Michigan Math. J. 52 (2004)

The Equivariant Cohomology Ring of Regular Varieties

Michel Brion & James B. Carrell

0. Introduction

Let $\mathfrak{B}$ denote the upper triangular subgroup of $\mathrm{SL}_2(\mathbb{C})$, $\mathfrak{T}$ its diagonal torus and $\mathfrak{U}$ its unipotent radical. A complex projective variety Y is called *regular* if it is endowed with an algebraic action of $\mathfrak{B}$ such that the fixed-point set $Y^{\mathfrak{U}}$ is a single point. Associated to any regular $\mathfrak{B}$-variety Y is a remarkable affine curve $\mathcal{Z}_Y$ with a $\mathfrak{T}$-action, which was studied in [7]. In this note, we show that the coordinate ring $\mathbb{C}[\mathcal{Z}_Y]$ is isomorphic with the equivariant cohomology ring $H^*_{\mathfrak{T}}(Y)$ with complex coefficients when Y is smooth or, more generally, is a $\mathfrak{B}$-stable subvariety of a regular smooth $\mathfrak{B}$-variety X such that the restriction map from $H^*(X)$ to $H^*(Y)$ is surjective. This isomorphism is obtained as a refinement of the localization theorem in equivariant cohomology; it applies for example to Schubert varieties in flag varieties and to the Peterson variety studied in [11]. Another application of our isomorphism is a natural algebraic formula for the equivariant push-forward.

1. Preliminaries

Let $\mathfrak{B}$ be the group of upper triangular 2×2 complex matrices of determinant 1. Let $\mathfrak{T}$ (resp. $\mathfrak{U}$) be the subgroup of $\mathfrak{B}$ consisting of diagonal (resp. unipotent) matrices. We have isomorphisms $\lambda\colon \mathbb{C}^* \to \mathfrak{T}$ and $\varphi\colon \mathbb{C} \to \mathfrak{U}$, where

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad \varphi(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

which together satisfy the relation

$$\lambda(t)\varphi(u)\lambda(t^{-1}) = \varphi(t^2u). \tag{1}$$

Consider the generators

$$\mathcal{V} = \dot{\varphi}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{W} = \dot{\lambda}(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the Lie algebras $\mathrm{Lie}(\mathfrak{U})$ and $\mathrm{Lie}(\mathfrak{T})$, respectively. Then $[\mathcal{W}, \mathcal{V}] = 2\mathcal{V}$, and

$$\mathrm{Ad}(\varphi(u))\mathcal{W} = \mathcal{W} - 2u\mathcal{V} \tag{2}$$

for all $u \in \mathbb{C}$.

Received November 27, 2002. Revision received May 29, 2003.

The second author was partially supported by the Natural Sciences and Engineering Research Council of Canada.

In this paper, X will denote a smooth complex projective algebraic variety endowed with an algebraic action of $\mathfrak{B}$ such that the fixed-point scheme $X^{\mathfrak{U}}$ consists of one point o. Both the $\mathfrak{B}$-variety X and the action will be called *regular.* Then $o \in X^{\mathfrak{T}}$, since $X^{\mathfrak{U}}$ is $\mathfrak{T}$-stable. Moreover, $X^{\mathfrak{T}}$ is finite by Lemma 1 of [7]. Thus we may write

$$X^{\mathfrak{T}} = \{\zeta_1 = o,\ \zeta_2, \dots, \zeta_r\}. \tag{3}$$

Clearly $r = \chi(X)$, the Euler characteristic of X.

Let $H^*_{\mathfrak{T}}(X)$ denote the $\mathfrak{T}$-equivariant cohomology ring of X with complex coefficients. To define it, let $\mathcal{E}$ be a contractible space with a free action of $\mathfrak{T}$ and let $X_{\mathfrak{T}} = (X \times \mathcal{E})/\mathfrak{T}$ (quotient by the diagonal $\mathfrak{T}$-action). Then

$$H^*_{\mathfrak{T}}(X) = H^*(X_{\mathfrak{T}}).$$

It is well known that the equivariant cohomology ring $H^*_{\mathfrak{T}}(\mathrm{pt})$ of a point is the polynomial ring $\mathbb{C}[z]$, where z denotes the linear form on the Lie algebra of $\mathfrak{T}$ such that $z(\mathcal{W}) = 1$. The degree of z is 2. Thus $H^*_{\mathfrak{T}}(X)$ is a graded algebra over the polynomial ring $\mathbb{C}[z] = H^*_{\mathfrak{T}}(\mathrm{pt})$ (via the constant map $X \to \mathrm{pt}$).

In our situation, the restriction map in cohomology

$$i^*_{\mathfrak{T}} \colon H^*_{\mathfrak{T}}(X) \to H^*_{\mathfrak{T}}(X^{\mathfrak{T}})$$

induced by the inclusion $i \colon X^{\mathfrak{T}} \hookrightarrow X$ is injective (see [10]). By (3),

$$H^*_{\mathfrak{T}}(X^{\mathfrak{T}}) = \bigoplus_{j=1}^{r} H^*_{\mathfrak{T}}(\zeta_j) \cong \bigoplus_{j=1}^{r} \mathbb{C}[z],$$

so each $\alpha \in H^*_{\mathfrak{T}}(X)$ defines an r-tuple of polynomials $(\alpha_{\zeta_1}, \dots, \alpha_{\zeta_r})$. That is,

$$i^*_{\mathfrak{T}}(\alpha) = (\alpha_{\zeta_1}, \dots, \alpha_{\zeta_r}). \tag{4}$$

We will define a refined version of this restriction map in Section 4.

2. The $\mathfrak{B}$-Stable Curves

Throughout this paper, a curve in X will be a purely one-dimensional closed subset of X; a curve that is stable under a subgroup G of $\mathfrak{B}$ is called a G-curve. The $\mathfrak{B}$-curves in X play a crucial role, so we next establish a few of their basic properties.

PROPOSITION 1. *If X is a regular $\mathfrak{B}$-variety, then every irreducible $\mathfrak{B}$-curve C in X has the form $C = \overline{\mathfrak{B} \cdot \zeta_j}$ for some index $j \geq 2$. Moreover, every $\mathfrak{B}$-curve contains o. In particular, there are only finitely many irreducible $\mathfrak{B}$-curves in X, and they all meet at o.*

Proof. It is clear that if $j \geq 2$ then $C = \overline{\mathfrak{B} \cdot \zeta_j}$ is a $\mathfrak{B}$-curve in X containing ζ_j that (by the Borel fixed-point theorem) also contains o, since o is the only $\mathfrak{B}$-fixed point. Conversely, every $\mathfrak{B}$-curve C in X contains o and at least one other $\mathfrak{T}$-fixed point. Indeed, since o has an affine open $\mathfrak{T}$-stable neighborhood X_o in X, the complement $C - X_o$ is nonempty and $\mathfrak{T}$-stable. It follows immediately that $C = \overline{\mathfrak{B} \cdot x}$ for some $x \in X^{\mathfrak{T}} - o$. $\square$

Consider the action of $\mathfrak{B}$ on the projective line $\mathbb{P}^1$ given by

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot z = \frac{z}{t(t+uz)}$$

(the inverse of the standard action). This action has $(\mathbb{P}^1)^{\mathfrak{T}} = \{0, \infty\}$ and $(\mathbb{P}^1)^{\mathfrak{U}} = \{0\}$ and is therefore regular. Note that, if $u \in \mathbb{C}^*$, then

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot \infty = \frac{1}{tu}$$

and so $\varphi(u) \cdot \infty = u^{-1}$.

