\alpha} = \omega - \tfrac{1}{2} D\tilde{\alpha}$$

(see Section 1.7) so that $p_1 \boxed{0}\, v = Dv$ for all $v \in V$. In general, for $n \ge 2$ we have

$$\begin{aligned} p_n = \frac{(-1)^n}{(n+1)!}\Bigg(\binom{n+1}{2}&(\cdots(\tilde{\alpha}\, \boxed{-2}\, \tilde{\alpha}) \overbrace{\boxed{-1}\, \tilde{\alpha} \cdots)}^{n-2 \text{ times}} \boxed{-1}\, \tilde{\alpha} \\ &- (\cdots(\tilde{\alpha} \overbrace{\boxed{-1}\, \tilde{\alpha}) \boxed{-1}\, \tilde{\alpha} \cdots) \boxed{-1}\, \tilde{\alpha}}^{n \text{ times}}\Bigg). \end{aligned} \tag{17}$$

3.3. *Inside the Heisenberg Vertex Algebra*

Here we investigate further the embedding $\hat{\mathfrak{W}} \subset V_0$ of the conformal algebra $\hat{\mathfrak{W}}$ into the vertex algebra V_0, constructed in Section 3.2.

THEOREM 1. *The conformal algebra $\hat{\mathfrak{W}} \subset V_0$ is a maximal conformal subalgebra of V_0.*

By Lemma 3, the space V_0 is a module over the Lie algebra $\widehat{\mathrm{gl}}_\infty$ and, in fact, over $\hat{M}$. For the proof of Theorem 1 we need to study the $\widehat{\mathrm{gl}}_\infty$-module structure of V_0. Recall that the Lie algebra $\widehat{\mathrm{gl}}_\infty$ has a structure quite similar to the structure of a Kac–Moody Lie algebra. Let $\mathfrak{d} \subset \widehat{\mathrm{gl}}_\infty$ be the maximal toral subalgebra of $\widehat{\mathrm{gl}}_\infty$ spanned by all diagonal matrices and c. Let $\mathfrak{d}' \subset \mathfrak{d}^*$ be the space of functionals on $\mathfrak{d}$ that take only finitely many nonzero values on E_{ii} for $i \in \mathbb{Z}$. Let $\lambda_i \in \mathfrak{d}'$ $(i \in \mathbb{Z})$ be the functional on $\mathfrak{d}$ such that $\lambda_i(E_{jj}) = \delta_{ij}$ and $\lambda_i(\mathsf{c}) = 0$, and let $\lambda_{\mathsf{c}} \in \mathfrak{d}'$ be such that $\lambda_{\mathsf{c}}(E_{jj}) = 0$ for all $j \in \mathbb{Z}$ and $\lambda_{\mathsf{c}}(\mathsf{c}) = 1$. The algebra $\widehat{\mathrm{gl}}_\infty$ is $\mathfrak{d}$-diagonalizable, and its root vectors are the elements E_{ij}, $i \neq j$, whose weights $\lambda_{ij} = \lambda_i - \lambda_j \in \mathfrak{d}'$ are called the roots of $\widehat{\mathrm{gl}}_\infty$. If $i < j$ then we call the root λ_{ij} positive; otherwise, we call it negative.

The element

$$\mathfrak{d} \ni H_{ij} = [E_{ij}, E_{ji}] = E_{ii} - E_{jj} + \begin{cases} \mathsf{c} & \text{if } j < 0 \text{ and } i \geq 0, \\ -\mathsf{c} & \text{if } j \geq 0 \text{ and } i < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

is called a *coroot*. Denote by $\Pi = \{\lambda \in \mathfrak{d}' \mid \lambda(H_{ij}) \in \mathbb{Z} \text{ for all coroots } H_{ij}\}$ the set of all *integral weights.*

Let U be a $\widehat{\mathrm{gl}}_\infty$-module. It is called $\mathfrak{d}$-*diagonalizable* if $U = \bigoplus_{\lambda\in\mathfrak{d}'} U_\lambda$, where $U_\lambda = \{v \in U \mid Hv = \lambda(H)v \ \forall H \in \mathfrak{d}\}$. The module U is called *integrable* if it is $\mathfrak{d}$-diagonalizable and if all E_{ij} for $i \neq j$ are locally nilpotent. Finally, U is called a *lowest weight* module with lowest weight $\lambda \in \mathfrak{d}'$ if (a) it is generated by a single vector $v \in U_\lambda$ such that $E_{ij}v = 0$ when $i > j$ and (b) for any $h \in \mathfrak{d}$ one has $hv = \lambda(h)v$. A lowest weight module U of lowest weight λ is integrable if and only if $\lambda \in \Pi$ and $\lambda(H_{ij}) \leq 0$ when $i < j$ (see [13, Chap. 10]). For a $\mathfrak{d}$-diagonalizable module U, denote by $\Xi(U) = \{\lambda \in \mathfrak{d}' \mid U_\lambda \neq 0\}$ the set of weights of U.

The $\widehat{\mathrm{gl}}_\infty$-module V_0 is an integrable irreducible lowest weight module generated by the lowest weight vector $v_0 = \mathbb{1}$ of weight λ_{c}. Using the general theory of integrable modules over Kac–Moody algebras [13], we can easily describe the set of weights $\Xi(V_0)$. Before doing so, we need a notion from combinatorics (see [24]).

Let $\kappa = (k_1 \geq k_2 \geq \cdots) \in \mathcal{P}(m)$ be a partition of an integer m and let $\kappa' = (k_1' \geq k_2' \geq \cdots) \in \mathcal{P}(m)$ be the dual partition (i.e., corresponding to the transposed Young diagram). Let $l = l(\kappa)$ be the number of rectangles at the main diagonal of the Young diagrams of κ and κ'. Then the pair of sequences

$$\xi = (k_1, k_2 - 1, k_3 - 2, \ldots, k_l - l + 1), \qquad \eta = (k_1' - 1, k_2' - 2, k_3' - 3, \ldots, k_l' - l)$$

are called Frobenius coordinates of κ. We have

$$\xi_1 > \xi_2 > \cdots > \xi_l > 0, \quad \eta_1 > \eta_2 > \cdots > \eta_l \geq 0, \quad \sum_{i=1}^{l} \xi_i + \sum_{i=1}^{l} \eta_i = m.$$

The Frobenius coordinates ξ, η of κ determine the partition κ uniquely. We will write $\kappa = \langle \xi | \eta \rangle$.

LEMMA 5. $\Xi(V_0) = \bigcup_{m\in\mathbb{Z}_+} \Xi_m$, *where*

$$\Xi_m = \left\{ \mu(\kappa) = \lambda_{\mathsf{c}} + \sum_{j=1}^{l(\kappa)} (\lambda_{-\xi_j} - \lambda_{\eta_j}) \;\middle|\; \kappa = \langle \xi | \eta \rangle \in \mathcal{P}(m) \right\}.$$

The homogeneous component $V_{0,m}$ of V_0 is decomposed into a direct sum of one-dimensional root spaces

$$V_{0,m} = \bigoplus_{\lambda\in\Xi_m} V_{0,\lambda}, \quad \dim V_{0,\lambda} = 1. \tag{19}$$

REMARK. Though this lemma could be proved using only the fact that V_0 is the irreducible lowest weight $\widehat{\mathrm{gl}}_\infty$-module of weight λ_{c}, in our case the proof is even simpler: We already know that $\dim V_{0,m} = P(m)$, so we only have to check that weights $\mu(\kappa)$ indeed appear in $\Xi(V_0)$.

