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Plurisubharmonic Lyapunov Functions

Małgorzata Stawiska

Stanisław Łojasiewicz in memoriam

1. Introduction

In the study of dynamics of a continuous map $f: X \mapsto X$ on a compact metric space X, one is often interested in f-invariant sets or measures. When $f: \mathbb{CP}^k \mapsto \mathbb{CP}^k$ is a holomorphic endomorphism of degree $d \geq 2$, such invariant objects can be constructed by means of the function

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log\|F^n(z)\|, \quad z \in \mathbb{C}^{k+1}$$

(cf. [HP; Ue]), where F is a lift of f to $\mathbb{C}^{k+1}$, that is, $\pi \circ F = f \circ \pi$ with π: $\mathbb{C}^{k+1} \setminus \{0\} \mapsto \mathbb{CP}^k$ the standard projection map. Each coordinate of F is a homogeneous polynomial of degree d and $F^{-1}(0) = 0$. It is easy to see that G is a plurisubharmonic (PSH) function on $\mathbb{C}^{k+1}$ that is not identically equal to $-\infty$, is continuous on $\mathbb{C}^{k+1}\setminus\{0\}$, and satisfies $G(F(z)) = d \cdot G(z)$ for $z \in \mathbb{C}^{k+1}$. Using G, one defines a positive closed (1, 1)-current T by $\pi^*T = dd^cG$, and subsequently $T^l = T \wedge \cdots \wedge T$ $(l = 2, \dots, k)$. Note that $\mu = T^k$ is a Borel finite measure on $\mathbb{CP}^k$. These currents and their supports satisfy the invariance conditions $f^*(T^l) = d^l \cdot T^l$ and $f^{-1}(\operatorname{supp} T^l) = \operatorname{supp} T^l = f(\operatorname{supp} T^l)$ for $l = 1, \dots, k$.

The function G has other properties of interest from the dynamical systems point of view. Note that 0 is an attracting fixed point for F. It was proven in [Ue] and [HP] that the basin of attraction $\mathcal{A}$ of 0, defined as $\mathcal{A} = \{z \in \mathbb{C}^{k+1} : F^n(0) \to 0$ as $z \to 0\}$, equals $\{z \in \mathbb{C}^{k+1} : G(z) < 0\}$. Also, $\mathcal{A}$ is a bounded domain. The equation $G \circ F = d \cdot G$ implies that in $\mathcal{A}$, $-G$ increases along the orbits of F (i.e., it is a Lyapunov function for F). Although G is commonly referred to as the "dynamical Green function", it seems that no proof has been given that it is indeed a Green function in any sense used in complex analysis. In fact, G is the pluricomplex Green function of $\mathcal{A}$ with logarithmic pole at the point 0 (see Proposition 3).

If the restriction of the holomorphic map $f: \mathbb{CP}^k \mapsto \mathbb{CP}^k$ to $\mathbb{C}^k \cong [z_1 : z_2 : \cdots : 1]$ is a regular polynomial endomorphism of $\mathbb{C}^k$ (i.e., if $f|_{\mathbb{C}^k} = (f_1, \dots, f_k)\colon \mathbb{C}^k \mapsto \mathbb{C}^k$ is a polynomial map with $\deg f_j = d$, $j = 1, \dots, k$, such that the homogeneous parts of f_j of degree d have a common zero only at the origin), then one obtains a continuous plurisubharmonic function by taking

$$g(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+\|f^n(z)\|, \quad z = (z_1, \dots, z_k) \in \mathbb{C}^k.$$

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The function g measures the rate of escape of a point in $\mathbb{C}^k$ to infinity under the iteration of f. We have $g(z) = G(z, 1)$ and $g(f(z)) = d \cdot g(z)$ for $z \in \mathbb{C}^k$. This implies that g is a Lyapunov function for f. By [K2], g equals the pluricomplex Green function with logarithmic pole at infinity for the compact set $K = \{z \in \mathbb{C}^k : \{f^n(z) : n = 1, 2, \dots\} \text{ is bounded}\}$. Namely, one has

$$g(z) = \sup\{u(z) : u \in \mathrm{PSH}(\mathbb{C}^k),\ u|_K \le 0,\ u(z) - \log\|z\| = O(1) \text{ as } z \to \infty\}.$$