The diagonal action of $\mathfrak{B}$ on $X \times \mathbb{P}^1$ is also regular. By Proposition 1, the irreducible $\mathfrak{B}$-curves in $X \times \mathbb{P}^1$ are of the form $\overline{\mathfrak{B} \cdot (x, \infty)}$ or $\overline{\mathfrak{B} \cdot (x, 0)}$, where $x \in X^{\mathfrak{T}}$. Only the first type will play a role here. Thus, we put $Z_j = \overline{\mathfrak{B} \cdot (\zeta_j, \infty)}$ and let $\pi_j : Z_j \to \mathbb{P}^1$ be the second projection. Clearly each π_j is bijective, hence $Z_j \cong \mathbb{P}^1$. In addition,

$$Z_j = \{(\varphi(u) \cdot \zeta_j, u^{-1}) \mid u \in \mathbb{C}^*\} \cup \{(\zeta_j, \infty)\} \cup \{(o, 0)\},$$

so $Z_i \cap Z_j = \{(o, 0)\}$ as long as $i \neq j$. Moreover, restricting π_j gives an isomorphism

$$p_j : Z_j - (\zeta_j, \infty) \to \mathbb{A}^1.$$

Finally, we put

$$Z = \bigcup_{1 \le j \le r} Z_j.$$

Thus, Z is the union of all irreducible $\mathfrak{B}$-stable curves in $X \times \mathbb{P}^1$ that are mapped onto $\mathbb{P}^1$ by the second projection $\pi : X \times \mathbb{P}^1 \to \mathbb{P}^1$.

3. The Fundamental Scheme $\mathcal{Z}$

Let $\mathcal{A}$ denote the vector field on $X \times \mathbb{A}^1$ defined by

$$\mathcal{A}_{(x,v)} = 2\mathcal{V}_x - v\mathcal{W}_x. \tag{5}$$

Obviously, $\mathcal{A}$ is tangent to the fibres of the projection to $\mathbb{A}^1$. By (2),

$$(\mathrm{Ad}(\varphi(u))\mathcal{W})_x = -u\mathcal{A}_{(x,u^{-1})}. \tag{6}$$

The contraction operator $i(\mathcal{A})$ defines a sheaf of ideals $i(\mathcal{A})(\Omega^1_{X \times \mathbb{A}^1})$ of the structure sheaf of $X \times \mathbb{A}^1$. Let $\mathcal{Z}$ denote the associated closed subscheme of $X \times \mathbb{A}^1$. In other words, $\mathcal{Z}$ is the zero scheme of $\mathcal{A}$.

Remark 1. The vector field $\mathcal{A}$ discussed here is a variant of the vector field studied in [7]. Both have the same zero scheme.

The properties of $\mathcal{Z}$ figure prominently in this paper. First put

$$X_o = \left\{ x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x = o \right\}. \tag{7}$$

Clearly, X_o is $\mathfrak{T}$-stable, and it follows easily from (1) that X_o is open in X (see [7, Prop. 1] for details). Hence, by the Bialynicki-Birula decomposition theorem [3], X_o is $\mathfrak{T}$-equivariantly isomorphic to the tangent space $T_o X$, where $\mathfrak{T}$ acts by its canonical representation at a fixed point. The weights of the associated action of λ on $T_o X$ are all negative. Thus we may choose coordinates $x_1, \dots, x_n$ on $X_o \cong T_o X$ that are eigenvectors of $\mathfrak{T}$; the weight a_i of x_i is a positive integer (it turns out to be even; see [2]). This identifies the positively graded ring $\mathbb{C}[X_o]$ with $\mathbb{C}[x_1, \dots, x_n]$, where $\deg x_i = a_i$. Now $X \times \mathbb{P}^1$ contains $X_o \times \mathbb{A}^1$ as a $\mathfrak{T}$-stable affine open subset with coordinate ring $\mathbb{C}[x_1, \dots, x_n, v]$, where v has degree 2.

PROPOSITION 2. *The scheme $\mathcal{Z}$ is reduced and is contained in $X_o \times \mathbb{A}^1$ as a $\mathfrak{T}$-curve. Its ideal in $\mathbb{C}[X_o \times \mathbb{A}^1] = \mathbb{C}[x_1, \dots, x_n, v]$ is generated by*

$$v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n). \tag{8}$$

These form a homogeneous regular sequence in $\mathbb{C}[x_1, \dots, x_n, v]$, and the degree of each $v\mathcal{W}(x_i) - 2\mathcal{V}(x_i)$ equals $a_i + 2$. The irreducible components of $\mathcal{Z}$ are the $\mathcal{Z}_j$ $(1 \le j \le r)$, where

$$\mathcal{Z}_j = Z_j - \{(\zeta_j, \infty)\}. \tag{9}$$

Each $\mathcal{Z}_j$ is mapped isomorphically to $\mathbb{A}^1$ by the second projection p. In particular, p is finite and flat of degree r, and $\mathcal{Z}$ has r irreducible components. Any two such components meet only at $(o, 0)$.

Proof. This follows from the results in [7, Sec. 3]. We provide direct arguments for the reader's convenience.

Since $[\mathcal{W}, \mathcal{V}] = 2\mathcal{V}$ and $\mathcal{W}(v) = 2v$ (on $\mathbb{A}^1$), it follows that $\mathcal{A}$ commutes with the vector field induced by the diagonal $\mathfrak{T}$-action on $X \times \mathbb{A}^1$. Hence, $\mathcal{Z}$ is $\mathfrak{T}$-stable.

Next we claim that (9) holds set-theoretically. Let $(x, v) \in X \times \mathbb{A}^1$ with $v \neq 0$. Then $(x, v) \in \mathcal{Z}$ if and only if $\mathcal{W}_x - 2v^{-1}\mathcal{V}_x = 0$, that is, iff x is a zero of $\mathrm{Ad}(\varphi(v^{-1}))\mathcal{W}$; equivalently, $\varphi(-v^{-1}) \cdot x \in X^{\mathfrak{T}}$. On the other hand, $(x, 0) \in \mathcal{Z}$ if and only if $\mathcal{V}_x = 0$, that is, iff $x = o$. Thus $\mathcal{Z} = Z \cap (X \times \mathbb{A}^1)$ (as sets). Further, $Z \cap (X \times \mathbb{A}^1)$ is an open affine $\mathfrak{T}$-stable neighborhood of $(o, 0)$ in Z and therefore equals $Z \cap (X_o \times \mathbb{A}^1)$. This implies our claim.