We will write $l(\mu)$ instead of $l(\kappa)$ if $\mu = \mu(\kappa) \in \Xi(V_0)$ is the weight of V_0 corresponding to a partition $\kappa \in \mathcal{P}(m)$.

Next we study in greater detail the action of the elementary matrices $E_{ij} \in \widehat{\mathrm{gl}}_\infty$ on V_0. Recall [13, Chap. 9] that there is a contravariant form $(\cdot|\cdot)$ on V_0 such that $(\mathbb{1}|\mathbb{1}) = 1$, $(V_{0,m}|V_{0,n}) = 0$ for $m \neq n$, and $(Au|v) = (u|A^t v)$ for any $u, v \in V_0$ and $A \in \hat{M}$, where A^t is the transposed matrix of A; in particular, $\alpha(n)^t = \alpha(-n)$ for $\alpha(n) \in H$. The following lemma is proved along the same lines as the analogous result about integrable modules for Kac–Moody algebras (see [13, Chap. 10]).

LEMMA 6. *Let $u \in V_{0,\mu}$ be a homogeneous vector of weight $\mu \in \Xi(V_0)$. If* $\operatorname{sign}(i - j)\mu(H_{ij}) > 0$ *then* $0 \neq E_{ij}u \in V_{0,\mu+\lambda_i-\lambda_j}$ *and* $(E_{ij}u|E_{ij}u) = (u|u)$; *otherwise,* $E_{ij}u = 0$.

Here H_{ij} is given by (18). Note that, by Lemma 5, $\mu(E_{ij}) \in \{0, \pm 1\}$.

We now investigate the structure of V_0 as a $\hat{\mathfrak{W}}$-module. By definition (see Section 1.3), this is the same as the action of $\hat{W}_+ \oplus \Bbbk D$ on V_0. Since $p_1(0)$ acts as D, the action of $\hat{\mathfrak{W}}$ on V_0 amounts only to the action of the Lie algebra $W_+ = \hat{W}_+ = \Bbbk\langle p, t \mid [t, p] = 1\rangle^{(-)}$.

LEMMA 7. (a) *Any W_+-submodule of V_0 is homogeneous with respect to the root space decomposition* (19).

(b) *Let $v \in V_{0,\lambda}$ for some $\lambda \in \Xi(V_0)$. Then*

$$W_+ v = \bigoplus_{\mu\in\Xi(V_0),\, l(\mu)\le l(\lambda)} V_{0,\mu}.$$

Proof. Recall that V_0 is a module over the Lie algebra $\hat{M}$, which is a central extension of the Lie algebra M of infinite matrices with finitely many nonzero diagonals, and that $\hat{W}$ is embedded in $\hat{M}$ by formulas (16). Let M_0 be subalgebra of diagonal matrices of M. Since $\phi(M_0, M_0) = 0$, it follows that M_0 is a subalgebra of

$\hat{M}$. Any functional from $\mathfrak{d}'$ takes values on M_0 as well, so we have $\mathfrak{d}' \subset M_0^*$. Let $W_0 = \mathrm{Span}\{p_m(m) \mid m \in \mathbb{Z}\}$ be the subalgebra of W consisting of all elements of degree 0. We have $W_0 \subset M_0$ under the mapping (16). To prove (a) it is enough to show that any two different weights $\lambda, \mu \in \Xi(V_0)$ remain different after restriction to W_0. But this follows from the fact that λ_i for $i \in \mathbb{Z}$ are linearly independent on M_0 because $\lambda_i(p_m(m)) = (-1)^m \binom{i+m}{m}$.

For the proof of (b) we note that, by (16), every element $p^m t^n \in W_+$ is an infinite linear combination of E_{ij} such that either $i \geq 0$ or $j < 0$. Therefore, by Lemma 6 and (a) we get that if $\mu \in \Xi(V_0)$ then, for any $i, j \in \mathbb{Z}$ such that either $i \geq 0$ or $j < 0$, the weight $\mu + \lambda_i - \lambda_j$ appears in $W_+ V_\mu$. Every weight λ such that $l(\lambda) \leq l(\mu)$ could be obtained by a successive application of this operation, but no weight of length more than $l(\mu)$ can be obtained. □

In particular, all weights of $\hat{\mathfrak{W}} \subset V_0$ are of length 1.

Proof of Theorem 1. Let $v \in V_0 \setminus \hat{\mathfrak{W}}$. We have to prove that the conformal algebra generated by v and $\hat{\mathfrak{W}}$ is the whole V_0. By Lemma 7 we can assume that v is homogeneous of some weight $\mu \in \Xi(V_0)$ such that $l(\mu) > 1$. The only weight of degree 4 that is not in $\Xi(\hat{\mathfrak{W}})$ is $\lambda_{-2} + \lambda_{-1} - \lambda_0 - \lambda_1$ (which is of length 2) and hence, by Lemma 7(b), $V_{0,4} \subset W_+ v + \hat{\mathfrak{W}}$. It is thus enough to prove that V_0 is generated as a conformal algebra by $\hat{\mathfrak{W}}$ and any element $u \in V_{0,4} \setminus \hat{\mathfrak{W}}$. Take $u = \tilde{\alpha} \boxed{-3}\, \tilde{\alpha}$.

Let $\mathfrak{L} \subset V_0$ be the conformal algebra generated by $\hat{\mathfrak{W}}$ and u. Assume on the contrary that $\mathfrak{L} \subsetneq V_0$. Then Lemma 7(b) implies that there is an integer $l \geq 3$ such that $\Xi(\mathfrak{L})$ does not contain weights of length l. Let l be the minimal possible among such integers. Let

$$\mu = \lambda_{-l} + \cdots + \lambda_{-3} + \lambda_{-2} - \lambda_2 - \lambda_3 - \cdots - \lambda_l + \lambda_{\mathsf{c}} \in \Xi(\mathfrak{L}),$$

$l(\mu) = l - 1$, and let $v \in \mathfrak{L}_\mu \subset V_{0,\mu}$. Let $\mu' = \mu + \lambda_{-1} - \lambda_0$ so that $l(\mu') = l$ and hence $\mu' \notin \Xi(\mathfrak{L})$. We prove that the projection of $u \boxed{2}\, v$ onto $V_{\mu'}$ is nonzero and hence $\mathfrak{L}_{\mu'} \neq 0$, arriving at a contradiction.

Using that $\tilde{\alpha}(k) = -t^k = -\sum_{i \in \mathbb{Z}} E_{i,i-k}$ and (7) yields

$$(\tilde{\alpha} \boxed{-3}\, \tilde{\alpha}) \boxed{2}\, v = \sum_{s \geq 0} \left(\binom{s}{2} + \binom{s+2}{2} \right) \sum_{i,j \in \mathbb{Z}} E_{i,i+s+1} E_{j,j-s} v.$$

The only pairs of E_{ij} in this expression that do not kill v and move v to $V_{\mu'}$ are $E_{-1,i} E_{i,0}$ for $2 \leq i \leq l$, $E_{i,0} E_{-1,i}$ for $-l \leq i \leq -2$, and $E_{-1,0} E_{i,i}$ for either $2 \leq i \leq l$ or $-l \leq i \leq -2$. Let $w = E_{-1,2} E_{2,0} v$. Using Lemma 6, we find that $E_{-1,i} E_{i,0} v = w$ for all $2 \leq i \leq l$ and $E_{i,0} E_{-1,i} v = -w$ for all $-l \leq i \leq -2$. Also, $E_{-1,0} E_{i,i} v = -E_{-1,0} E_{-i,-i} v$. Summing up:

$$\begin{aligned} \mathrm{Pr}_{V_{\mu'}}(\tilde{\alpha} \boxed{-3}\, \tilde{\alpha}) \boxed{2}\, v &= \sum_{s=2}^{l} \left(\binom{s}{2} + \binom{s+2}{2} \right) w - \sum_{s=1}^{l-1} \left(\binom{s}{2} + \binom{s+2}{2} \right) w \\ &= \left(\binom{l}{2} + \binom{l+2}{2} - 3 \right) x \neq 0. \end{aligned}$$

□

3.4. Representation Theory of $\hat{\mathfrak{W}}$

The technique of Section 3.3 allows us to investigate the $\hat{\mathfrak{W}}$-module structure of other spaces V_λ. This section diverges from the main exposition and will not be used later in this paper. We assume here that $\Bbbk$ is an algebraically closed field of characteristic 0.