The preceding examples have two features in common. First, the holomorphic map in question has an invariant set (resp., a point or a hyperplane) that is attracting. Second, the Green function with logarithmic pole at this attracting set gives a Lyapunov function for the map. (Lyapunov functions play an important role in dynamics, e.g., in the study of chain recurrent sets and attractor–repeller decomposition of a manifold; for this purpose they were introduced by Conley [Co].) The question then arises: Are there other examples like those just discussed? More specifically, suppose a holomorphic endomorphism f of $\mathbb{CP}^k$ has an invariant attracting hypersurface A. One can define the pluricomplex Green function with logarithmic pole along A for the dual repeller K of A (for details, see Sections 2 and 3 and the references). Can one obtain a Lyapunov function for f out of this Green function?

In this paper we give an answer to this question when $k = 2$. We assume that a holomorphic map $f : \mathbb{CP}^2 \mapsto \mathbb{CP}^2$ has an invariant nonsingular quadratic curve A contained in the critical set of f (A must then be attracting). Then we proceed as follows: in Section 2 we collect some known facts about attracting sets, in particular those for holomorphic endomorphisms of $\mathbb{CP}^2$. In Section 3 we review the theory of pluricomplex Green functions with logarithmic poles in a Stein manifold according to Zeriahi [Ze] and also introduce a parabolic potential on $\mathbb{CP}^2 \setminus A$ and examine how it behaves on $f(\mathbb{CP}^2 \setminus A) \setminus A$. In Section 4 we first prove an estimate for $\mathrm{dist}(f(x), A)$ and then use this estimate to prove that the pluricomplex Green function G_K for the repeller K dual to A, with logarithmic pole along A, is a PSH Lyapunov function for f in $\mathbb{CP}^2 \setminus (A \cup f^{-1}(A))$. Our main result is the following (cf. Theorem 5).

MAIN THEOREM. *G_K is a continuous plurisubharmonic function satisfying $G_K \le G_K \circ f$ in $\mathbb{CP}^2 \setminus (A \cup f^{-1}(A))$.*

Finally, we show that construction of a Lyapunov function for f by applying to G_K a standard procedure (due to Conley and Franks; see [FM]) also yields G_K.

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2. Attracting and Repelling Sets in Holomorphic Dynamics

First let us recall some general background in topological dynamics. Let (X, dist) be a metric space and let f be a closed relation on X (i.e., a closed subset of $X \times X$).

Definition 1 [Ak, discussion before Prop. 2.9]. A *Lyapunov function for* f is a continuous function $L: X \mapsto \mathbb{R}$ such that $f \subset \{(x, y) : L(x) \leq L(y)\}$. In particular, if $f: X \mapsto X$ is a continuous map then a Lyapunov function for f is a continuous real-valued function on X that is nondecreasing along the orbits of f.

Definition 2 [Ak, discussion before Prop. 1.8]. Let $\varepsilon \geq 0$ and let $x, y \in X$. An ε*-chain* for f from x to y is a sequence $\{x_1, \dots, x_N\}$ such that $\text{dist}(x_{n+1}, f(x_n)) < \varepsilon$ for $n = 1, \dots, N-1$ (assuming $f(x_n) \neq \emptyset$).

We can associate the following definition with a closed relation f on X.

Definition 3 [Ak, formula (1.11)]. A pair (x, y) is in $\mathcal{C}_f \subset X \times X$ if and only if, for every $\varepsilon > 0$, there is an ε-chain for f from x to y.

Definition 4 [Ak, formula (1.15)]. A point $x \in X$ is *chain recurrent* if $(x, x) \in \mathcal{C}_f$. The set of all such points will be denoted by C.

By [Ak, Prop. 1.8], $\mathcal{C}_f$ is a closed relation whenever f is. This implies that C is a closed subset of X. Note also that $\mathcal{C}_f$ is transitive.

Let us now recall the notions of invariance.

Definition 5. Let X and f be as before. A subset $B \subset X$ is called f*-invariant* if $f(B) = B$, and B is *totally* f*-invariant* if it is both f-invariant and f^{-1}-invariant.

Let now A be a closed subset of X.

Definition 6 [FM, remarks after Def. 2.4]. A is an *attracting set for* f if there is a closed neighborhood W of A such that $f(W) \subset \text{int}\, W$ and $\bigcap_{n \geq 0} f^n(W) = A$.

Note that an attracting set for f is f-invariant.