It follows that $\mathcal{Z} \subseteq X_o \times \mathbb{A}^1$ (as schemes), so that the ideal of $\mathcal{Z}$ is generated by $v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$. These polynomials are homogeneous of degrees $a_1 + 2, \dots, a_n + 2$; together with v, they have only the origin as their common zero (since o is the unique zero of $\mathcal{V}$). Hence

$$v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n), v$$

form a regular sequence in $\mathbb{C}[x_1, \dots, x_n, v]$, and v is a nonzero divisor in $\mathbb{C}[\mathcal{Z}]$. As a consequence, the $\mathbb{C}[v]$-module $\mathbb{C}[\mathcal{Z}]$ is finitely generated and free. In other words, $p\colon \mathcal{Z} \to \mathbb{A}^1$ is finite and flat.

Fix $v_0 \neq 0$ and consider the scheme-theoretic intersection $\mathcal{Z} \cap (X \times v_0)$. This identifies with the zero scheme of $2\mathcal{V}_x - v_0\mathcal{W}_x$ in X, that is, to the zero scheme of $\mathrm{Ad}(\varphi(v_0^{-1})\mathcal{W})$. The latter consists of $r = \chi(X)$ distinct points, so that $\mathcal{Z} \cap (X \times v_0)$ is reduced. Since $\mathbb{C}[\mathcal{Z}]$ is a free module over $\mathbb{C}[v]$, it follows easily that $\mathcal{Z} \cap (v \neq 0)$ is reduced. But the open subset $\mathcal{Z} \cap (v \neq 0)$ is dense in $\mathcal{Z}$ by (9), and $\mathcal{Z}$ is a complete intersection in $\mathbb{A}^{n+1}$. Thus, $\mathcal{Z}$ is reduced. This completes the proof. □

4. The Refined Restriction

We now define our refined restriction on equivariant cohomology. Let $\alpha \in H^*_{\mathfrak{T}}(X)$. Recall from (4) that $i^*_{\mathfrak{T}}(\alpha) = (\alpha_{\zeta_1}, \ldots, \alpha_{\zeta_r})$, where each $\alpha_{\zeta_j} \in \mathbb{C}[z]$. We regard each α_{ζ_j} as a polynomial function on $\mathcal{Z}_j$ (isomorphic to $\mathbb{A}^1$ via p) and hence on $\mathcal{Z}_j - (o, 0)$. Since $\mathcal{Z} - (o, 0)$ is the disjoint union of the $\mathcal{Z}_j - (o, 0)$, this yields an algebra homomorphism

$$\rho\colon H^*_{\mathfrak{T}}(X) \to \mathbb{C}[\mathcal{Z} - (o, 0)]$$

such that $\rho(\alpha)(x, v) = \alpha_{\zeta_j}(v)$ whenever $(x, v) \in \mathcal{Z}_j - (o, 0)$. In particular, $\rho(z)(x, v) = v$, so that $\rho(z) = v$. And since $i^*_{\mathfrak{T}}$ preserves the grading, the same holds for ρ.

Note that the value $\alpha_{\zeta_j}(0)$ at the origin is independent of the index j. (For $X_{\mathfrak{T}} = (X \times \mathcal{E})/\mathfrak{T}$ is connected since both X and $\mathcal{E}$ are, so $H^0(X_{\mathfrak{T}})$ is by definition the set of constant functions on $X_{\mathfrak{T}}$; consequently, the component of α in degree 0 gives the same value at each fixed point.) Now let $\mathbb{C}_0[\mathcal{Z}]$ denote the subalgebra of $\mathbb{C}[\mathcal{Z} - (0, o)]$ consisting of all elements that extend continuously to $\mathcal{Z}$ in the classical topology. Then

$$\rho(H^*_{\mathfrak{T}}(X)) \subseteq \mathbb{C}_0[\mathcal{Z}].$$

We will demonstrate that $\rho(H^*_{\mathfrak{T}}(X))$ is in fact the coordinate ring $\mathbb{C}[\mathcal{Z}]$. Toward this end, we compute the image under ρ of an equivariant Chern class.

5. Equivariant Chern Theory

Suppose Y is an algebraic variety with an action of an algebraic group G. Then recall that a vector bundle E on Y is said to be *G-linearized* if there is an action of G on E lifting that on Y and such that each $g \in G$ defines a linear map from E_y to $E_{g\cdot y}$ for any $y \in Y$. In particular, if $y \in Y^G$ then we have a representation of G in E_y, and hence a representation of $\mathrm{Lie}(G)$. Thus, each $\xi \in \mathrm{Lie}(G)$ acts on E_y by $\xi_y \in \mathrm{End}(E_y)$.

Also recall that the kth equivariant Chern class $c_k^G(E) \in H_G^{2k}(Y)$ is defined to be the kth Chern class of the vector bundle

$$E_G = (E \times \mathcal{E})/G \to X_G = (X \times \mathcal{E})/G,$$

where $\mathcal{E}$ is a contractible space with a free action of G. For $y \in Y^G$, the restriction $c_k^G(E)_y$ lies in $H_G^*(\mathrm{pt})$. The latter identifies with a subring of the coordinate ring of $\mathrm{Lie}(G)$, and we have

$$c_k^G(E)_y(\xi) = \mathrm{Tr}_{\wedge^k E_y}(\xi_y)$$

for any $\xi \in \mathrm{Lie}(G)$.

Returning to our previous situation, let E be a $\mathfrak{B}$-linearized vector bundle on X. Then each $(x, v) \in \mathcal{Z}$ is a zero of $v\mathcal{W} - 2\mathcal{V} \in \mathrm{Lie}(\mathfrak{B})$. This yields an element $(v\mathcal{W} - 2\mathcal{V})_x \in \mathrm{End}(E_x)$.

LEMMA 1. *Let E be a $\mathfrak{B}$-linearized vector bundle on X, and let k be a nonnegative integer. Then, for any $(x, v) \in \mathcal{Z}$ we have*

$$\rho(c_k^T(E))(x, v) = \mathrm{Tr}_{\wedge^k E_x}((v\mathcal{W} - 2\mathcal{V})_x). \tag{10}$$

As a consequence, $\rho(c_k^T(E)) \in \mathbb{C}[\mathcal{Z}]$.

Proof. It suffices to check (10) for $v \neq 0$. Let j be the index such that $x = \varphi(v^{-1}) \cdot \zeta_j$. Letting $\mathcal{W}_{\zeta_j} \in \mathrm{End}(E_{\zeta_j})$ denote the lift of $\mathcal{W}$ at ζ_j, we have

$$\rho(c_k^T(E))(x, v) = c_k^T(E)_{\zeta_j}(v) = v^k \, \mathrm{Tr}_{\wedge^k E_{\zeta_j}}(\mathcal{W}_{\zeta_j}). \tag{11}$$

Since E is $\mathfrak{B}$-linearized, this equals

$$\begin{aligned} v^k \, \mathrm{Tr}_{\wedge^k E_x}((\mathrm{Ad}(\varphi(v^{-1}))\mathcal{W})_x) &= v^k \, \mathrm{Tr}_{\wedge^k E_x}((\mathcal{W} - 2v^{-1}\mathcal{V})_x) \\ &= \mathrm{Tr}_{\wedge^k E_x}((v\mathcal{W} - 2\mathcal{V})_x), \end{aligned}$$

which proves (10). $\square$

We now obtain some of the main properties of ρ.