Let $\Lambda = \mathbb{Z}\beta$ be a lattice of rank 1 generated by a single vector β. As before, let $\mathfrak{h} = \Lambda \otimes \Bbbk$. Let $V = V_\Lambda = \bigoplus_{i\in\mathbb{Z}} V_{i\beta}$ be the corresponding vertex algebra constructed in Section 1.7. Take $\alpha = \beta/\sqrt{(\beta|\beta)} \in \mathfrak{h}$. Then $(\alpha|\alpha) = 1$, and the vertex operator $\tilde{\alpha} = \alpha(-1)\mathbb{1} \in V_0$ generates a copy of the Heisenberg conformal algebra $\mathfrak{H}$ and also gives an embedding of $\hat{\mathfrak{W}}$ in V_0 by formulas (17). So $V_{i\beta}$ becomes a module over $\hat{\mathfrak{W}}$ and over W_+.

If $(\beta|\beta) \neq 1$ then it is not difficult to see that $V_{i\beta}$ is an irreducible W_+-module. In fact, in this case we have $W_+ v_{i\beta} = V_{i\beta}$. If $(\beta|\beta) = 1$ (in this case $\Lambda = \mathbb{Z}$) then, as in Section 3.3, we have an action of $\widehat{\mathrm{gl}}_\infty$ on $V_{i\beta} = V_i$. The module V_i is an irreducible integrable lowest weight $\widehat{\mathrm{gl}}_\infty$-module of the lowest weight $\lambda_{\mathsf{c}} - \lambda_0 - \lambda_1 - \cdots - \lambda_{i-1}$ if $i \geq 0$ and $\lambda_{\mathsf{c}} + \lambda_{-1} + \cdots + \lambda_i$ if $i < 0$.

We need to introduce the biased Frobenius coordinates of a partition. Let $\kappa = (k_1 \geq k_2 \geq \cdots) \in \mathcal{P}(m)$ be a partition of an integer m and let $\kappa' = (k_1' \geq k_2' \geq \cdots) \in \mathcal{P}(m)$ be the dual partition. Denote by $l_i(\kappa)$ the number of squares on the ith diagonal of the Young diagram of κ, so that $l_i(\kappa) = l_{-i}(\kappa')$. The biased Frobenius coordinates of κ are sequences

$$\begin{aligned} \xi &= (k_1 - i, k_2 - i - 1, \ldots, k_{l_i} - i - l_i + 1), \\ \eta &= (k_1' + i - 1, k_2' + i - 2, \ldots, k'_{l_i+i} - l_i). \end{aligned}$$

They determine κ uniquely, and we will write $\kappa = \langle \xi|\eta\rangle_i$.

Let $\Xi(V_i) \subset \mathfrak{d}'$ be the set of weights of the module V_i. In analogy with Lemma 5 we have that $\Xi(V_i) = \bigcup_{m\in\mathbb{Z}_+} \Xi_m(V_i)$, where

$$\Xi_m(V_i) = \left\{ \mu_i(\kappa) = \lambda_{\mathsf{c}} + \sum_{j=1}^{l_i(\kappa)} \lambda_{-\xi_j} - \sum_{j=1}^{l_i(\kappa)+i} \lambda_{\eta_j} \;\middle|\; \kappa = \langle \xi|\eta\rangle_i \in \mathcal{P}(m) \right\}$$

is the set of weights appearing in the homogeneous component of V_i consisting of elements of degree $m + \frac{1}{2}i^2$. As before, we have $\dim V_{i,\mu} = 1$ for every $\mu \in \Xi(V_i)$.

Both statements (a) and (b) of Lemma 7 remain without changes. In particular, dimensions of homogeneous components of W_+-submodules of V_i grow polynomially.

QUESTION. Is it true that all lowest weight W_+-modules of polynomial growth are obtained in this way?

4. Conformal Subalgebras of Lattice Vertex Algebras

4.1. Conformal Subalgebras of Lattice Vertex Superalgebras

Let Λ be an integer lattice and let $V_\Lambda = \bigoplus_{\lambda\in\Lambda} V_\lambda$ be the lattice vertex superalgebra constructed in Section 1.7. Recall that V_0 contains the Heisenberg conformal algebra $\mathfrak{H}$, which is spanned over $\Bbbk[D]$ by elements of the form $\tilde{h}$ for $h \in \mathfrak{h} = \Bbbk \otimes \Lambda$ and $\mathbb{1}$. The left regular action of this element is given by $Y(\tilde{h}) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$, where the operators $h(n)$ define a representation of the Heisenberg algebra H on V_Λ.

Let $\mathfrak{L} \subset V_\Lambda$ be a conformal subalgebra of V_Λ. Assume that $\mathfrak{L}$ is homogeneous with respect to the grading by Λ; that is, $\mathfrak{L} = \bigoplus_{\lambda \in \Lambda} \mathfrak{L}_\lambda$, where $\mathfrak{L}_\lambda = \mathfrak{L} \cap V_\lambda$. The set $\Delta = \{\lambda \in \Lambda \setminus 0 \mid \mathfrak{L}_\lambda \neq 0\}$ is called the *root system* of $\mathfrak{L}$. We will always assume that $\Delta = -\Delta$. This happens, for example, if $\mathfrak{L}$ is closed under the involution $\sigma : V_\Lambda \to V_\Lambda$, corresponding to the automorphism $\lambda \mapsto -\lambda$ of the lattice Λ.

We will also assume that $\mathfrak{L}$ is closed under the action of the conformal algebra $\mathfrak{H}$. In this case it is easy to show that $\mathfrak{L}$ must contain the elements v_λ for each $\lambda \in \Delta$. Therefore, the Lie conformal superalgebra $\mathfrak{L}' \subset V_\Lambda$ generated by the set $\{v_\lambda \mid \lambda \in \Delta\}$ will be a subalgebra of $\mathfrak{L}$ and will be graded by the same root system. It follows from formula (9) that, for the products in V_Λ, if $\alpha, \beta \in \Delta$ are such that $k = (\alpha|\beta) < 0$ then $v_\alpha \boxed{-k-1}\, v_\beta = \pm v_{\alpha+\beta} \in \mathfrak{L}'$ and hence $\alpha + \beta \in \Delta$.

The following proposition summarizes properties of root systems.