Definition 7. A closed set K is a *repeller* for f if it is an attracting set for f^{-1}.

The following proposition shows how one can associate attracting sets with repellers.

Proposition 1 ([Ak, Prop. 3.9]; cf. [FM, Lemma 2.8]). *For each attracting set* A_+ *for a closed relation* f*, there is a unique repeller* A_-*, called the dual repeller, such that* $A_+ \cap A_- = \emptyset$ *and the chain recurrent set* $C \subset A_+ \cup A_-$*. The repeller* A_- *is given by* $A_- = (\mathcal{C}_f)^{-1}(C \setminus A_+)$.

Now we will give some examples of attracting sets for holomorphic maps $f\colon \mathbb{CP}^2 \mapsto \mathbb{CP}^2$ of degree $d \geq 2$. An attracting periodic orbit is an attracting set for f. If f restricts to a regular polynomial endomorphism on $\mathbb{C}^2$, then the hyperplane at infinity is an attracting set. More generally, the following theorem holds.

THEOREM 1 [FS1, Lemma 7.9]. *Suppose that a holomorphic map $f\colon \mathbb{CP}^2 \mapsto \mathbb{CP}^2$ of degree d maps a compact complex hypersurface A to itself and that A is contained in the critical set of f. Then* $\operatorname{dist}(f(x), A) = o(\operatorname{dist}(x, A))$.

In particular, the quadratic curve $A = \{[z : w : t] \in \mathbb{CP}^2 : zw - t^2 = 0\}$ is an attracting set for a family of maps $f\colon \mathbb{CP}^2 \mapsto \mathbb{CP}^2$, $f([z : w : t]) = [\lambda(z + 4w - 4t)^3 : (1/\lambda)z^3 : (z - 2t)^3 + 6(z - 2t)(zw - t^2)]$, where $\lambda \in \mathbb{C}$ is such that the map $z \mapsto \lambda(1 - 2/z)^3$ is critically finite on $\mathbb{CP}^1$. The curve A is also an attracting set for a family $h_\delta([z : w : t]) = [(z + 4w - 4t)^2 : z^2 : z(z + 4w - 4t) + \delta(t^2 - zw)]$ with small $\delta \neq 0$. (These examples come from [FW, Sec. 5].) For other examples of holomorphic maps with invariant algebraic varieties (not necessarily attracting ones), see [BD].

Note that, for a regular polynomial map, the hyperplane at infinity is a totally invariant set. (It is an attracting set whose dual repeller is the filled-in Julia set.) Note also that a holomorphic endomorphism on $\mathbb{CP}^k$ ($k \geq 2$) of degree $d \geq 2$ cannot have a totally invariant nonsingular hypersurface of degree ≥ 2 (by Théorème 1 and 2 in [CL]).

In Section 3 we will give an estimate for the growth of f near its attracting curve A that is sharper than Theorem 1.

The following proposition concerns the f-invariant sets mentioned at the beginning of Section 1.

PROPOSITION 2 [FS2, Prop. 2.16]. *Let $f\colon \mathbb{CP}^k \mapsto \mathbb{CP}^k$ be a holomorphic map of degree $d \geq 2$ and let μ be the measure defined in Section 1. If $A \neq \mathbb{CP}^k$ is an attracting set for f, then $A \cap \operatorname{supp}\mu = \emptyset$.*

COROLLARY 1. *Let f and A be as in Proposition 2. Then* $\operatorname{supp}\mu \subset K$, *where K is the repeller dual to A.*

Proof. The Borel measure μ is f-invariant, so $\operatorname{supp}\mu \subset C$ (see e.g. [Ak, remark preceding Prop. 8.8]). By Proposition 1, $C \subset A \cup K$. By Proposition 2, $\operatorname{supp}\mu \cap A = \emptyset$ and hence $\operatorname{supp}\mu \subset K$. □

3. Pluricomplex Green Function with Logarithmic Singularity

When introducing Lyapunov functions occuring in holomorphic dynamics, we pointed out their relation with Green functions. For example, for a homogeneous polynomial map $F\colon \mathbb{C}^{k+1} \mapsto \mathbb{C}^{k+1}$ with $F^{-1}(0) = 0$, the function $-G$ (see Section 1) is a plurisubharmonic Lyapunov function on the basin of attraction $\mathcal{A}$ for 0. Thus we have our next proposition.