PROPOSITION 3. *The image of the morphism $\rho \colon H^*_{\mathfrak{T}}(X) \to \mathbb{C}[\mathcal{Z} - (o, 0)]$ is contained in $\mathbb{C}[\mathcal{Z}]$.*

Proof. Since X is smooth and projective, we have an exact sequence

$$0 \to zH^*_{\mathfrak{T}}(X) \to H^*_{\mathfrak{T}}(X) \to H^*(X) \to 0. \tag{12}$$

By [7, Prop. 3], the algebra $H^*(X)$ is generated by Chern classes of $\mathfrak{B}$-linearized vector bundles on X; it then follows by Nakayama's lemma that $H^*_{\mathfrak{T}}(X)$ is generated (as an algebra) by their equivariant Chern classes. This together with Lemma 1 now implies that $\rho(H^*_{\mathfrak{T}}(X)) \subseteq \mathbb{C}[\mathcal{Z}]$. $\square$

6. The First Main Result

THEOREM 1. *For a smooth projective regular variety X, the homomorphism*

$$\rho \colon H^*_{\mathfrak{T}}(X) \to \mathbb{C}[\mathcal{Z}]$$

is an isomorphism of graded algebras.

Proof. To see that ρ is injective, suppose $\rho(\alpha) = 0$. Then each α_{ζ_j} is 0 and so $i^*_{\mathfrak{T}}(\alpha) = 0$, where $i^*_{\mathfrak{T}}$ is the restriction (see (4)). But, as noted previously, $i^*_{\mathfrak{T}}$ is injective, so $\alpha = 0$.

To complete the proof, it suffices to check that the Poincaré series of $H^*_{\mathfrak{T}}(X)$ and $\mathbb{C}[\mathcal{Z}]$ coincide (since both algebras are positively graded, and ρ preserves the grading). Denote the former by $F_X(t)$ and the latter by $F_{\mathcal{Z}}(t)$. By (12), we have an isomorphism

$$H^*_{\mathfrak{T}}(X)/zH^*_{\mathfrak{T}}(X) \cong H^*(X). \tag{13}$$

Since $H^*_{\mathfrak{T}}(X)$ is a free $\mathbb{C}[z]$-module and z has degree 2, (13) implies

$$F_X(t) = \frac{P_X(t)}{1-t^2}, \tag{14}$$

where $P_X(t)$ is the Poincaré polynomial of $H^*(X)$. On the other hand, since $p\colon \mathcal{Z} \to \mathbb{A}^1$ is finite, flat, and $\mathfrak{T}$-equivariant (where $\mathfrak{T}$ acts linearly on $\mathbb{A}^1$ with weight 2), it follows that

$$F_{\mathcal{Z}}(t) = \frac{P_{\mathcal{Z}}(t)}{1-t^2},$$

where $P_{\mathcal{Z}}(t)$ is the Poincaré polynomial of the finite-dimensional graded algebra $\mathbb{C}[\mathcal{Z}]/(v)$. We now use the fact that the cohomology ring of X is isomorphic as a graded algebra to the coordinate ring of the zero scheme of $\mathcal{V}$ with the principal grading [7]. So

$$H^*(X) \cong \mathbb{C}[x_1, \dots, x_n]/(\mathcal{V}(x_1), \dots, \mathcal{V}(x_n)) \cong \mathbb{C}[\mathcal{Z}]/(v), \tag{15}$$

where the second isomorphism follows from Proposition 2. Thus, $P_X(t) = P_{\mathcal{Z}}(t)$, whence $F_X(t) = F_{\mathcal{Z}}(t)$. $\square$

For a simple example, let $X = \mathbb{P}^n$. Let $e\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ denote the nilpotent linear transformation defined by

$$v_n \to v_{n-1} \to \cdots \to v_1 \to v_0 \to 0,$$

where $(v_0, v_1, \dots, v_n)$ is the standard basis of $\mathbb{C}^{n+1}$. Also, let $h = \operatorname{diag}(n, n-2, \dots, -n+2, -n)$. Then $[h, e] = 2e$, so we obtain a $\mathfrak{B}$-action on $\mathbb{P}^n$. This action is regular with unique fixed point $[v_0] = [1, 0, \dots, 0]$; its neighborhood X_o is the standard affine chart centered at $[v_0]$. Let $(x_1, \dots, x_n)$ be the usual affine coordinates at $[v_0]$. That is, $x_j = z_j/z_0$, where $[z_0, z_1, \dots, z_n]$ are the homogeneous coordinates on $\mathbb{P}^n$. Then each x_j is homogeneous of degree $2j$ and, by Proposition 2,

$$e(x_1) = x_2 - x_1^2,\ e(x_2) = x_3 - x_1x_2,\ \dots,$$
$$e(x_{n-1}) = x_n - x_1x_{n-1},\ e(x_n) = -x_1x_n.$$

Thus, the ideal of $\mathcal{Z}$ in $\mathbb{C}[x_1, \dots, x_n, v]$ is generated by $-x_2 + x_1(x_1 + v)$, $-x_3 + x_2(x_1 + 2v)$, $\dots$, $-x_n + x_{n-1}(x_1 + (n-1)v)$, $x_n(x_1 + nv)$. After we eliminate $x_2, x_3, \dots, x_{n-1}$, Theorem 1 says that

$$H^*_{\mathfrak{T}}(\mathbb{P}^n) \cong \mathbb{C}[x_1, v]/\big(\textstyle\prod_{m=0}^n (x_1 + mv)\big).$$

By equivariant Chern theory, $x_1 = -c_1^{\mathfrak{T}}(L)$, where L is the tautological line bundle on $\mathbb{P}^n$. This presentation of $H^*_{\mathfrak{T}}(\mathbb{P}^n)$ can be derived directly and is probably well known.