PROPOSITION 1. (a) *A set $\Delta \subset \Lambda$ is a root system if and only if the Lie conformal superalgebra generated by the set $\{v_\lambda \mid \lambda \in \Delta\}$ does not contain any homogeneous components other than $\mathfrak{L}_\lambda$ for $\lambda \in \Delta$ and $\mathfrak{L}_0$.*

(b) *If $\Delta \subset \Lambda$ is a root system, then Δ is closed under the negation $\lambda \mapsto -\lambda$ and under the partial summation*:

$$\alpha, \beta \in \Delta,\ (\alpha|\beta) < 0 \implies \alpha + \beta \in \Delta. \tag{20}$$

We are mostly interested in the case when the root system Δ is finite. If Δ contains a vector λ such that $(\lambda|\lambda) < 0$, then by Proposition 1(b) we have $k\lambda \in \Delta$ for all $k = 1, 2, \ldots$. The following lemma suggests that we should restrict ourselves to the case when the form $(\cdot|\cdot)$ is semi–positive definite.

LEMMA 8. *Let $\Delta \in \Lambda$ be a set closed under the partial summation* (20) *such that $\Delta = -\Delta$ and* $\mathrm{Span}_{\mathbb{Z}} \Delta = \Lambda$.

(a) *Assume that the form $(\cdot|\cdot)$ is not semi–positive definite*; *that is, assume there exists an $\alpha \in \Lambda$ such that $(\alpha|\alpha) < 0$. Then there is some $\delta \in \Delta$ such that $(\delta|\delta) < 0$.*

(b) *Assume that the form $(\cdot|\cdot)$ is semi–positive definite but not positive definite. Then there exists some $\delta \in \Delta$ such that $(\delta|\delta) = 0$.*

Proof. Both statements are proved by a standard argument. We shall prove (a); the proof of (b) is identical. Assume on the contrary that $(\lambda|\lambda) \geq 0$ for every $\lambda \in \Delta$. Let $\alpha \in \Lambda$ be such that $(\alpha|\alpha) < 0$. Then α could be expressed as a linear combination $\alpha = k_1\alpha_1 + \cdots + k_l\alpha_l$ of elements $\alpha_i \in \Delta$ with integer coefficients. Since $\Delta = -\Delta$, we can assume that all $k_i > 0$. Assume that this combination has minimal $\sum_i k_i$ among all representations of α as a linear combination of the elements of Δ. Then, since $(\alpha_i|\alpha_i) \geq 0$ for all i and since $(\alpha|\alpha) < 0$, we must have $(\alpha_i|\alpha_j) < 0$ for some $i \neq j$. But then $\alpha_i + \alpha_j \in \Delta$, and we could make the expression for α with a smaller sum of coefficients. $\square$

The sections that follow will be dedicated to the classification of finite root systems. When the bilinear form has a nontrivial kernel, we allow for a bigger class of root systems. Let $\Lambda_0 = \{\lambda \in \Lambda \mid (\lambda|\Lambda) = 0\} \subset \Lambda$ be the sublattice of isotropic vectors. Let $\bar{\Lambda} = \Lambda/\Lambda_0$ be the positive definite quotient. We will denote the projection of an element $\lambda \in \Lambda$ to $\bar{\Lambda}$ by $\bar{\lambda}$. A set $\Sigma \in \Lambda$ is called *almost finite* if $\bar{\Sigma} \subset \bar{\Lambda}$ is finite.

We will adopt the following notation. Let Λ be a semi–positive definite integer lattice and let Λ_0 and $\bar{\Lambda}$ be as just described. Let $\Delta \subset \Lambda$ be a root system. Denote by $\Delta_0 = \Delta \cap \Lambda_0$ the set of isotropic roots and by $\Delta^\times = \Delta \setminus \Delta_0$ the set of all real roots. Let $\bar{\Delta} = (\Delta/\Lambda_0) \setminus \{0\}$ be the positive definite root system in $\bar{\Lambda}$ obtained from the projection of Δ to $\bar{\Lambda}$.

We also need the following definition. A root system $\Delta \subset \Lambda$ is called *decomposable* if it can be represented as a disjoint union $\Delta = \Delta_1 \sqcup \Delta_2$ such that, for any $\alpha \in \Delta_1$ and $\beta \in \Delta_2$, we have $\alpha + \beta \notin \Delta$. Otherwise Δ is called *indecomposable*. By Proposition 1(b), if $\Delta = \Delta_1 \sqcup \Delta_2$ is decomposable then $(\Delta_1|\Delta_2) = 0$ and the Lie conformal superalgebra $\mathfrak{L}$ splits into a direct sum $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ such that $[\mathfrak{L}_1, \mathfrak{L}_2] = 0$. In the other direction, however, if $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ so that $[\mathfrak{L}_1, \mathfrak{L}_2] = 0$ and if we set Δ_i to be the root system of $\mathfrak{L}_i$, then even though $(\Delta_1|\Delta_2) = 0$, in general we have $(\Delta_1 + \Delta_2) \cap \Delta \neq \emptyset$ unless the form is positive definite.

4.2. Rank-1 Case

Let $\alpha \in \Lambda$ be a vector in an integer lattice Λ. Let $\mathfrak{L} \subset V_\Lambda$ be the conformal superalgebra generated by v_α and $v_{-\alpha}$ in the lattice vertex superalgebra V_Λ (constructed in Section 1.7). The algebra $\mathfrak{L} = \bigoplus_{\lambda\in\Lambda} \mathfrak{L}_\lambda$ is graded by the lattice Λ. We start with determining all cases when $\mathfrak{L}$ does not contain any homogeneous components other than $\mathfrak{L}_{-\alpha} \ni v_{-\alpha}$, $\mathfrak{L}_0 \ni \mathbb{1}$, and $\mathfrak{L}_\alpha \ni v_\alpha$.

Clearly, if $(\alpha|\alpha) < 0$ then we can take $n = -(\alpha|\alpha) - 1 \geq 0$ and get $v_\alpha \boxed{n}\, v_{-\alpha} = v_{2\alpha} \in \mathfrak{L}$. Hence all $v_{j\alpha} \in \mathfrak{L}$ for $j \in \mathbb{Z}$. Also, if $(\alpha|\alpha) = 0$ then all products in $\mathfrak{L}$ are zero, so this case is not interesting. Therefore, without a loss of generality we assume that $(\alpha|\alpha) > 0$.

PROPOSITION 2. *If* $(\alpha|\alpha) = 1$ *then* $\mathfrak{L} = \mathfrak{Cl}$ *is the Clifford conformal superalgebra.*

If $(\alpha|\alpha) = 2$ *then* $\mathfrak{L} = \mathfrak{Aff}(\mathrm{sl}_2)$ *is the affine conformal algebra* $\widehat{\mathrm{sl}}_2$.

If $(\alpha|\alpha) = 3$ *then* $\mathfrak{L}$ *is the central extension of the* $N = 2$ *simple conformal superalgebra.*

If $(\alpha|\alpha) = 4$ *then* $\mathfrak{L}$ *is the conformal algebra* $\hat{\mathfrak{K}}$ *constructed in Section 2.3.*

Finally, if $(\alpha|\alpha) \geq 5$ *then* $\mathfrak{L} = V_{\mathbb{Z}\alpha}$.

Proof. First we show that if $(\alpha|\alpha) = 1, 2, 3,$ or 4 then the conformal algebra $\mathfrak{L} \subset V_\Lambda$ generated by v_α and $v_{-\alpha}$ is as claimed.

As was remarked at the end of Section 1.7, all calculations in vertex algebra V_Λ are very explicit. So we just have to read off the defining relations of conformal superalgebras from the formula (9) for the products in V_Λ. Of course, the case when $(\alpha|\alpha)$ is 1 or 2 is well-known. Let us do the most difficult case when $(\alpha|\alpha) = 4$. In this case we can identify $\mathbb{Z}\alpha$ with $2\mathbb{Z} \subset \mathbb{Z}$ by letting $\alpha = 2$.

