Michigan Math. J. 52 (2004)

Ideals of Operators, Approximability in the Strong Operator Topology, and the Approximation Property

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1. Introduction

Let us recall that a linear subspace F of a Banach space E is an *ideal* in E if $F^{\perp}$ is the kernel of a norm-1 projection in E^*. The notion of an ideal was introduced and studied by Godefroy, Kalton, and Saphar in [3].

A linear operator $\phi\colon F^* \to E^*$ is called a *Hahn–Banach extension operator* if $(\phi x^*)(x) = x^*(x)$ and $\|\phi x^*\| = \|x^*\|$ for all $x \in F$ and all $x^* \in F^*$. Let us denote the set of all Hahn–Banach extension operators $\phi\colon F^* \to E^*$ by $\mathrm{HB}(F, E)$. It is well known (and straightforward to verify) that $\mathrm{HB}(F, E) \neq \emptyset$ if and only if F is an ideal in E.

Let X be a Banach space. In [10, Thm. 3.1], Lima, Nygaard, and Oja proved that, for all Banach spaces Z, the space of compact operators $\mathcal{K}(Z, X)$ from Z to X is an ideal in $\mathcal{W}(Z, X)$, the space of weakly compact operators from Z to X, if and only if $\mathcal{K}(Z, X)$ is an ideal in $\mathcal{W}(Z, X)$ for all reflexive Banach spaces Z. They also showed (see [10, Thm. 4.1]) that these conditions can be equivalently characterized through the approximability of weakly compact operators in the strong operator topology as follows: for every Banach space Z and every $T \in \mathcal{W}(Z, X)$, there exists a net (T_α) in $\mathcal{K}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to T$ in the strong operator topology.

Let X now be a closed subspace of a Banach space Y. In [13, Thm. 4.4] the authors proved that $\mathcal{K}(Z, X)$ is an ideal in $\mathcal{W}(Z, Y)$ for all Banach spaces Z if and only if $\mathcal{K}(Z, X)$ is an ideal in $\mathcal{W}(Z, Y)$ for all reflexive Banach spaces Z, thus extending the first-mentioned result by Lima, Nygaard, and Oja. The main purpose of this article is to characterize also this more general situation by means of the approximability of weakly compact operators in the strong operator topology. The key result that enables us to achieve this purpose comes from the main theorem of [12] (see Theorem 2.4 and its proof in [12]) and reads as follows.

Lemma 1.1. *Let X be a closed subspace of a Banach space Y. Let $\mathcal{A}$ and $\mathcal{B}$ be closed operator ideals satisfying $\mathcal{A} \subset \mathcal{B}$. If $\mathcal{A}(Z, X)$ is an ideal in $\mathcal{B}(Z, Y)$ for*

Received October 31, 2002. Revision received January 15, 2004.
The research of the second-named author was partially supported by Estonian Science Foundation Grant no. 4400.

all reflexive Banach spaces Z, then there exists $\phi \in \mathrm{HB}(X,Y)$ such that, for every reflexive Banach space Z, there exists $\Phi \in \mathrm{HB}(\mathcal{A}(Z,X),\mathcal{B}(Z,Y))$ such that

$$\Phi(x^* \otimes z) = (\phi x^*) \otimes z \quad \forall x^* \in X^*,\ \forall z \in Z.$$

By $\mathcal{F}$, $\bar{\mathcal{F}}$, $\mathcal{K}$, $\mathcal{W}$, and $\mathcal{L}$, we shall denote the operator ideals (in the sense of Pietsch [19]) of finite-rank operators, approximable operators (i.e., norm limits of finite-rank operators), compact operators, weakly compact operators, and bounded operators, respectively.

In Section 2, relying on Lemma 1.1 and the uniform isometric version of the Davis–Figiel–Johnson–Pełczyński (DFJP) factorization theorem in [10], we prove, in the case when $\mathcal{A} = \mathcal{F}$ or $\mathcal{A} = \mathcal{K}$ and $\mathcal{B} = \mathcal{K}$ or $\mathcal{B} = \mathcal{W}$, that $\mathcal{A}(Z,X)$ is an ideal in $\mathcal{B}(Z,Y)$ for all (reflexive) Banach spaces Z if and only if there exists a Hahn–Banach extension operator $\phi \colon X^* \to Y^*$ such that, for every Banach space Z and every $T \in \mathcal{B}(Z,Y)$, there exists a net (T_α) in $\mathcal{A}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^* \circ \phi$ in the strong operator topology (see Theorem 2.3).

Results of Section 2 are applied in Sections 3 and 4 to characterize the approximation property of X and X^* in terms of the approximability of weakly compact operators in the strong operator topology, a problem that goes back to the fundamental work of Grothendieck [6]. Here the main result is Theorem 3.2, which lists four conditions that are equivalent to the approximation property of X^* and are expressed through the approximability of weakly compact or compact operators by finite-rank operators in the strong operator topology. The final application (Corollary 4.3) asserts that, if X is an M-ideal in its bidual, then X^* has the approximation property if and only if $\overline{\mathcal{F}(Z,X)}$ is an M-ideal in $\mathcal{K}(Z,X^{**})$ for all Banach spaces Z or, equivalently, iff $\mathcal{F}(Z,X)$ is an ideal in $\mathcal{K}(Z,X^{**})$ for all reflexive Banach spaces Z.