7. The General Case

In this section we will make some comments on regular $\mathfrak{B}$-varieties, dropping the smoothness assumption. Let Y be a complex projective variety endowed with a $\mathfrak{B}$-action such that $Y^{\mathfrak{U}}$ is a unique point o. If Y is singular then it is necessarily singular at o, so the $\mathfrak{T}$-stable neighborhood Y_o of o defined in (7) is singular. Hence the results in Section 3 on the structure of $\mathcal{Z}$ (relative to Y) do not necessarily obtain. To be able to conclude something, let us assume that Y is equivariantly embedded into a smooth projective regular $\mathfrak{B}$-variety X. Thus the curve $\mathcal{Z}$ (relative to X) is well-defined and enjoys all the properties derived previously. We can therefore define

$$\mathcal{Z}_Y = \mathcal{Z} \cap (Y_o \times \mathbb{A}^1), \tag{16}$$

taking the intersection to be reduced. In other words, $\mathcal{Z}_Y$ is the union of the components of $\mathcal{Z}$ lying in $Y_o \times \mathbb{A}^1$. Now the construction of Section 4 yields a graded homomorphism

$$\rho_Y \colon H^*_{\mathfrak{T}}(Y) \to \mathbb{C}_0[\mathcal{Z}_Y].$$

If we make the additional assumption that $H^*(Y)$ is generated by Chern classes of $\mathfrak{B}$-linearized vector bundles, then the odd cohomology of Y is trivial, so that $H^*_{\mathfrak{T}}(Y)$ is a free module over $\mathbb{C}[z]$, and the restriction $i^*_{\mathfrak{T}} \colon H^*_{\mathfrak{T}}(Y) \to H^*_{\mathfrak{T}}(Y^{\mathfrak{T}})$ is injective [10]. Hence, ρ_Y is injective. By exactly the same argument, $\rho_Y(H^*_{\mathfrak{T}}(Y)) \subseteq \mathbb{C}[\mathcal{Z}_Y]$. The obstruction to ρ_Y being an isomorphism is therefore the equality of Poincaré series of these graded algebras.

There is a natural situation where this assumption is satisfied, so we can generalize Theorem 1. Assume that the inclusion $j \colon Y \to X$ induces a surjection $j^* \colon H^*(X) \to H^*(Y)$. Then $H^*(Y)$ is generated by Chern classes of $\mathfrak{B}$-linearized vector bundles, and from the Leray spectral sequence we see that $j^*_{\mathfrak{T}} \colon H^*_{\mathfrak{T}}(X) \to H^*_{\mathfrak{T}}(Y)$ is surjective. The surjectivity of j^* holds, for example, in the case of Schubert varieties in flag varieties. See also Section 8.

Proceeding as just described, we obtain an extension of Theorem 1.

THEOREM 2. *Suppose that X is a smooth projective variety with a regular $\mathfrak{B}$-action and that Y is a closed $\mathfrak{B}$-stable subvariety for which the restriction map $H^*(X) \to H^*(Y)$ is surjective. Then the map $\rho_Y \colon H^*_{\mathfrak{T}}(Y) \to \mathbb{C}_0[\mathcal{Z}_Y]$ yields a graded algebra isomorphism*

$$\rho_Y \colon H^*_{\mathfrak{T}}(Y) \to \mathbb{C}[\mathcal{Z}_Y]$$

that fits into a commutative diagram

$$\begin{array}{ccc} H^*_{\mathfrak{T}}(X) & \xrightarrow{\rho} & \mathbb{C}[\mathcal{Z}] \\ \downarrow & & \downarrow \\ H^*_{\mathfrak{T}}(Y) & \xrightarrow{\rho_Y} & \mathbb{C}[\mathcal{Z}_Y], \end{array} \tag{17}$$

where the vertical maps are the natural restrictions. Both vertical maps are surjections.

Proof. By Theorem 1 we know that ρ is an isomorphism, and by our previous remarks we know that ρ_Y is injective. Since the restriction map $\mathbb{C}[\mathcal{Z}] \to \mathbb{C}[\mathcal{Z}_Y]$ is clearly surjective, it therefore follows that ρ_Y is also. □

As a corollary, we obtain one of the main results of [7].

COROLLARY 1. *With the notation and assumptions of Theorem 2, there exists a commutative diagram*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{Z}]/(v) & \xrightarrow{\psi} & H^*(X) \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathcal{Z}_Y]/(v) & \xrightarrow{\psi_Y} & H^*(Y), \end{array} \tag{18}$$

where ψ and ψ_Y are graded algebra isomorphisms and the vertical maps are the natural restrictions.

Note that $\mathbb{C}[\mathcal{Z}_Y]/(v)$ is the coordinate ring of the schematic intersection of $\mathcal{Z}_Y$ and $X \times 0$ in $X \times \mathbb{A}^1$.

8. Equivariant Cohomology of the Peterson Variety

Let G be a complex semi-simple linear algebraic group. Fix a pair of opposite Borel subgroups B and B^- and let $T = B \cap B^-$, a maximal torus of G. Denote the corresponding Lie algebras by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{b}^-$, and $\mathfrak{t}$, and let Φ^+ and Φ^- be the roots of the pair (G, T) that arise from $\mathfrak{b}$ and $\mathfrak{b}^-$, respectively.

Let M be a B-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$. Then

$$M = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Omega(M)} \mathfrak{g}_\alpha, \quad \text{where } \Omega(M) = \{\alpha \in \Phi^- \mid \mathfrak{g}_\alpha \subset M\}.$$

Hence $\Omega(M)$ is the set of weights of the quotient $M/\mathfrak{b}$. The B-module $M \subseteq \mathfrak{g}$ yields a homogeneous vector bundle $G \times^B M$ over G/B together with a morphism $G \times^B M \to \mathfrak{g}$ induced by $(g, m) \mapsto gm$. The fiber of this morphism at an arbitrary $x \in \mathfrak{g}$ identifies with

$$Y_M(x) = \{gB \in G/B \mid g^{-1}x \in M\}.$$

This is a closed subvariety of G/B that is stable under the action of $G_{\mathbb{C}x}$ (the isotropy group of the line $\mathbb{C}x$).

If x is regular and semi-simple, then $Y_M(x)$ is a nonsingular variety known as a *Hessenberg variety.* The Poincaré polynomial of $Y_M(x)$ was determined in [8].

On the other hand, if x is nilpotent then the $Y_M(x)$ give a broad class of projective varieties containing, for example, the variety $\mathcal{B}_x \subset G/B$ of Borel subgroups of G whose Lie algebra contains x. If x is also regular then it defines a canonical $\mathfrak{B}$-action on G/B stablizing $Y_M(x)$. The cohomology ring of $Y_M(x)$ has been obtained by Dale Peterson (unpublished). In fact, according to Peterson's result, the $Y_M(x)$ satisfy the hypotheses of Theorem 2.

To describe this cohomology ring, we first make a convenient choice for x. Let $\alpha_1, \dots, \alpha_l \in \Phi^+$ denote the simple roots, and fix a nonzero e_{α_j} in each T-weight space $\mathfrak{g}_{\alpha_j} \subset \mathfrak{b}$. Then for x choose the principal nilpotent $e = \sum_{j=1}^{l} e_{\alpha_j}$. Let $h \in \mathfrak{t}$ be the unique element for which $\alpha_j(h) = 2$ for each index j. Then the pair $(e, h) \in \mathfrak{b} \times \mathfrak{t}$ determines a regular $\mathfrak{B}$-action on G/B with unique $\mathfrak{B}$-fixed point $o = B$ that stabilizes $Y_M(e)$ for any M. Note that the semi-simple element h is also regular.