PROPOSITION 3. *G is the pluricomplex Green function of $\mathcal{A}$ with* (*logarithmic*) *pole at* 0.

Proof. The statement means that

$$G(z) = \sup\{u(z) : u \in \mathrm{PSH}(\mathcal{A}),\ u \le 0,\ u(z) - \log\|z\| \le \mathcal{O}(1) \text{ as } z \to 0\}$$

(see [K1, remarks before Prop. 6.1.1]). Note that $\mathcal{A}$ is hyperconvex. Indeed, G is a negative PSH exhaustion function on $\mathcal{A}$, that is, $\{z \in \mathcal{A} : G(z) < c\} \subset\subset \mathcal{A}$ for all $c < 0$ (this follows easily from continuity of G and the characterization of $\mathcal{A}$). Théorème 4.3 in [De] (see also [K1, Thm. 6.3.6]) states that the pluricomplex Green function of a hyperconvex bounded domain $\mathcal{A} \subset \mathbb{C}^{k+1}$ with pole at a point $a \in \mathcal{A}$ (we take $a = 0$ here) is the unique solution of the following problem:

$$h \in \mathcal{C}(\mathcal{A} \setminus \{a\}) \cap \mathrm{PSH}(\mathcal{A}), \tag{1}$$

$$h(z) \to 0 \text{ as } z \to \partial\mathcal{A}, \tag{2}$$

$$h(z) - \log\|z - a\| = \mathcal{O}(1) \text{ as } z \to a, \tag{3}$$

$$(dd^c h)^{k+1} = (2\pi)^{k+1}\delta_a. \tag{4}$$

We already mentioned that G satisfies (1). Continuity of G and $\mathcal{A} = \{G < 0\}$ imply (2). Part (3) was proven as Theorem 2.1(c) in [HP]. To prove (4), note that each iterate F^j has its only (isolated) zero at 0, and $\deg_0 F^j = (d^j)^{k+1}$, so $(dd^c \log\|F^j\|)^{k+1} = (2\pi d^j)^{k+1}\delta_0$ (cf. [BT1, beginning of Sec. 4). The convergence $(1/d^j)\log\|F^j\| \to G$ as $j \to \infty$ is uniform on $\mathbb{C}^{k+1} \setminus \{0\}$, so by [BT1, Prop. 2.3] we have $((1/d^j)dd^c \log\|F^j\|)^{k+1} \to (dd^c G)^{k+1}$ as $j \to \infty$ on $\mathbb{C}^{k+1} \setminus \{0\}$. Finally, there exists an $r > 0$ such that $\|F(z)\| < (1/2)\|z\|$ for $\|z\| < r$. Hence also $(1/d^{j+1})\log\|F^{j+1}(z)\| < (1/d^j)\log\|F^j(z)\|$ for $\|z\| < r$ and $j = 0, 1, 2, \dots$. By [BT2, Thm. 2.1] we have $((1/d^j)dd^c \log\|F^j\|)^{k+1} \to (dd^c G)^{k+1}$ in the ball $\{\|z\| < r\}$ as $j \to \infty$. (See [K1] for a good overview of convergence theorems for the Monge–Ampère operator.) Hence $(dd^c G)^{k+1} = (2\pi)^{k+1}\delta_0$. □

Looking for new examples of plurisubharmonic Lyapunov functions, we will consider Green functions with logarithmic pole along a hypersurface rather than at an isolated point. Our arguments will resemble (and, in fact, will generalize) the analysis in [K2] of the function g associated with a regular polynomial endomorphism of $\mathbb{C}^k$. We will study holomorphic endomorphisms of $\mathbb{CP}^2$ with an invariant nonsingular quadratic curve A, so we let $A = \{[z : w : t] \in \mathbb{CP}^2 : zw - t^2 = 0\}$. In order to define the pluricomplex Green function with logarithmic pole along A for a relatively compact set $E \subset \mathbb{CP}^2 \setminus A$, we will introduce a parabolic potential on $\mathbb{CP}^2 \setminus A$—that is, a continuous plurisubharmonic exhaustion function satisfying the homogeneous Monge–Ampère equation outside the set where it equals $-\infty$. We begin by stating the following proposition.