Let us fix some more notation. In a linear normed space X, we denote the closed unit ball by B_X and the open ball with center x and radius r by $B(x,r)$. The closure of a set $A \subset X$ is denoted by $\bar{A}$ and its linear span by $\operatorname{span} A$. The identity operator on X is denoted by I_X. We shall always regard X as a subspace of X^{**}.

2. Ideals of Operators and Approximability in the Strong Operator Topology

In order to prove the main result of this article (Theorem 2.3, as described in Section 1), we shall also use the next two lemmas.

LEMMA 2.1. *Let X be a closed subspace of a Banach space Y, and let Z be a Banach space. Let $\mathcal{A} \subset \mathcal{L}(Z,X)$ and $\mathcal{B} \subset \mathcal{L}(Z,Y)$ be subspaces such that $\mathcal{F}(Z,X) \subset \mathcal{A}$, $\mathcal{F}(Z,Y) \subset \mathcal{B}$, and $\mathcal{A} \subset \mathcal{B}$. Assume that $\phi \in \mathrm{HB}(X,Y)$ and $\Phi \in \mathrm{HB}(\mathcal{A},\mathcal{B})$ and that*

$$Z_{\Phi\phi} = \{z \in Z : \Phi(x^* \otimes z) = (\phi x^*) \otimes z\ \forall x^* \in X^*\}$$

is reflexive. Then, for every operator $T \in \mathcal{B}$, there exists a net $(T_\alpha) \subset \mathcal{A}$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $(T_\alpha|_{Z_{\Phi\phi}})^* \to (T|_{Z_{\Phi\phi}})^* \circ \phi$ in the strong operator topology of $\mathcal{L}(X^*, Z_{\Phi\phi}^*)$.

Proof. We follow the idea of the proof of [10, Lemma 1.4]. Let P be the norm-1 projection on $\mathcal{B}^*$ with $\ker P = \mathcal{A}^\perp$ defined by Φ; that is,

$$Pf = \Phi(f|_{\mathcal{A}}), \quad f \in \mathcal{B}^*.$$

Since $P^*(T) \in \mathcal{A}^{\perp\perp} \subset \mathcal{B}^{**}$ and $\|P^*(T)\| \le \|T\|$, there exists a net $(T_\alpha) \subset \mathcal{A}$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to P^*(T)$ weak* in $\mathcal{B}^{**}$. In particular, for all $y^* \in Y^*$ and $z \in Z_{\Phi\phi}$, we have

$$\begin{aligned} y^*(T_\alpha z) &= (y^* \otimes z)(T_\alpha) \to (y^* \otimes z)(P^*(T)) = (P(y^* \otimes z))(T) \\ &= (\Phi(y^*|_X \otimes z))(T) = (\phi(y^*|_X))(Tz). \end{aligned}$$

This means that

$$x^*(T_\alpha z) \to (\phi x^*)(Tz) \quad \forall x^* \in X^*,\ \forall z \in Z_{\Phi\phi},$$

or, equivalently, that $(T_\alpha|_{Z_{\Phi\phi}})^* \to (T|_{Z_{\Phi\phi}})^* \circ \phi$ in the weak operator topology of $\mathcal{L}(X^*, Z^*_{\Phi\phi})$. Since the weak and strong operator topologies yield the same dual space (cf. e.g. [1, Thm. VI.1.4]), after passing to convex combinations we may assume that the desired convergence is in the strong operator topology. □

The following local formulation of ideals is well known. A proof can be found in [9].

LEMMA 2.2. *Let F be a closed subspace of a Banach space E. Then the following statements are equivalent.*

(a) *F is an ideal in E.*

(b) *F is locally 1-complemented in E; that is, for every finite-dimensional subspace G of E and for all $\varepsilon > 0$, there is an operator $U : G \to F$ such that $\|U\| \le 1 + \varepsilon$ and $Ux = x$ for all $x \in G \cap F$.*

REMARK 2.1. It is straightforward to verify that the condition $Ux = x$ for all $x \in G \cap F$ in Lemma 2.2 can be replaced by $\|Ux - x\| \le \varepsilon$ for all $x \in B_{G\cap F}$.

Let us recall the easy observation that, if F is a linear subspace of a Banach space E, then F is an ideal in E if and only if $\bar{F}$ is an ideal in E.

THEOREM 2.3. *Let X be a closed subspace of a Banach space Y. Let $\mathcal{A} = \mathcal{F}$ or $\mathcal{A} = \mathcal{K}$ and let $\mathcal{B} = \mathcal{K}$ or $\mathcal{B} = \mathcal{W}$. Then the following assertions are equivalent.*

(a) *$\mathcal{A}(Z, X)$ is an ideal in $\mathcal{B}(Z, Y)$ for all Banach spaces Z.*

(b) *$\mathcal{A}(Z, X)$ is an ideal in $\mathcal{B}(Z, Y)$ for all reflexive Banach spaces Z.*

(c) *For every reflexive Banach space Z and every operator $T \in \mathcal{B}(Z, Y)$, there exist a Hahn–Banach extension operator $\phi : X^* \to Y^*$ and a net $(T_\alpha) \subset \mathcal{A}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $x^*(T_\alpha z) \to (\phi x^*)(Tz)$ for all $x^* \in X^*$ and all $z \in Z$.*