By Lemma 4, the elements $p_m = v_{-1} \boxed{-m-1}\, v_1 \in V_0$ and $\mathbb{1}$ span a copy of $\hat{\mathfrak{W}}$ over $\Bbbk[D]$ such that the products are given by (4) and (6). Set $u_m = v_{-1} \boxed{-m-1}\, v_{-1} \in V_{-2}$ for $m \geq 1$. Recall that we then have an involution $\sigma : V_\Lambda \to V_\Lambda$ induced by the involution $\lambda \mapsto -\lambda$ of the lattice Λ, so that $\sigma(u_m) = v_1 \boxed{-m-1}\, v_1 \in V_2$. We have to show that the formulas (10)–(14) hold for p_m, u_m, that $\sigma(u_m) \in V_{\mathbb{Z}}$ and $\mathsf{c} = \mathbb{1}$, and also that $u_m \boxed{k}\, u_n = \sigma(u_m) \boxed{k}\, \sigma(u_n) = 0$ for all integer $m, n \geq 1$ and $k \geq 0$.

Let us, for example, check (12). First we note that

$$\begin{aligned} v_{-1} \boxed{i}\, (v_1 \boxed{-n-1}\, v_1) &= -v_1 \boxed{-n-1}\, (v_{-1} \boxed{i}\, v_1) + [v_{-1}(s), v_1(-n-1)]v_1 \\ &= -\delta_{i,0} v_1(-n-1)\mathbb{1} + \mathbb{1}(i-n-1)v_1 \\ &= -\delta_{i,0} \frac{1}{n!} D^n v_1 + \delta_{i,n} v_1. \end{aligned}$$

Using this, we calculate

$$\begin{aligned} p_m \boxed{k}\, \sigma(u_n) &= (v_{-1} \boxed{-m-1}\, v_1) \boxed{k}\, (v_1 \boxed{-n-1}\, v_1) \\ &= \sum_{s \leq -m-1} \binom{-s-1}{m} v_{-1}(s) v_1(k-m-1-s) v_1(-n-1) v_1 \\ &\quad - (-1)^m \sum_{s \geq 0} \binom{m+s}{m} v_{-1}(k-m-1-s) v_{-1}(s) v_1(-n-1) v_1. \end{aligned}$$

The first sum here is zero because $k - m - 1 - s \geq 0$, hence $v_1(k-m-1-s)$ commutes with $v_1(-n-1)$ and $v_1(k-m-1-s)v_1 = v_1 \boxed{k-m-1-s}\, v_1 = 0$. The second sum gives

$$\frac{(-1)^m}{n!} v_1 \boxed{k-m-1}\, (D^n v_1) - (-1)^m \binom{m+n}{m} v_1 \boxed{k-m-1-n}\, v_1,$$

and (12) follows. The other formulas are checked in the same way. Instead of checking (10) directly, we note that by Section 3.4 the $\hat{\mathfrak{W}}$-modules $U(W_+)v_{\pm 2}$ and $\mathfrak{J}$ are both irreducible highest weight W_+-modules corresponding to the same highest weight; hence they must be isomorphic.

The verification of the conformal cocycle formula (14) is done in the same way.

We are left to show that if $n = (\alpha|\alpha) \geq 5$ then $\mathfrak{L} = V_\Lambda$. Recall from Section 3.2 that we have an embedding $\hat{\mathfrak{W}} \subset V_0$ given by (17) such that

$$\hat{\mathfrak{W}}_i = \operatorname{Span}_\Bbbk\{p_{i-1}, Dp_{i-2}, \ldots, D^{i-1}p_0\} \subset V_{0i}$$

for $i \geq 1$, so that $\hat{\mathfrak{W}}_i = V_{0i}$ for $0 \leq i \leq 3$. Simple calculations now show that $v_\alpha \boxed{n-i}\, v_\alpha \in \hat{\mathfrak{W}}_{i-1}$ for $1 \leq i \leq 4$ and also $\{\mathbb{1}, p_0, p_1, p_2\} \subset \operatorname{Span}_\Bbbk\{v_{-\alpha} \boxed{n-i}\, v_\alpha \mid 1 \leq i \leq 4\}$. Since $\hat{\mathfrak{W}}$ is generated by $\{\mathbb{1}, p_0, p_1, p_2\}$, we have that $\hat{\mathfrak{W}} \subset \mathfrak{L}$. However, $v_\alpha \boxed{n-5}\, v_\alpha \in V_{04} \setminus \hat{\mathfrak{W}}_4$ and so, since $\hat{\mathfrak{W}}$ is a maximal conformal subalgebra in V_0 by Theorem 1, we must have $\mathfrak{L} = V_0$. □

It follows that all possible finite root systems Δ of rank 1 are from the following list:

A_1: $\Delta = \{\pm\alpha\}$, $(\alpha|\alpha) = 2$;
B_1: $\Delta = \{\pm\alpha\}$, $(\alpha|\alpha) = 1$;
C_1: $\Delta = \{\pm\alpha\}$, $(\alpha|\alpha) = 4$;
BC_1: $\Delta = \{\pm\alpha, \pm 2\alpha\}$, $(\alpha|\alpha) = 1$;
B_1': $\Delta = \{\pm\alpha\}$, $(\alpha|\alpha) = 3$.

We will generalize this result for the case of a higher-ranking integer positive definite lattice Λ and finite root system $\Delta \in \Lambda$ in Section 4.4 (see Theorem 2).

Of special interest is the case when we take the root system $\Delta = \{\pm 1, \pm 2\} \subset \mathbb{Z}$ to be of type BC_1. Then the conformal superalgebra $\mathfrak{L} \subset V_{\mathbb{Z}}$ generated by the set $\{v_{\pm 1}, v_{\pm 2}\}$ is isomorphic to an extension of $\mathfrak{K}$ by the Clifford conformal superalgebra $\mathfrak{Cl}$ such that $\mathfrak{L}^{\bar{0}} = \mathfrak{L}_{-2} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_2 = \hat{\mathfrak{K}}$ and $\mathfrak{L}_{-1} \oplus \mathfrak{L}_1 \oplus \mathbb{1} = \mathfrak{Cl}$.

We also remark that $\mathfrak{L}$ and $\mathfrak{K}$ are maximal among conformal subalgebras of $V_{\mathbb{Z}}$ graded by the corresponding root system.

4.3. Rank-2 Case

Consider now two vectors $\alpha, \beta \in \Lambda$, where Λ is an integer lattice as before. Let V_Λ be the vertex superalgebra corresponding to Λ, and let $\mathfrak{L} \subset V_\Lambda$ be the conformal superalgebra generated by $v_{\pm\alpha}$ and $v_{\pm\beta}$. Since the generators of $\mathfrak{L}$ are homogeneous, $\mathfrak{L} = \bigoplus_{\lambda\in\Lambda} \mathfrak{L}_\lambda$ is graded by Λ. Let $\Delta = \{\lambda \in \Lambda \setminus \{0\} \mid \mathfrak{L}_\lambda \neq 0\}$ be the root system of $\mathfrak{L}$. If $(\alpha|\beta) = 0$ then $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ is decomposed into a direct sum of ideals ($\mathfrak{L}_1 = \langle v_{\pm\alpha}\rangle$ and $\mathfrak{L}_2 = \langle v_{\pm\beta}\rangle$) and $\Delta = \Delta_1 \sqcup \Delta_2$ for $\Delta_1 \subset \mathbb{Z}\alpha$ and $\Delta_2 \subset \mathbb{Z}\beta$, so everything is reduced to the case of Section 4.2. Therefore, without a loss of generality we assume that $(\alpha|\beta) < 0$.