Let $\mathfrak{T} \subset \mathfrak{B}$ denote the maximal torus in $\mathfrak{B}$ with Lie algebra $\mathbb{C}h$. Since h is regular, $Y_M(e)^{\mathfrak{T}} = (G/B)^T \cap Y_M(e)$. It is well known that $(G/B)^T = \{n_w B \mid w \in W\}$, where $W = N_G(T)/T$ is the Weyl group of (G, T) and n_w is a representative of w. But $n_w B \in Y_M(e)^{\mathfrak{T}}$ if and only if $n_w^{-1} e \in M$; that is, $w^{-1}(\alpha_j) \in \Omega(M) \cup \Phi^+$ for each j.

Formulated in these terms, Peterson's result can be stated as follows.

THEOREM 3. *For any B-submodule M of $\mathfrak{g}$ containing $\mathfrak{b}$, the restriction map*

$$H^*(G/B) \to H^*(Y_M(e))$$

is surjective. Hence, by Theorem 2,

$$H^*_{\mathfrak{T}}(Y_M(e)) \cong \mathbb{C}[\mathcal{Z}_{Y_M(e)}].$$

Moreover, $\mathcal{Z}_{Y_M(e)}$ is a complete intersection. Thus, the cohomology ring $H^*(Y_M(e))$ satisfies Poincaré duality. Its Poincaré polynomial is given by the product formula

$$P(Y_M(e), t) = \prod_{-\alpha \in \Omega(M)} \frac{1 - t^{ht(\alpha)+1}}{1 - t^{ht(\alpha)}}, \tag{19}$$

where $ht(\alpha)$ denotes the sum of the coefficients of $\alpha \in \Phi^+$ over the simple roots.

If $M = \mathfrak{g}$ then (19) is, of course, a well-known product formula for the Poincaré polynomial of the flag variety.

The ideals $I(\mathcal{Z}_{Y_M(e)})$ admit explicit expressions related to the geometry of the nilpotent variety of $\mathfrak{g}$. Let $U^- \subset G$ denote the unipotent radical of B^-. Since the natural map $\mu\colon U^- \to G/B$, $\mu(u) = uB$, is T-equivariant with respect to the conjugation action of T on U^-, we may equivariantly identify U^- and the open cell $(G/B)_o$. Let $\mathfrak{u}^-$ denote the Lie algebra of U^-, and let $\Pi_*\colon \mathfrak{g} \to \mathfrak{u}^-$ denote the projection. To get this explicit picture, we first note that $L_{u^{-1}*}\mathcal{V}_u = \Pi_*(u^{-1}e)$, where L_u denotes left translation by $u \in U^-$ and L_{u*} is its differential (cf. [6, Sec. 2.1]). Consequently,

$$L_{u^{-1}*}\mathcal{A}_{(u,v)} = 2\Pi_*(u^{-1}e) - vL_{u^{-1}*}\mathcal{W}_u. \tag{20}$$

Defining $F_\alpha(u)$ as the component of $L_{u^{-1}*}\mathcal{A}_{(u,v)}$ in $\mathfrak{g}_\alpha$, it follows that

$$I(\mathcal{Z}) = (F_\alpha \mid \alpha \in \Phi^-).$$

More precisely, we can write

$$u^{-1}e = e + k(u) + \sum_{\alpha \in \Phi^-} v_\alpha(u) e_\alpha,$$

where $k \in \mathfrak{t} \otimes \mathbb{C}[U^-]$ and all $v_\alpha \in \mathbb{C}[U^-]$. Then $F_\alpha = 2v_\alpha - vw_\alpha$, where we have put $w_\alpha(u) = (L_{u^{-1}*}\mathcal{W}_u)_\alpha$. Hence,

$$H^*_{\mathfrak{T}}(G/B, \mathbb{C}) \cong \mathbb{C}[U^- \times \mathbb{A}^1]/(2v_\alpha - vw_\alpha \mid \alpha \in \Phi^-).$$

Since the functions v_β ($\beta \in \Phi^- - \Omega(M)$) cut out $Y_M(e)_o \times \mathbb{A}^1$, it follows that $I(\mathcal{Z}_{Y_M(e)})$ is the ideal of the variety cut out by the $2v_\alpha - vw_\alpha$ (where $\alpha \in \Phi^-$) and the v_β (where $\beta \in \Phi^- - \Omega(M)$).

The case where

$$M = \mathfrak{b} \oplus \bigoplus_{\substack{\alpha \in \Phi^+ \\ \alpha(h) > 2}} \mathfrak{g}_{-\alpha} \tag{21}$$

is of particular interest. Then, as shown by Kostant (see [11, (9)]), the Givental–Kim and Peterson formulas for the flag variety quantum cohomology [9; 12] may be interpreted as asserting an isomorphism of graded rings

$$\mathbb{C}[Y_M(e)_o] \cong QH^*(G^\vee/B^\vee),$$

where $G^\vee$ and $B^\vee$ denote the Langlands duals of G and B (respectively) and $QH^*(G^\vee/B^\vee)$ is the complex quantum cohomology ring of $G^\vee/B^\vee$. This led Kostant to call $Y_M(e)$ the *Peterson variety*. Notice that, by Theorem 3, $P(Y_M(e), t) = (1+t)^{n-1}$.

More recently, Tymoczko [13] has shown that the varieties $Y_M(e)$ admit affine cell decompositions when G is of classical type.

9. An Alternative Proof of Theorem 1

We now give another proof of Theorem 1 that is independent of (15). Hence we will also obtain another proof of (15). Denote by $[X^{\mathfrak{T}}]_{\mathfrak{T}} \in H^*_{\mathfrak{T}}(X)$ the equivariant cohomology class of the $\mathfrak{T}$–fixed-point set. We compute its image under $\rho\colon H^*_{\mathfrak{T}}(X) \to \mathbb{C}[\mathcal{Z}]$.

LEMMA 2. *$\rho([X^{\mathfrak{T}}]_{\mathfrak{T}})$ is the restriction to $\mathcal{Z}$ of the Jacobian determinant of the polynomial functions $v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$ in the variables $x_1, \dots, x_n$.*

Proof. Let T_X be the tangent bundle to X. Then $\mathcal{W}$ yields a $\mathfrak{T}$-invariant global section of T_X with zero scheme $X^{\mathfrak{T}}$. Thus, we have $c_n^{\mathfrak{T}}(T_X) = [X^{\mathfrak{T}}]_{\mathfrak{T}}$ in $H^*_{\mathfrak{T}}(X)$. On the other hand, T_X carries a $\mathfrak{B}$-linearization and so, by Lemma 1, we have

$$\rho(c_n^{\mathfrak{T}}(T_X))(x, v) = \mathrm{Tr}_{\wedge^n T_x X}(v\mathcal{W}_x - 2\mathcal{V}_x).$$

This is the Jacobian determinant of $v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$. □

We will also need the following easy result of commutative algebra; we will provide a proof, for lack of a reference.