(d) *There exists a Hahn–Banach extension operator $\phi : X^* \to Y^*$ such that, for every Banach space Z and every operator $T \in \mathcal{B}(Z, Y)$, there exists a net $(T_\alpha) \subset \mathcal{A}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T^*_\alpha x^* \to T^*\phi x^*$ for all $x^* \in X^*$.*

Proof. (a) $\Leftrightarrow$ (b) As mentioned in the Introduction, the equivalence was established in [13, Thms. 4.2 & 4.4] for $\mathcal{A} = \mathcal{K}$ and for $\mathcal{B} = \mathcal{K}$ or $\mathcal{B} = \mathcal{W}$. The remaining two cases when $\mathcal{A} = \mathcal{F}$ can be proved by the same reasoning as in [13, Thm. 4.2] using the local formulation of ideals (see Lemma 2.2). (This actually yields the claim for $\mathcal{A} = \bar{\mathcal{F}}$ instead of $\mathcal{A} = \mathcal{F}$, but the former is clearly equivalent to the latter.)

(b) $\Rightarrow$ (d) By Lemma 1.1, there exists $\phi \in \mathrm{HB}(X, Y)$ such that, for every reflexive Banach space Z, there exists $\Phi \in \mathrm{HB}(\mathcal{A}(Z, X), \mathcal{B}(Z, Y))$ such that $Z = Z_{\Phi\phi}$.

Let W be an arbitrary Banach space and let $T \in \mathcal{B}(W, Y)$. Using the isometric version of the DFJP factorization theorem for weakly compact operators [10, Lemma 1.1 & Thm. 2.2], we find a reflexive Banach space Z, a norm-1 operator $J \in \mathcal{B}(Z, Y)$, and an operator $S \in \mathcal{B}(W, Z)$ with $\|S\| = \|T\|$ such that $T = J \circ S$.

By Lemma 2.1, for $J \in \mathcal{B}(Z, Y)$ there exists a net $(J_\alpha) \subset \mathcal{A}(Z, X)$ with $\sup_\alpha \|J_\alpha\| \le 1$ such that $J_\alpha^* x^* \to J^* \phi x^*$ for all $x^* \in X^*$. Put $T_\alpha = J_\alpha \circ S$. Then $(T_\alpha) \subset \mathcal{A}(W, X)$, $\sup_\alpha \|T_\alpha\| \le \|S\| = \|T\|$, and $T_\alpha^* x^* = S^* J_\alpha^* x^* \to S^* J^* \phi x^* = T^* \phi x^*$.

(d) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (b) Let Z be a reflexive Banach space. We shall apply the local formulation of the notion of an ideal (Lemma 2.2 together with Remark 2.1) to prove that $\mathcal{A}(Z, X)$ is an ideal in $\mathcal{B}(Z, Y)$. (In particular, the claim for $\mathcal{A} = \mathcal{F}$ will be obtained in the equivalent form of $\mathcal{A} = \bar{\mathcal{F}}$.)

Let $G \subset \mathcal{B}(Z, Y)$ be a finite-dimensional subspace and let $\varepsilon > 0$. Using the uniform isometric version of the DFJP factorization [10, Thm. 2.3 & Lemma 2.1(iv)], we can find a reflexive Banach space W, a norm-1 operator $J \in \mathcal{B}(W, Y)$, and a linear isometry $\Phi\colon G \to \mathcal{W}(Z, W)$ such that $T = J \circ \Phi(T)$ for all $T \in G$.

By (c), there exist a Hahn–Banach extension operator $\phi\colon X^* \to Y^*$ and a net $(J_\alpha) \subset \mathcal{A}(W, X)$ with $\sup_\alpha \|J_\alpha\| \le 1$ such that $x^*(J_\alpha w) \to (\phi x^*)(Jw)$ for all $x^* \in X^*$ and all $w \in W$. We now show that, for every $S \in G \cap \mathcal{A}(Z, X)$, after passing to convex combinations of (J_α) we may suppose that

$$\|J_\alpha \circ \Phi(S) - S\| \to 0.$$

In fact, we have that $(J_\alpha \circ \Phi(S) - S) \subset \mathcal{A}(Z, X) \subset \mathcal{K}(Z, X)$ is a bounded net in $\mathcal{K}(Z, X)$, the space Z is reflexive, and (letting $w := \Phi(S)z$)

$$\begin{aligned} x^*((J_\alpha \circ \Phi(S) - S)z) &= x^*(J_\alpha w) - x^*(Jw) \\ &= x^*(J_\alpha w) - (\phi x^*)(Jw) \to 0 \end{aligned}$$

for all $x^* \in X^*$ and all $z \in Z$. Hence, by the description of the weak convergence in $\mathcal{K}(Z, X)$ due to Feder and Saphar [2, Cor. 1.2], the net $(J_\alpha \circ \Phi(S) - S)$ converges to zero weakly in $\mathcal{K}(Z, X)$. Therefore a net of its convex combinations converges to zero in the norm as desired.