Now we formulate an analogue of Proposition 2 for the case of two vectors.

PROPOSITION 3. *Let $\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\beta$ be an integer lattice of rank* 2. *Assume that the conformal superalgebra $\mathfrak{L} \subset V_\Lambda$ generated by $v_{\pm\alpha}$ and $v_{\pm\beta}$ is graded by a finite or an almost finite root system $\Delta \subset \Lambda$. Then there are only the following possibilities*:

(i) $(\alpha|\alpha) = (\beta|\beta) = 2$, $(\alpha|\beta) = -1$;
(ii) $(\alpha|\alpha) = 2$, $(\beta|\beta) = 1$, $(\alpha|\beta) = -1$;
(iii) $(\alpha|\alpha) = 4$, $(\beta|\beta) = 2$, $(\alpha|\beta) = -2$;
(iv) $(\alpha|\alpha) = (\beta|\beta) = 1$, $(\alpha|\beta) = -1$;
(v) $(\alpha|\alpha) = (\beta|\beta) = 2$, $(\alpha|\beta) = -2$;
(vi) $(\alpha|\alpha) = (\beta|\beta) = 3$, $(\alpha|\beta) = -3$;
(vii) $(\alpha|\alpha) = (\beta|\beta) = 4$, $(\alpha|\beta) = -4$;
(viii) $(\alpha|\alpha) = 4$, $(\beta|\beta) = 1$, $(\alpha|\beta) = -2$.

In cases (i)–(iii), *Λ is positive definite. In* (iv)–(viii), *Λ is semi–positive definite with the kernel of the bilinear form spanned by single vector δ, given by $\delta = \alpha + \beta$ in cases* (iv)–(vii) *and by $\delta = \alpha + 2\beta$ in case* (viii). *The root system is*:

$$\Delta = \begin{cases} \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\} & \textit{in cases } \text{(i), (ii), } \textit{and } \text{(iv)}; \\ \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta)\} & \textit{in case } \text{(iii)}; \\ \{k\delta, \pm\alpha + k\delta, \pm\beta + k\delta \mid k \in \mathbb{Z}\} & \textit{in cases } \text{(v)–(viii)}. \end{cases}$$

If $(\lambda|\lambda) = 1$ *or* 2 *for* $\lambda \in \Delta$, *then* $\mathfrak{L}_\lambda \cong \Bbbk[D]v_\lambda$; *if* $(\lambda|\lambda) = 3$ *or* 4 *then* $\mathrm{rk}_{\Bbbk[D]}\mathfrak{L}_\lambda = \infty$. *For an isotropic* $\lambda \in \Delta$ *we have that* $\mathrm{rk}_{\Bbbk[D]}\mathfrak{L}_\lambda = 1, 2, 3, \infty, \infty$ *in the cases* (iv)–(viii), *respectively. Finally,* $\mathrm{rk}_{\Bbbk[D]}\mathfrak{L}_0 = 2, 1, \infty, 0, 2, 4, \infty, \infty$ *in the cases* (i)–(viii), *respectively.*

Proof. By Proposition 2, the square lengths of α and β could be only 1, 2, 3 or 4. Also, $(\alpha|\beta) \geq -\frac{1}{2}((\alpha|\alpha) + (\beta|\beta))$, because otherwise we would have $(\alpha+\beta|\alpha+\beta) < 0$. So we have only finitely many possibilities. One can easily check using formulas (9) that every choice of vectors α and β *not* listed in (i)–(viii) will give an infinite root system. So it remains to show that in each of the cases (i)–(viii) the conformal algebra $\mathfrak{L}$ generated by $\{v_{\pm\alpha}, v_{\pm\beta}\}$ will, in fact, be graded by the corresponding root system Δ. Let us check this for the most difficult case (iii). In this case $\mathfrak{L}_0 = \hat{\mathfrak{W}}_1 \oplus \hat{\mathfrak{W}}_2$ is a direct sum of two copies of $\hat{\mathfrak{W}}$, corresponding to vectors $\frac{1}{2}\alpha$ and $\frac{1}{2}\alpha + \beta$, so that the subalgebras $\mathfrak{L}_\alpha \oplus \hat{\mathfrak{W}}_1 \oplus \mathfrak{L}_{-\alpha}$ and $\mathfrak{L}_{\alpha+2\beta} \oplus \hat{\mathfrak{W}}_2 \oplus \mathfrak{L}_{-\alpha-2\beta}$ of $\mathfrak{L}$ are isomorphic to the conformal algebra $\hat{\mathfrak{K}}$ constructed in Section 2. Then the claim follows from Proposition 2. □

4.4. *Case When the Bilinear Form is Positive Definite*

THEOREM 2. *Assume* Λ *is a positive definite integer lattice and* $\Delta \subset \Lambda$ *is a finite indecomposable root system of some conformal superalgebra* $\mathfrak{L} = \bigoplus_{\lambda\in\Delta}\mathfrak{L}_\lambda \subset V_\Lambda$ *such that* $\Delta = -\Delta$ *and* $\mathfrak{L}$ *is generated by the set* $\{v_\lambda \mid \lambda \in \Delta\}$. *Then there are only the following possibilities.*

- *A-D-E*: Δ *is a simply laced finite Cartan root system of type* A_n $(n \geq 1)$, D_n $(n \geq 4)$, *or* E_n $(n = 6, 7, 8)$ *such that* $(\lambda|\lambda) = 2$ *for all roots* $\lambda \in \Delta$.
- *B*: Δ *is a finite Cartan root system of type* B_n $(n \geq 1)$; *the short roots have square length* 1 *and the long roots have square length* 2. (*When* $n = 1$, $\Delta = \{\pm 1\} \subset \Lambda = \mathbb{Z}$.)
- *C*: Δ *is a finite Cartan root system of type* C_n $(n \geq 1)$; *the short roots have square length* 2 *and the long roots have square length* 4. (*When* $n = 1$, $\Delta = \{\pm 2\} \subset \Lambda = \mathbb{Z}$.)
- *BC*: Δ *is the union of* B_n *and* C_n *for* $n \geq 1$.
- B^0: Δ *is a subset of* B_n *consisting of all the short roots of* B_n *and half of the long roots*: *If* $\alpha_1, \ldots, \alpha_n$ *is the basis of* Λ *consisting of short roots, so that* $(\alpha_1|\alpha_j) = 0$, *then all the long roots of* Δ *are of the form* $\alpha_i - \alpha_j$ $(i \neq j)$.
- B_1': $\Delta = \{\pm 3\} \subset \Lambda = \mathbb{Z}$.

We will call a vector of square length 1 *short,* of square length 2 *long,* and of square length 4 *extra-long.*

Proof. Let $\alpha, \beta \in \Delta$ be a pair of roots. The root system that they generate must be of one of the types (i)–(iii) from Proposition 3. It follows that the Cartan number $\langle\alpha, \beta\rangle = 2(\alpha|\beta)/(\alpha|\alpha)$ is an integer, hence Δ must lay inside a Cartan root system Φ. Moreover, it follows from the structure of root systems of rank 2 in cases (i)–(iii) that Φ cannot be of type F_4 or G_2, and the condition for the square lengths of the root vectors holds.