LEMMA 3. *Let $P_1, \dots, P_n$ be polynomial functions in $x_1, \dots, x_n$ that are weighted homogeneous for a positive grading defined by $\deg x_j = a_j$. If the origin is the* *unique common zero to $P_1, \dots, P_n$, then the Jacobian determinant $J(P_1, \dots, P_n)$ is not in the ideal $(P_1, \dots, P_n)$.*

Proof. We begin with the special case where $P_1, \dots, P_n$ are homogeneous. We argue by induction on n, the result being evident for $n = 1$. Let $d_1, \dots, d_n$ be the degrees of $P_1, \dots, P_n$. We have the Euler identities

$$x_1 \frac{\partial P_1}{\partial x_1} + \cdots + x_n \frac{\partial P_1}{\partial x_n} = d_1 P_1,$$
$$\vdots$$
$$x_1 \frac{\partial P_n}{\partial x_1} + \cdots + x_n \frac{\partial P_n}{\partial x_n} = d_n P_n,$$

which we view as a system of linear equalities in $x_1, \dots, x_n$. The determinant of this system is the Jacobian $J(P_1, \dots, P_n) = J$. For $1 \le i, j \le n$, let $J_{i,j}$ be its maximal minor associated with the ith line and the jth column. Then we have

$$x_i J = \sum_{j=1}^{n} (-1)^{j-1} d_j P_j J_{ij}. \tag{22}$$

Assume that $J \in (P_1, \dots, P_n)$ and write

$$J = f_1 P_1 + \cdots + f_n P_n, \quad \text{with } f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n].$$

Using (22) for $i = 1$, it follows that $(x_1 f_1 - d_1 J_{11}) P_1$ is in the ideal $(P_2, \dots, P_n)$. But $P_1, \dots, P_n$ form a regular sequence in $\mathbb{C}[x_1, \dots, x_n]$, since they are homogeneous and the origin is their unique common zero. Therefore, $x_1 f_1 - d_1 J_{11} \in (P_2, \dots, P_n)$. In other words, $J_{11} \in (x_1, P_2, \dots, P_n)$.

After a linear change of coordinates, we can assume that $(x_1, P_2, \dots, P_n)$ is a regular sequence. For $2 \le i \le n$, let

$$Q_i(x_2, \dots, x_n) = P_i(0, x_2, \dots, x_n).$$

Then, in $\mathbb{C}[x_2, \dots, x_n]$ we have

$$J_{11}(0, x_2, \dots, x_n) \in (Q_2, \dots, Q_n).$$

Now $J_{11}(0, x_2, \dots, x_n) = J(Q_2, \dots, Q_n)$, and $Q_2, \dots, Q_n$ are homogeneous polynomial functions of $x_2, \dots, x_n$ having the origin as their unique common zero. But this contradicts the inductive assumption, which completes the proof in the homogeneous case.

Consider now the case where $P_1, \dots, P_n$ are quasi-homogeneous for the weights $a_1, \dots, a_n$. Let $y_1, \dots, y_n$ be indeterminates; then the functions

$$(y_1, \dots, y_n) \mapsto P_i(y_1^{a_1}, \dots, y_n^{a_n})$$

$(1 \le i \le n)$ are homogeneous polynomials with the origin as their unique common zero. Their Jacobian determinant is

$$\left(\prod_{i=1}^{n} a_i y^{a_i - 1}\right) J(P_1, \dots, P_n)(y_1^{a_1}, \dots, y_n^{a_n}).$$

By the first step of the proof, this function of $(y_1, \dots, y_n)$ is not in the ideal generated by the $P_i(y_1^{a_1}, \dots, y_n^{a_n})$. Thus, $J(P_1, \dots, P_n)(x_1, \dots, x_n)$ cannot be in $(P_1, \dots, P_n)$. $\square$

Proof of Theorem 1. The functions $\mathcal{V}(x_1), \dots, \mathcal{V}(x_n)$ satisfy the assumption of Lemma 3 because (a) they are quasi-homogeneous for the grading defined by the action of $\mathfrak{T}$ and (b) o is the unique zero of $\mathcal{V}$. Thus, the Jacobian determinant of these functions is not in the ideal that they generate. By Lemma 2, it follows that $\rho([X^{\mathfrak{T}}]_{\mathfrak{T}})$ is not divisible by v in $\mathbb{C}[\mathcal{Z}]$ for $\mathbb{C}[\mathcal{Z}]/(v) = \mathbb{C}[x_1, \dots, x_n]/(\mathcal{V}(x_1), \dots, \mathcal{V}(x_n))$.

The $\mathbb{C}[z]$-linear map $\rho\colon H^*_{\mathfrak{T}}(X) \to \mathbb{C}[\mathcal{Z}]$ defines a map

$$\bar{\rho}\colon H^*(X) \cong H^*_{\mathfrak{T}}(X)/(z) \to \mathbb{C}[\mathcal{Z}]/(v),$$

a graded ring homomorphism. It suffices to prove that $\bar{\rho}$ is an isomorphism. Note that the spaces $H^*(X)$ and $\mathbb{C}[\mathcal{Z}]/(v)$ have dimension r, so it suffices to check the injectivity of $\bar{\rho}$.

The image in $H^*(X)$ of $[X^{\mathfrak{T}}]_{\mathfrak{T}}$ is $r[\mathrm{pt}]$, where $[\mathrm{pt}]$ denotes the cohomology class of a point. Thus, $\bar{\rho}([\mathrm{pt}]) \neq 0$. Let $(\alpha_1, \dots, \alpha_r)$ be a basis of $H^*(X)$ consisting of homogeneous elements, and let $(\beta_1, \dots, \beta_r)$ be the dual basis for the intersection pairing $(\alpha, \beta) \mapsto \int_X (\alpha \cup \beta) \cap [X]$. Then the homogeneous component of degree $2n$ in each product $\alpha_i \cup \beta_j$ equals $[\mathrm{pt}]$ if $i = j$ and is zero otherwise. Assume that $\bar{\rho}(t_1\alpha_1 + \cdots + t_r\alpha_r) = 0$ for some complex numbers $t_1, \dots, t_r$. Multiplying by $\bar{\rho}(\beta_j)$ and taking the homogeneous component of degree $2n$, we obtain $t_j\bar{\rho}([\mathrm{pt}]) = 0$, whence $t_j = 0$. Thus $\bar{\rho}$ is injective, and the proof is complete. $\square$

10. The Equivariant Push-Forward

Next we describe the equivariant push-forward map

$$\int_X \colon H^*_{\mathfrak{T}}(X) \to H^*_{\mathfrak{T}}(\mathrm{pt}) = \mathbb{C}[z]$$

associated to the map $X \to \mathrm{pt}$. Note that $\int_X$ is $\mathbb{C}[z]$-linear and homogeneous of degree $-2n$. Denote by J the restriction to $\mathcal{Z}$ of the Jacobian determinant of the polynomial functions

$$v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$$

in the variables $x_1, \dots, x_n$. Then J is homogeneous of degree $2n$, as follows from Lemma 2.