Let us fix a finite set $\{S_1, \dots, S_n\}$ in $G \cap \mathcal{A}(Z, X)$ so that

$$B_{G \cap \mathcal{A}(Z,X)} \subset \bigcup_{k=1}^{n} B(S_k, \varepsilon/3).$$

After passing to convex combinations of (J_α), we may assume that, for some fixed α and all $k = 1, \dots, n$,

$$\|J_\alpha \circ \Phi(S_k) - S_k\| < \varepsilon/3.$$

Define $U(T) = J_\alpha \circ \Phi(T)$, $T \in G$. Then U maps G into $\mathcal{A}(Z, X)$, $\|U\| \leq 1$, and for every $S \in B_{G \cap \mathcal{A}(Z,X)}$ we have

$$\begin{aligned}\|U(S) - S\| &\leq \|U(S - S_k)\| + \|U(S_k) - S_k\| + \|S_k - S\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,\end{aligned}$$

where S_k has been chosen to satisfy $\|S - S_k\| < \varepsilon/3$. □

It is interesting to observe that, even in the particular case when $X = Y$ (which was thoroughly studied in [10]), Theorem 2.3 yields a new characterization in terms of convergence in the strong operator topology (namely, condition (d) in our next corollary).

COROLLARY 2.4. *Let X be a Banach space, and let $\mathcal{A} = \mathcal{F}$ or $\mathcal{A} = \mathcal{K}$. Then the following assertions are equivalent.*

- (a) *$\mathcal{A}(Z, X)$ is an ideal in $\mathcal{W}(Z, X)$ for all Banach spaces Z.*
- (b) *$\mathcal{A}(Z, X)$ is an ideal in $\mathcal{W}(Z, X)$ for all reflexive Banach spaces Z.*
- (c) *For every Banach space Z and every operator $T \in \mathcal{W}(Z, X)$, there exists a net $(T_\alpha) \subset \mathcal{A}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\|$ such that $T_\alpha \to T$ in the strong operator topology.*
- (d) *For every Banach space Z and every operator $T \in \mathcal{W}(Z, X)$, there exists a net $(T_\alpha) \subset \mathcal{A}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\|$ such that $T_\alpha^* \to T^*$ in the strong operator topology.*

Proof. Consider the following assertion.

- (c′) For every reflexive Banach space Z and every operator $T \in \mathcal{W}(Z, X)$, there exists a net $(T_\alpha) \subset \mathcal{A}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\|$ such that $T_\alpha \to T$ in the weak operator topology.

Because $\mathrm{HB}(X, X) = \{I_{X^*}\}$, the equivalence of conditions (a), (b), (c′), and (d) is immediate from Theorem 2.3. Condition (c) is equivalent to them because, obviously, (c) ⇒ (c′) and (d) ⇒ (c) by the convex combinations argument (used already in the proof of Lemma 2.1). □

REMARK 2.2. The equivalences (a) ⇔ (b) ⇔ (c) of Corollary 2.4 were proved in [10, Cor. 1.5 & Thms. 3.1, 3.3, 4.1].

3. Approximability of Weakly Compact Operators in the Strong Operator Topology and the Approximation Property

Grothendieck was the first to show that the approximability of weakly compact operators by finite-rank operators in the strong operator topology is closely related to the approximation property of Banach spaces. Namely, in [6, Cor. 2, p. 141] he proved that, if the dual space X^* of a Banach space X has the approximation property, then for every Banach space Z it follows that $B_{\mathcal{F}(Z,X)}$ is dense in $B_{\mathcal{W}(Z,X)}$ in the strong operator topology. This result was strengthened by [10, Cor. 1.5]; the latter condition is, in fact, equivalent to the approximation property of the space X itself.

In [10, Thm. 3.3] it was also proved that X has the approximation property if and only if $\mathcal{F}(Z,X)$ is an ideal in $\mathcal{W}(Z,X)$ for all Banach spaces Z. Previously, the authors showed in [11, Thm. 5.1] that X has the approximation property if and only if $\mathcal{F}(Z,X)$ is an ideal in $\mathcal{K}(Z,X)$ for all Banach spaces Z. These two results, together with Theorem 2.3 and Corollary 2.4, immediately yield the following criteria of the approximation property expressed in terms of approximability of operators in the strong operator topology.

THEOREM 3.1. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X has the approximation property.*
(b) *For every Banach space Z and every operator $T \in \mathcal{W}(Z,X)$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to T$ in the strong operator topology.*
(c) *For every Banach space Z and every operator $T \in \mathcal{W}(Z,X)$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*$ in the strong operator topology.*
(d) *For every reflexive Banach space Z and every operator $T \in \mathcal{K}(Z,X)$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*$ in the strong operator topology.*

REMARK 3.1. Similarly to Theorem 3.1 (but by a different method), the approximation property is characterized in [18, Thm. 3] in terms of the approximability of weak*-weak continuous operators. This also yields an alternative proof of Theorem 3.1.

REMARK 3.2. Condition (b) clearly means that, for every Banach space Z, $B_{\mathcal{F}(Z,X)}$ is dense in $B_{\mathcal{W}(Z,X)}$ in the strong operator topology.

REMARK 3.3. Grothendieck [6, Cor. 1, p. 184] conjectured that the approximation property of a Banach space implies condition (b) of Theorem 3.1. But the proof (see [6, proof of Thm. 15, pp. 183–184]) only goes through in the particular case when Z is complemented in Z^{**} by a norm-1 projection. This was made clear by Reinov (see [20, proof of Thm. 4 and subsequent remark]).