Next we want to prove that if Δ is a finite Cartan root system of a type other than G_2 and F_4 and if the length of roots are as prescribed by the theorem, then Δ is indeed the root system of the conformal superalgebra $\mathfrak{L} \subset V_\Lambda$ generated by the set $\{v_\lambda \mid \lambda \in \Delta\}$. If Δ is a simply laced root system of type A-D-E then $\mathfrak{L}$ is the affine Kac–Moody conformal algebra, as is well known. If Δ is of type B then it is equally easy to check that the space $\mathfrak{L} = \bigoplus_{\alpha\in\Delta} \Bbbk[D]v_\alpha \oplus \mathfrak{H} \subset V_\Lambda$ will be closed under the products (9), hence it is the desired subalgebra.

Let Δ be of type C_n. Let $\alpha_1, \dots, \alpha_n \in \Delta$ be the basis of Λ consisting of pairwise orthogonal extra-long roots. Take $\mathfrak{L}_0 = \hat{\mathfrak{W}}_1 \oplus \cdots \oplus \hat{\mathfrak{W}}_n \subset V_0$, where $\hat{\mathfrak{W}}_i$ is the Weyl conformal algebra spanned by $v_{\alpha_i} \boxed{n}\, v_{-\alpha_i}$ $(n \in \mathbb{Z})$; see Lemma 4. Take $\mathfrak{L}_{\alpha_i}$ to be equal to the conformal Jordan triple system constructed in Section 2, so that the subalgebra $\mathfrak{L}_{-\alpha_i} \oplus \hat{\mathfrak{W}}_i \oplus \mathfrak{L}_{\alpha_i}$ is isomorphic to the conformal algebra $\mathfrak{K}$. If $\beta \in \Delta$ is a short root then take $\mathfrak{L}_\beta = \Bbbk[D]v_\beta$. Using calculations of Proposition 3, it is easy to see that $\mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{L}_\alpha$ is closed under the products (9).

Finally, if Δ is of type BC then the corresponding conformal superalgebra $\mathfrak{L}$ is easily obtained by combining the conformal superalgebras corresponding to the subsystems of Δ of types B and C.

So let Δ be a finite root system, and let $\Phi \supset \Delta$ be a minimal Cartan root system containing Δ. We prove that if Φ is either simply laced or is of type C or BC then $\Delta = \Phi$.

Assume first that Φ is simply laced. Let $\alpha \in \Phi \setminus \Delta$. Since Δ spans Λ over $\mathbb{Z}$, we can write α as a linear combination of elements of Δ with integer coefficients. Let $\alpha = \sum_i k_i\alpha_i$ $(k_i \in \mathbb{Z}, \alpha_i \in \Delta)$ be such a linear combination of the minimal length. Because $\Delta = -\Delta$, we can assume that all $k_i > 0$. But then, since $(\alpha|\alpha) = (\alpha_i|\alpha_i) = 2$, we must have $(\alpha_i|\alpha_j) < 0$ for some pair $\alpha_i \neq \alpha_j$; hence $\alpha_i + \alpha_j \in \Delta$ and we can make the combination shorter, contrary to our assumption. Thus, $\Delta = \Phi$. The same argument shows that if Φ is of type B or BC then Δ contains all short roots, and if Φ is of type C then Δ contains all long roots.

Let Φ be of type C. Then Δ contains all long roots of Φ. Since Φ is a minimal Cartan root system containing Δ, the latter must contain at least one extra-long root α. Let β be a long root such that $(\alpha|\beta) = -2$. Then, by Proposition 3(iii), Δ contains the whole root system of type C_2 generated by α and β, and so the extra-long root $\alpha + 2\beta$ also belongs to Δ. Continuing this argument, we get that all extra-long roots lay in Δ and hence $\Delta = \Phi$.

Assume now that Φ is of type B_n. When $n = 1$ or 2 we refer to Proposition 2 and Proposition 3, so assume that $n \geq 3$. Let $\alpha_1, \dots, \alpha_n \in \Delta \subset \Phi$ be a basis of Λ consisting of pairwise orthogonal short roots. Let Φ_l be the set of all long roots in Φ; they form a simply laced Cartan root system of type D_n (A_3 if $n = 3$). The root system Δ must contain some long roots, too. Let $\Delta_l \subset \Phi_l$ be the set of long roots of Δ; they must form a root system as well. It is not too difficult to see that, in order for Δ to be indecomposable, the root system Δ_l must be equal either to the whole Φ_l or to $\{\alpha_i - \alpha_j \mid i \neq j\}$, in which case Δ_l is of type A_{n-1}. This choice of Δ_l is unique up to the action of the Weyl group. Therefore, Δ is either equal to Φ or is of type B_n^0.

Finally, let Φ be of type BC_n. Then, by the result of the previous paragraph, the set Δ_l of long roots must at least contain the set $\{\alpha_i - \alpha_j \mid i \neq j\}$. Also, Δ must contain at least one extra-long root. So, as in the case of type C, using (iii) of Proposition 3 yields $\Delta = \Phi$. $\square$

If the root system Δ is of type A-D-E, then the corresponding conformal algebra $\mathfrak{L} = \bigoplus_{\lambda\in\Delta} \mathfrak{L}_\lambda \subset V_\Lambda$ generated by the set $\{v_\lambda \mid \lambda \in \Delta\}$ is the affine Kac–Moody conformal algebra and V_Λ is the Frenkel–Kac–Segal construction of its basic representation (see [10; 30]). In this case, $\mathfrak{L}$ is a central extension of a simple loop algebra (see §1.4.1). If Δ is of type C then $\mathfrak{L}$ is also a central extension of a simple conformal algebra, which is a generalization of the algebra $\mathfrak{K}$ constructed in Section 2.3.

4.5. Finite Root Systems

In this section we describe all possible finite root systems. Let $\Delta \subset \Lambda$ be an indecomposable root system and assume that $|\Delta| < \infty$. As we saw in Section 4.1, the bilinear form $(\cdot|\cdot)$ on Λ must be either positive definite or semi–positive definite. Assume that it is semi–positive definite. Let $\pi : \Lambda \to \bar{\Lambda}$ be the projection of Λ onto the positive definite lattice $\bar{\Lambda}$, and let $\bar{\Delta} = \pi(\Delta) \setminus \{0\} \subset \bar{\Lambda}$ be the positive definite finite root system obtained from the projection of Δ (see Section 4.1 for definitions). The root system $\bar{\Delta}$ decomposes into a disjoint union $\bar{\Delta} = \bar{\Delta}_1 \sqcup \cdots \sqcup \bar{\Delta}_l$ of indecomposable root systems, each of them must be of one of the types described in Theorem 2. Denote $\Delta_i = \pi^{-1}(\bar{\Delta}_i) \cap \Delta$.

If for some $\bar{\Delta}_i$ we have $\#\pi^{-1}(\alpha) = 1$ for all $\alpha \in \bar{\Delta}_i$, then Δ decomposes as $\Delta_i \sqcup \left(\bigcup_{j\neq i} \Delta_j\right)$, which is a contradiction. So we assume that each Δ_i is a semi–positive definite root system.