THEOREM 4. *For any $f \in \mathbb{C}[\mathcal{Z}]$, the function*

$$v \mapsto \sum_{(x,v)\in\mathcal{Z}} \frac{f(x,v)}{J(x,v)}$$

is polynomial. Furthermore, for any $\alpha \in H^*_{\mathfrak{T}}(X)$,

$$\left(\int_X \alpha\right)(v) = \sum_{(x,v)\in\mathcal{Z}} \frac{\rho(\alpha)(x,v)}{J(x,v)}. \tag{23}$$

Proof. Since $\mathbb{C}[\mathcal{Z}]$ is a graded free $\mathbb{C}[v]$-module of rank r, we may choose a homogeneous basis $f_1, \dots, f_r$ with $f_1 = 1$. As $J \notin (v)$, we may also assume $f_r = J$. If $f \in \mathbb{C}[\mathcal{Z}]$, let $\varphi(f)$ be the rth coordinate of f in this basis. Then the map

$$\mathbb{C}[\mathcal{Z}] \times \mathbb{C}[\mathcal{Z}] \to \mathbb{C}[v], \quad (f, g) \mapsto \varphi(fg)$$

is a nondegenerate bilinear form, since it reduces modulo (v) to the duality pairing on $\mathbb{C}[\mathcal{Z}]/(v) = \mathbb{C}[x_1, \dots, x_n]/(\mathcal{V}(x_1), \dots, \mathcal{V}(x_n))$. If $(g_1, \dots, g_r)$ is the dual basis with respect to $(f_1, \dots, f_r)$ for this bilinear form, then $g_1, \dots, g_r$ are homogeneous and satisfy $\deg(f_i) + \deg(g_i) = 2n$ for all i. As a consequence, the kernel of φ is generated by $f_1, \dots, f_{r-1}$ and also by $g_2, \dots, g_r$ as a $\mathbb{C}[z]$-module. Since $J = f_r$, we have $\varphi(Jg_1) = \cdots = \varphi(Jg_{r-1}) = 0$, so that $\varphi(Jf_2) = \cdots = \varphi(Jf_r) = 0$ whereas $\varphi(Jf_1) = \varphi(J) = 1$.

Let

$$\mathrm{Tr}\colon \mathbb{C}[\mathcal{Z}] \to \mathbb{C}[v]$$

be the trace map for the (finite, flat) morphism $p\colon \mathcal{Z} \to \mathbb{A}^1$. Then

$$\mathrm{Tr}(f)(v) = \sum_{(x,v) \in \mathcal{Z}} f(x, v)$$

for all $v \in \mathbb{A}^1$; in particular, $\mathrm{Tr}(1) = r$. Since Tr is homogeneous of degree 0 and $\mathbb{C}[z]$-linear, its kernel is a graded complement of the $\mathbb{C}[v]$-module $\mathbb{C}[v] = \mathbb{C}[v] f_1$ in $\mathbb{C}[\mathcal{Z}]$. It follows that this kernel is generated by $f_2, \dots, f_r$. Thus, we have

$$\mathrm{Tr}(g) = r\varphi(Jg)$$

for all $g \in \mathbb{C}[\mathcal{Z}]$. This equality holds then for all $g \in \mathbb{C}[\mathcal{Z}][v^{-1}]$ and hence for all rational functions on $\mathcal{Z}$. Note that J restricts to a nonzero function on any component of $\mathcal{Z}$, so $1/J$ is a rational function on $\mathcal{Z}$. We thus have

$$\mathrm{Tr}(f/J) = r\varphi(f)$$

for any $f \in \mathbb{C}[\mathcal{Z}]$; this implies the first assertion. The second assertion follows from the localization theorem in equivariant cohomology. □

11. Some Concluding Remarks

If $\alpha \in H^*_{\mathfrak{T}}(X)$ is a product of equivariant Chern classes, then (23) is the equivariant Bott residue formula for regular $\mathfrak{B}$-varieties (cf. [4]).

Using similar methods, one can also extend (23) to obtain a formula for the equivariant Gysin homomorphism associated to an equivariant morphism of regular smooth projective B-varieties (cf. [1] for the case of flag varieties).

More generally, consider a smooth projective variety X with an action of an arbitrary torus T such that X^T is finite. Then the precise version of the localization theorem given in [10] yields a reduced affine scheme whose coordinate ring is the equivariant cohomology ring of X (see [5] for details and applications). The scheme $\mathcal{Z}$ in this paper gives a more explicit picture, but the requirement of a regular $\mathfrak{B}$-action is harder to satisfy. It would be nice to be able to relax the regularity assumption and so allow $X^{\mathfrak{U}}$ to have positive dimension.

Acknowledgments. The second author thanks Shrawan Kumar for asking the question that led to Theorem 1 and Dale Peterson for describing his result (Theorem 3).

References

[1] E. Akyildiz and J. B. Carrell, *An algebraic formula for the Gysin homomorphism from G/B to G/P*, Illinois J. Math. 31 (1987), 312–320.
[2] ———, *A generalization of the Kostant–Macdonald identity,* Proc. Nat. Acad. Sci. U.S.A. 86 (1989), 3934–3937.
[3] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups,* Ann. of Math. (2) 98 (1973), 480–497.
[4] R. Bott, *Vector fields and characteristic numbers,* Michigan Math. J. 14 (1967), 231–244.
[5] M. Brion, *Poincaré duality and equivariant (co)homology,* Michigan Math. J. 48 (2000), 77–92.
[6] J. B. Carrell, *Bruhat cells in the nilpotent variety and the intersection rings of Schubert varieties,* J. Differential Geom. 37 (1993), 651–668.
[7] ———, *Deformation of the nilpotent zero scheme and the intersection ring of invariant subvarieties,* J. Reine Angew. Math. 460 (1995), 37–54.
[8] F. DeMari, C. Procesi, and M. Shayman, *Hessenberg varieties,* Trans. Amer. Math. Soc. 332 (1992), 529–534.
[9] A. Giventhal and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices,* Comm. Math. Phys. 168 (1995), 609–641.
[10] R. M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem,* Invent. Math. 131 (1998), 25–84.
[11] B. Kostant, *Flag manifold quantum cohomology, the Toda lattice and the representation with highest weight,* New Series Selecta Math. 2 (1996), 43–91.
[12] D. Peterson, *Quantum cohomology of G/P,* unpublished manuscript.
[13] J. Tymoczko, *Decomposing nilpotent Hessenberg varieties over classical groups,* preprint, arXiv:math.AG/0211226.

M. Brion
Université de Grenoble I
Département de Mathématiques
Institut Fourier
UMR 5582 du CNRS
38402 Saint-Martin d'Hères Cedex
France

Michel.Brion@ujf-grenoble.fr

J. B. Carrell
Department of Mathematics
University of British Columbia
Vancouver, BC V6T 1Z2
Canada

carrell@math.ubc.ca