In the next theorem we characterize the approximation property of the dual space X^* in terms of the approximability of operators in the strong operator topology. In particular, we shall prove that by replacing $\mathcal{W}(Z,X)$ with $\mathcal{W}(Z,X^{**})$ in condition (c) of Theorem 3.1 (or, equivalently, $\mathcal{K}(Z,X)$ with $\mathcal{K}(Z,X^{**})$ in condition (d)) we may obtain a condition that is equivalent to the approximation property of X^*. Comparing these two conditions, it becomes quite evident that the approximation property of X^* implies the approximation property of X (a well-known fact, of course, due to Grothendieck [6]).

THEOREM 3.2. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X^* has the approximation property.*
(b) *For every Banach space Y containing X as an ideal, every Hahn–Banach extension operator $\phi \in \mathrm{HB}(X,Y)$, every Banach space Z, and every operator $T \in \mathcal{W}(Z,Y)$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^* \circ \phi$ in the strong operator topology.*
(c) *For every Hahn–Banach extension operator $\phi \in \mathrm{HB}(X,X^{**})$, every Banach space Z, and every operator $T \in \mathcal{W}(Z,X^{**})$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^* \circ \phi$ in the strong operator topology.*
(d) *For every Banach space Z and every operator $T \in \mathcal{W}(Z,X^{**})$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*|_{X^*}$ in the strong operator topology.*
(e) *For every reflexive Banach space Z and every operator $T \in \mathcal{K}(Z,X^{**})$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*|_{X^*}$ in the strong operator topology.*

Proof. (a) $\Rightarrow$ (b) As in the proof of Theorem 2.3 (see (b) $\Rightarrow$ (d)), we find a reflexive Banach space W, a norm-1 operator $J\colon W \to Y$, and an operator $S\colon Z \to W$ with $\|S\| = \|T\|$ such that $T = J \circ S$. Let $\phi \in \mathrm{HB}(X,Y)$. Since W is reflexive, we can use a result by the authors (see [12, Thm. 4.6]) asserting that the approximation property of X^* implies the existence of $\Phi \in \mathrm{HB}(\mathcal{F}(W,X), \mathcal{L}(W,Y))$ such that $W = W_{\Phi\phi}$. By Lemma 2.1, there exists a net $(J_\alpha) \subset \mathcal{F}(W,X)$ with $\sup_\alpha \|J_\alpha\| \le 1$ such that $J_\alpha^* \to J^* \circ \phi$ in the strong operator topology. It is clear that $T_\alpha = J_\alpha \circ S$ is the desired net.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious if one notices that the canonical embedding from X^* to X^{***} is a Hahn–Banach extension operator.

(d) $\Rightarrow$ (e) is even more obvious.

(e) $\Rightarrow$ (a) Let us note, first of all, that X must have the approximation property (this is clear from Theorem 3.1). By a well-known criterion (due to Grothendieck [6]; see e.g. [15, p. 32]) this means that, for all sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$ such that $\sum_{n=1}^\infty \|x_n^*\| \|x_n\| < \infty$ and $\sum_{n=1}^\infty x_n^*(x)x_n = 0$, whenever $x \in X$ one has $\sum_{n=1}^\infty x_n^*(x_n) = 0$. We shall make use of this condition in the sequel, but for now we apply the criterion just cited to show that X^* has the approximation property.

Let $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ satisfy $\sum_{n=1}^\infty \|x_n^{**}\| \|x_n^*\| < \infty$, and let $\sum_{n=1}^\infty x_n^{**}(x^*)x_n^* = 0$ for all $x^* \in X^*$. We may assume without loss of generality (this is well known and straightforward to verify; see e.g. [15, p. 31]) that $\sum_{n=1}^\infty \|x_n^*\| < \infty$ and $\|x_n^{**}\| \to 0$. Moreover, we may assume that $\sup_n \|x_n^{**}\| \le 1$.

Let K be the closed absolutely convex hull of (x_n^{**}) in X^{**}. Then $K \subset B_{X^{**}}$. Noting that K is compact and using the isometric version of the DFJP construction [10, Lemmas 1.1 & 2.1], we find a reflexive Banach space Z, which is also a linear subspace of X^{**}, such that $K \subset Z$ and the identity embedding J of Z into X^{**} has norm 1 and is compact.

Let us denote x_n^{**} by z_n when looking at x_n^{**} as an element of the Banach space Z. Thus we have $Jz_n = x_n^{**}$ for all n. By [10, Lemma 2.1], $\|z_n\| = \mathcal{O}(\|x_n^{**}\|^{1/2}) = \mathcal{O}(1)$. Put

$$M = \sup_n \|z_n\|.$$

Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ so that

$$\sum_{n>N} \|x_n^*\| < \frac{\varepsilon}{4M}.$$

Since $J \in \mathcal{K}(Z, X^{**})$, by (e) there exists an operator $J_\alpha \in \mathcal{F}(Z, X)$ with $\|J_\alpha\| \leq 1$ such that

$$\|J_\alpha^* x_n^* - J^* x_n^*\| < \frac{\varepsilon}{2NM}, \quad n = 1, \dots, N.$$