LEMMA 9. *Let Δ be an indecomposable semi–positive definite root system such that $\bar{\Delta}$ is a positive definite indecomposable root system of type other than B or B^0. Then $|\Delta| = \infty$.*

Proof. The root system Δ must contain some isotropic roots for otherwise it would be positive definite. At least some isotropic root δ must be of the form $\delta = \alpha + \beta$, where α and β are real roots. If Δ is not of type BC, then α and β have square lengths exceeding 1 and hence we are in the situation of (v), (vi), or (vii) of Proposition 3; therefore, $k\delta \in \Delta$ for all integer k and Δ is infinite. If Δ is of type BC, then it might happen that α and β have length 1. If this is the case, let α' and β' be real roots such that $\overline{\alpha'} = 2\bar{\alpha}$ and $\overline{\beta'} = 2\bar{\beta}$. Then, by Proposition 3(vii), $\delta' = \alpha' + \beta'$ is an isotropic root such that $k\delta'$ is also a root for all integer k. $\square$

Return now to our finite root system Δ. The lemma implies that all indecomposable components $\bar{\Delta}_i$ of $\bar{\Delta}$ are of type either B or B^0. On the other hand, assume we are given a positive definite root system $\bar{\Delta} = \bar{\Delta}_1 \sqcup \cdots \sqcup \bar{\Delta}_l \subset \bar{\Lambda}$ such that all components $\bar{\Delta}_i$ are of type either B or B^0. Let $\bar{\Delta}_{\mathrm{S}}$ (resp., $\bar{\Delta}_{\mathrm{L}}$) be the set of short (resp., long) roots of $\bar{\Delta}$. There are many degrees of freedom in reconstructing the finite semi–positive definite root system Δ. First we choose an arbitrary lattice Λ_0 and set $\Lambda = \bar{\Lambda} \oplus \Lambda_0$. Then, in each $\bar{\Delta}_i$ of type B, we choose a subsystem $\bar{\Delta}'_i$ of type B_0. Denote by $\Omega \subset \bar{\Delta}_{\mathrm{L}}$ be the set of all long roots in $\bar{\Delta}$ that do not get into any of the root systems $\bar{\Delta}_i$ or $\bar{\Delta}'_i$ of type B^0. For each $\alpha \in \Omega$ we choose an arbitrary isotropic vector $\delta(\alpha) \in \Lambda_0$ such that $\delta(\alpha) = -\delta(-\alpha)$, and for each short root $\beta \in \bar{\Delta}_{\mathrm{S}}$ we choose an arbitrary finite set $\Sigma(\beta) \subset \Lambda_0$ such that $\Sigma(\beta) = -\Sigma(\beta) = \Sigma(-\beta)$. We impose the following restriction: If $\alpha \in \Omega$ and β is a short root such that $(\alpha|\beta) \neq 0$, then $\delta(\alpha) \in \Sigma(\beta)$. Now we set

$$\Delta = \{\beta + \delta \mid \beta \in \bar{\Delta}_{\mathrm{S}},\ \delta \in \Sigma(\beta)\} \cup \{\alpha + \delta(\alpha) \mid \alpha \in \Omega\} \cup (\bar{\Delta}_{\mathrm{L}} \setminus \Omega).$$

We may summarize as follows.

THEOREM 3. *Let $\Delta \subset \Lambda$ be a finite indecomposable root system. Then*: *either Λ is positive definite, in which case Δ is of one of the types described in Theorem 2*; *or Λ is semi–positive definite, in which case the positive definite quotient of Δ decomposes into a disjoint union $\bar{\Delta} = \bar{\Delta}_1 \sqcup \cdots \sqcup \bar{\Delta}_l$ of finite positive definite root systems of type either B or B^0, and Δ could be reconstructed from $\bar{\Delta}$ by the foregoing procedure.*

4.6. Connection to Extended Affine Root Systems

In this section we point out the relations with the theory of extended affine root systems (EARS; see e.g. [1]). By Proposition 3, for any two vectors $\alpha, \beta \in \Delta$ such that $(\alpha|\alpha) \neq 0$, the Cartan number $\langle \alpha, \beta \rangle = 2(\alpha|\beta)/(\alpha|\alpha)$ is an integer. This already makes Δ look similar to an EARS. To make the similarity even more complete, we must impose the following indecomposability assumption:

$$\forall \delta \in \Delta_0,\ \exists \alpha \in \Delta^\times \text{ such that } \delta + \alpha \in \Delta^\times. \tag{21}$$

It is easy to see that, if a root system Δ satisfies (21) and $\bar{\Delta}$ is indecomposable, then Δ lays inside an EARS (as defined in [1, p. 1]. The following theorem shows that, when there are no short roots, the structure of Δ is much simpler than the structure of a general EARS.

THEOREM 4. *Assume that $\bar{\Delta}$ is an indecomposable root system of one of the types A_n, D_n, E_n, B'_1, or C_n*; *that is, Δ does not contain short roots. Assume also that condition* (21) *holds. Then, for any $\delta \in \Delta_0$ and $\alpha \in \Delta$ we have $\delta + \alpha \in \Delta$.*

The theorem asserts that Δ_0 is a sublattice in Λ_0 and that, for any $\alpha \in \bar{\Delta}$, the whole equivalence class $\alpha + \Delta_0$ belongs to Δ.

Proof. Let $\mathfrak{L} = \bigoplus_{\alpha \in \Delta} \mathfrak{L}_\alpha \subset V_\Lambda$ be the conformal superalgebra generated by the set $\{v_\alpha \mid \alpha \in \Delta\}$. We claim that for any $h \in \mathfrak{h} = \Bbbk \otimes \Lambda$ we have $h(-1)v_\delta \in \mathfrak{L}_\delta$.

Let us first show that the theorem follows from this claim. Let $\delta \in \Delta_0$ and $\alpha \in \Delta$. If $\alpha \in \Delta^\times$, then using (3) and (9) yields

$$(\alpha(-1)v_\delta)\ \boxed{0}\ v_\alpha = \pm(\alpha|\alpha)v_{\alpha+\delta} \in \mathfrak{L},$$

hence $\alpha + \delta \in \Delta^\times$. If $\alpha \in \Delta_0$, take some $\beta \in \Delta^\times$ and then use (3) and (9) as before to obtain

$$(\beta(-1)v_\delta) \boxed{0} (\beta(-1)v_\alpha) = \pm(\beta|\beta)\delta(-1)v_{\alpha+\delta} \in \mathfrak{L},$$

hence $\alpha + \delta \in \Delta_0$.

Let us now prove the claim. The condition (21) assures that any isotropic root $\delta \in \Delta_0$ is obtained as a sum $\delta = \alpha + \beta$ of two real roots $\alpha, \beta \in \Delta^\times$. Since Δ does not have any short roots, the pair α, β must generate a root system of type (v), (vi), or (vii) of Proposition 3. Then, Proposition 3 implies that $\alpha(-1)v_\delta, \beta(-1)v_\delta \in \mathfrak{L}_\delta$.

Assume now that $\lambda \in \Delta^\times$ is such that $\lambda(-1)v_\delta \in \mathfrak{L}_\delta$ and $\mu \in \Delta^\times$ satisfies $(\lambda|\mu) \neq 0$. Then we have

$$(\lambda(-1)v_\delta) \boxed{0} v_\mu = \pm(\lambda|\mu)v_{\mu+\lambda} \in \mathfrak{L},$$

hence $\mu + \delta \in \Delta^\times$. Therefore, the real roots μ and $\mu + \delta$ form a root system of type (v), (vi), or (vii) of Proposition 3, so we get that $\mu(-1)\, v_\delta \in \mathfrak{L}_\delta$.

It follows that, for every real root $\lambda \in \Delta^\times$ that is not orthogonal to either α or β, we have $\lambda(-1)v_\delta \in \mathfrak{L}_\delta$; therefore, since $\bar{\Delta}$ is indecomposable, $\lambda(-1)v_\delta \in \mathfrak{L}_\delta$ for all $\lambda \in \Delta^\times$. Condition (21) implies that $\mathfrak{h} = \mathrm{Span}_{\Bbbk}\, \Delta^\times$, and the claim follows. $\square$

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