We now prove that

$$\sum_{n=1}^{\infty} x_n^*(J_\alpha z_n) = 0.$$

Since X has the approximation property, it is sufficient to show that

$$\sum_{n=1}^{\infty} x_n^*(x) J_\alpha z_n = 0$$

for all $x \in X$. But this is so because J is injective and

$$\sum_{n=1}^{\infty} x_n^*(x) J_\alpha z_n = J_\alpha \left(\sum_{n=1}^{\infty} x_n^*(x) z_n \right),$$

$$J\left(\sum_{n=1}^{\infty} x_n^*(x) z_n \right) = \sum_{n=1}^{\infty} x_n^*(x) x_n^{**} = 0;$$

indeed,

$$\left(\sum_{n=1}^{\infty} x_n^*(x) x_n^{**} \right)(x^*) = \left(\sum_{n=1}^{\infty} x_n^{**}(x^*) x_n^* \right)(x) = 0 \quad \forall x^* \in X^*.$$

Finally, we have

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} x_n^{**}(x_n^*) \right| &= \left| \sum_{n=1}^{\infty} (Jz_n)(x_n^*) \right| = \left| \sum_{n=1}^{\infty} (J^* x_n^*)(z_n) \right| \\
&= \left| \sum_{n=1}^{\infty} (J^* x_n^*)(z_n) - \sum_{n=1}^{\infty} x_n^*(J_\alpha z_n) \right| \\
&\leq \sum_{n=1}^{N} \|J^* x_n^* - J_\alpha^* x_n^*\| \|z_n\| + \sum_{n>N} (\|J^*\| + \|J_\alpha^*\|) \|z_n\| \|x_n^*\| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} x_n^{**}(x_n^*) = 0$ and X^* has the approximation property. □

REMARK 3.4. For comparison with conditions (c) and (d) of Theorem 3.2, let us mention that X^* has the approximation property if and only if, for every Banach space Z and every operator $T \in \mathcal{W}(X,Z)$, there exists a net $(T_\alpha) \subset \mathcal{F}(X,Z)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*$ in the strong operator topology (see [10, Thm. 3.4 & proof of Cor. 4.3]).

REMARK 3.5. Let us recall that the metric approximation property of X (respectively, of X^*) is equivalent to the existence of a net $(I_\alpha) \subset \mathcal{F}(X,X)$ with $\|I_\alpha\| \le 1$ such that $I_\alpha \to I_X$ (respectively, $I_\alpha^* \to I_{X^*}$) in the strong operator topology. It is therefore clear that, if we replace the operator ideal $\mathcal{W}$ by $\mathcal{L}$ in condition (b) of Theorem 3.1, then we obtain a condition that is equivalent to the metric approximation property of X. Further, replacing $\mathcal{W}$ by $\mathcal{L}$ in conditions (b), (c), and (d) of Theorem 3.2 yields conditions that are equivalent to the metric approximation property of X^* (here one uses that $T_\alpha = I_\alpha^{**} \circ \phi^*|_Y \circ T \in \mathcal{F}(Z,X)$ if $T \in \mathcal{L}(Z,Y)$).

In particular, we also see that X^* has the metric approximation property if and only if, for every Banach space Z and every operator $T \in \mathcal{L}(Z, X^{**})$, there exists a net (T_α) in $\mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*|_{X^*}$ in the strong operator topology. This improves the result by Sharir [21, Cor. 3.3] asserting the existence of a net (T_α) in $\mathcal{L}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $x^*(T_\alpha z) \to (Tz)(x^*)$ for $x^* \in X^*$ and $z \in Z$ whenever X^* has the metric approximation property.

In [6, proof of Thm. 15, pp. 183–184], Grothendieck showed (this was made explicit by Reinov in [20, Cor. 2 of Thm. 4]) that if X^* has the approximation property then, for every Banach space Z and every operator $T \in \mathcal{W}(X^*, Z)$, there exists a net (T_α) in $\mathcal{F}(X^*, Z)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to T$ in the strong operator topology. As an application of Theorem 3.2, we obtain that the latter condition is actually equivalent to the approximation property of X^*.

COROLLARY 3.3. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X^* has the approximation property.*
(b) *For every Banach space Z and every operator $T \in \mathcal{W}(X^*, Z)$, there exists a net $(T_\alpha) \subset X \otimes Z$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha x^* \to Tx^*$ for all $x^* \in X^*$.*
(c) *For every Banach space Z and every operator $T \in \mathcal{W}(X^*, Z)$, there exists a net $(T_\alpha) \subset \mathcal{F}(X^*, Z)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha x^* \to Tx^*$ for all $x^* \in X^*$.*

Proof. (a) $\Leftrightarrow$ (b) To show this equivalence, we prove that (b) is equivalent to condition (d) of Theorem 3.2.

Assume that (b) holds and let $T \in \mathcal{W}(Z, X^{**})$. Since $T^*|_{X^*} \in \mathcal{W}(X^*, Z^*)$, there exists a net (S_α) in $X \otimes Z^*$ with $\sup_\alpha \|S_\alpha\| \le \|T\|$ such that $S_\alpha x^* \to T^* x^*$ for all $x^* \in X^*$. But $S_\alpha = T_\alpha^*$ for some $T_\alpha \in \mathcal{F}(Z,X)$ $\big($if $S_\alpha = \sum_{i=1}^n x_i \otimes z_i^*$, then $T_\alpha = \sum_{i=1}^n z_i^* \otimes x_i\big)$.

Assume, conversely, that Theorem 3.2(d) holds. It is then immediate that Corollary 3.3(b) holds for reflexive Banach spaces Z $\big($because $\big(\sum_{i=1}^n z_i \otimes x_i\big)^* = \sum_{i=1}^n x_i \otimes z_i \in X \otimes Z\big)$. If Y is a Banach space and $T \in \mathcal{W}(X^*, Y)$ then, as in the proof of Theorem 2.3 (see (b) $\Rightarrow$ (d)), we find a reflexive Banach space Z, a norm-1 operator $J\colon Z \to Y$, and an operator $S\colon X^* \to Z$ with $\|S\| = \|T\|$ such that $T = J \circ S$. Then we find (S_α) in $X \otimes Z$ with $\sup_\alpha \|S_\alpha\| \le \|S\|$ such that $S_\alpha x^* \to Sx^*$ for all $x^* \in X^*$. It is clear that $T_\alpha = J \circ S_\alpha$ is the desired net.

(b) $\Rightarrow$ (c) is more than obvious.

(c) $\Rightarrow$ (b) by the principle of local reflexivity. In fact, assume that $T \in \mathcal{W}(X^*, Z)$ and that a net (S_α) in $\mathcal{F}(X^*, Z)$ with $\sup_\alpha \|S_\alpha\| \le \|T\|$ satisfies $S_\alpha x^* \to Tx^*$ for all $x^* \in X^*$. For $S_\alpha = \sum_{i=1}^n x_i^{**} \otimes z_i$, we denote $F_\alpha = \operatorname{span}\{x_1^{**}, \dots, x_n^{**}\} \subset X^{**}$.

Let us order the set (α, G, ε), where $\varepsilon > 0$ and G runs through the finite-dimensional subspaces of X^*, by

$$(\alpha, G, \varepsilon) \ge (\tilde\alpha, \tilde G, \tilde\varepsilon) \iff \alpha \ge \tilde\alpha,\ G \supset \tilde G,\ \varepsilon < \tilde\varepsilon.$$

For each (α, G, ε), choose an operator $S_{(\alpha,G,\varepsilon)}\colon F_\alpha \to X$ with $\|S_{(\alpha,G,\varepsilon)}\| \le 1 + \varepsilon$ such that $x^*(S_{(\alpha,G,\varepsilon)} x^{**}) = x^{**}(x^*)$ for all $x^* \in G$ and $x^{**} \in F_\alpha$. Considering S_α as an element of $F_\alpha \otimes Z$, we see that $S_{(\alpha,G,\varepsilon)}(S_\alpha) \in X \otimes Z$ and $S_\alpha x^* = (S_{(\alpha,G,\varepsilon)}(S_\alpha))x^*$ for all $x^* \in G$. Put

$$T_{(\alpha,G,\varepsilon)} = \frac{S_{(\alpha,G,\varepsilon)}(S_\alpha)}{1+\varepsilon}.$$

Then clearly $(T_{(\alpha,G,\varepsilon)}) \subset X \otimes Z$, $\sup_{(\alpha,G,\varepsilon)} \|T_{(\alpha,G,\varepsilon)}\| \le \|T\|$, and $T_{(\alpha,G,\varepsilon)} x^* \to Tx^*$ for all $x^* \in X^*$. □

REMARK 3.6. Concerning the Grothendieck–Reinov condition (c) of Corollary 3.3, we remark that the convergence $T_\alpha \to T$ in the strong operator topology cannot be replaced by the convergence $T_\alpha^* \to T^*$ in the strong operator topology. This would yield a condition that is equivalent to the approximation property of X^{**} (see Remark 3.4). Also, it is well known that there are Banach spaces X such that X^* has the approximation property but X^{**} does not have the approximation property (see e.g. [15, p. 35]).

4. The Approximation Property of Dual Spaces and the Unique Extension Property

We know (see [11, Thm. 5.2]) that the dual space X^* of a Banach space X has the approximation property if and only if $\mathcal{F}(X, Z)$ is an ideal in $\mathcal{K}(X, Z)$ for all Banach spaces Z. We also know (see [13, Cor. 4.7]) that $\mathcal{F}(Z, X)$ is an ideal in $\mathcal{K}(Z, X^{**})$ for all Banach spaces Z whenever X^* has the approximation property. As an application of Theorems 2.3 and 3.2, we shall show in this section that the converse assertion is true for Banach spaces X having the unique extension property. It is not true in general, as the following example demonstrates.

EXAMPLE 4.1. *There is a Banach space X with the approximation property* (*in fact, having a boundedly complete basis*) *such that X^* is separable and does not have the approximation property. But $\mathcal{F}(Z, X)$ is an ideal in $\mathcal{K}(Z, X^{**})$ for all Banach spaces Z.*

Proof. Let us take a closed subspace W of c_0 that does not have the approximation property and use the famous James–Lindenstrauss construction [14] just as in [15, p. 35]. This gives us a Banach space Y such that W is isomorphic to Y^{**}/Y and such that $X = Y^{**}$ has all the properties described in the first half of the example. Since X is a dual space, there is a norm-1 projection P from X^{**} onto X. Then $\Phi\colon \mathcal{K}(Z,X)^* \to \mathcal{K}(Z,X^{**})^*$, defined by

$$(\Phi f)(T) = f(P \circ T), \quad f \in \mathcal{K}(Z,X)^*,\ T \in \mathcal{K}(Z,X^{**}),$$

is clearly a Hahn–Banach extension operator and therefore $\mathcal{K}(Z,X)$ is an ideal in $\mathcal{K}(Z,X^{**})$. Since $\mathcal{K}(Z,X) = \overline{\mathcal{F}(Z,X)}$ because X has the approximation property (this is well known and easy to verify; see e.g. [15, p. 32]), we also have that $\mathcal{F}(Z,X)$ is an ideal in $\mathcal{K}(Z,X^{**})$. □

Recall that a Banach space X is said to have the *unique extension property* if the only operator $T \in \mathcal{L}(X^{**},X^{**})$ such that $\|T\| \le 1$ and $T|_X = I_X$ is $T = I_{X^{**}}$. This property was introduced and deeply studied by Godefroy and Saphar in [4] (using the term "X is uniquely decomposed") and [5]. For instance, the following Banach spaces have the unique extension property (cf. [5]): spaces that have Phelps's uniqueness property U in their biduals (Hahn–Banach smooth spaces)—in particular, spaces that are M-ideals in their biduals; spaces with a Fréchet-differentiable norm; separable polyhedral Lindenstrauss spaces; and spaces of compact operators $\mathcal{K}(Z,X)$ for reflexive Z and X.

It is easy to verify (see [3]) that X has the unique extension property if and only if the canonical embedding from X^* to X^{***} is the only Hahn–Banach extension operator from X^* to X^{***}. This fact will be used in the proof of our next corollary.

COROLLARY 4.2. *Let X be a Banach space having the unique extension property. Then the following assertions are equivalent.*

(a) *X^* has the approximation property.*
(b) *$\mathcal{F}(Z,X)$ is an ideal in $\mathcal{W}(Z,X^{**})$ for all Banach spaces Z.*
(c) *$\mathcal{F}(Z,X)$ is an ideal in $\mathcal{K}(Z,X^{**})$ for all reflexive Banach spaces Z.*

Proof. (a) ⇒ (b) has been proved in [13, Cor. 4.7], and (b) ⇒ (c) is obvious. By Theorem 2.3, assertion (c) means that, for every Banach space Z and every $T \in \mathcal{K}(Z,X^{**})$, there exists a net $(T_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha^* \to T^*|_{X^*}$ in the strong operator topology. This implies (a) by Theorem 3.2. □

There is an extensive literature on a special subclass of spaces having the unique extension property: Banach spaces that are M-ideals in their biduals (see e.g. [7]). Recall that a closed subspace F of a Banach space E is an *M-ideal* in E if there exists a linear projection P on E^* with $\ker P = F^\perp$ such that $\|f\| = \|Pf\| + \|f - Pf\|$ for all $f \in E^*$.

It has been proved in [13, Cor. 4.8] that, if X is an M-ideal in its bidual X^{**} and if X^* has the approximation property, then $\overline{\mathcal{F}(Z,X)}$ is an M-ideal in $\mathcal{K}(Z,X^{**})$ for all Banach spaces Z. This, together with Corollary 4.2, immediately yields the following characterization.

COROLLARY 4.3. *Let X be an M-ideal in its bidual X^{**}. Then the following assertions are equivalent.*

(a) *X^* has the approximation property.*
(b) *$\overline{\mathcal{F}(Z,X)}$ is an M-ideal in $\mathcal{K}(Z,X^{**})$ for all Banach spaces Z.*
(c) *$\mathcal{F}(Z,X)$ is an ideal in $\mathcal{K}(Z,X^{**})$ for all reflexive Banach spaces Z.*

REMARK 4.1. Corollary 4.3 improves the following result of Werner [22, Cor. 2.4]: If X is an M-ideal in its bidual X^{**} and if the bidual X^{**} has the approximation property, then condition (b) of Corollary 4.3 holds.

REMARK 4.2. Corollary 4.3 can be extended from M-ideals to more general classes of ideals (for example, to ideals F in E with respect to an ideal projection P satisfying $\|af + bPf\| + c\|Pf\| \leq \|f\|$ for given numbers a, b, c and for all $f \in E^*$; these ideals were recently studied in [16] and [17]) under the assumption that X has the unique extension property.

It is well known that all closed subspaces of c_0 are M-ideals in their biduals. We can therefore apply Corollary 4.3 to the Johnson–Schechtman subspace of c_0 (see [8, Cor. JS]). This yields the following example.

EXAMPLE 4.4. *There is a closed subspace X of c_0 with the approximation property (in fact, having a basis) such that X^* does not have the approximation property. There also is a separable reflexive Banach space Z such that $\overline{\mathcal{F}(Z,X)} = \mathcal{K}(Z,X)$ is not an ideal in $\mathcal{K}(Z,X^{**})$.*

Proof. We need only show that a reflexive Banach space Z (given by Corollary 4.3) can be chosen separable. This is clear from [13, Thm. 2.7], which asserts that, for arbitrary Banach spaces X and Z, $\mathcal{K}(Z,X)$ is an ideal in $\mathcal{K}(Z,X^{**})$ if and only if $\mathcal{K}(Y,X)$ is an ideal in $\mathcal{K}(Y,X^{**})$ for every separable ideal Y in Z. □

REMARK 4.3. Example 4.4 shows, in particular, that the assumption "X^* has the compact approximation property with conjugate operators" is essential in [13, Cor. 4.8]; it cannot even be replaced by the assumption "X has a basis".

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