PROPOSITION 4. *$\mathbb{CP}^2 \setminus A$ is an affine algebraic variety.*

Proof. $\mathbb{CP}^2 \setminus A$ is a Zariski open subset of $\mathbb{CP}^2$, so we need to show it is isomorphic to an algebraic subset of some $\mathbb{C}^N$. Consider the mapping

$$\Phi\colon \mathbb{CP}^2 \ni [z:w:t] \mapsto [\phi_1 : \cdots : \phi_6] = [z^2 : w^2 : zw - t^2 : t^2 : zt : wt] \in \mathbb{CP}^5$$

(this is the standard Veronese map, after a linear change of coordinates). By [Sh, Ex. 4.4.2], $\Phi(\mathbb{CP}^2)$ is an algebraic set in $\mathbb{CP}^5$. It is straightforward to check that $\Phi|_{(\mathbb{CP}^2\setminus A)}$ is 1-to-1. The quadric A is mapped onto the intersection of $\Phi(\mathbb{CP}^2)$ with the hyperplane $\phi_3 = 0$, so $\Phi(\mathbb{CP}^2 \setminus A)$ can be regarded as a subset of $\mathbb{C}^5$. □

Proposition 4 can be obtained as a special case of Proposition 6.3.5 in [Fu]. Instead, we have given a proof that does not make extensive use of algebra and also introduces a map that will be important throughout the remaining part of the paper. Specifically, let Φ be as in Proposition 4 and let $\phi = [\phi_1 : \phi_2 : \phi_3] : \mathbb{CP}^2 \mapsto \mathbb{CP}^2$. Observe that $\phi|_{(\mathbb{CP}^2\setminus A)}$ is a proper holomorphic map onto $\mathbb{C}^2$. Therefore, $g = \log\|\phi\|$ can be taken to be a parabolic potential in $\mathbb{CP}^2 \setminus A$ (see [Ze, opening discussion in Sec. 5]).

In a Stein manifold X endowed with a parabolic potential g, we define the class $\mathcal{L}$ of plurisubharmonic functions with minimal growth with respect to g as

$$\mathcal{L} = \{v \in \mathrm{PSH}(X) : v \le c_v + g^+\},$$

where c_v is a constant dependent only on v and $g^+ = \max\{g, 0\}$.

We will say that a set $E \subset X$ is *$\mathcal{L}$-polar* if there is a function $u \in \mathcal{L}$ such that u is identically $-\infty$ on E.

For a relatively compact set $E \subset X$, define the pluricomplex Green function with logarithmic singularity as

$$G_E(x) = \sup\{v(x) : v \in \mathcal{L},\ v|_E \le 0\}, \quad x \in X$$

(cf. [Ze, (3.1) and (3.2)]). (When we take $X = \mathbb{CP}^2 \setminus A$ with g as before, G_E will be referred to as the pluricomplex Green function for E with logarithmic pole along A.)

For a locally bounded function u with values in $[-\infty, \infty)$, we define its upper semicontinuous regularization as

$$u^*(x) = \overline{\lim_{y\to x}}\, u(y)$$

(cf. [K1]).

We will use the following facts.

THEOREM 2 [Ze, Lemma 3.10; K1, Prop. 5.2.1]. *Let $\mathcal{U} \subset \mathcal{L}$ be a nonempty family and let $v = \sup\{u : u \in \mathcal{U}\}$. If the set $\{x : v(x) < +\infty\}$ is not $\mathcal{L}$-polar, then the family $\mathcal{U}$ is locally uniformly bounded above and $v \in \mathcal{L}$.*

COROLLARY 2. *If E is not $\mathcal{L}$-polar, then $G_E^* \in \mathcal{L}$.*

We now consider the following special class of L-regular subsets of X.

DEFINITION 8 [Ze, Def. 3.13]. Let E be a compact subset of a Stein manifold X with a parabolic potential g, and let $x_0 \in E$. We say that E is *L-regular* at x_0 if $G_E^*(x_0) = 0$. We call E L-regular if it is L-regular at every point $x_0 \in E$.

An important example of an L-regular set is given by the following theorem.

THEOREM 3 [Ze, Thm. 3.6]. *Consider the set* $B_R = \{x \in X : g(x) \leq R\}$. *Then*

$$G_{B_R}(x) = (g - R)^+(x), \quad x \in X.$$

We also have the following.

PROPOSITION 5. *Let* $f : \mathbb{CP}^2 \mapsto \mathbb{CP}^2$ *be a holomorphic map with* $f(A) = A$. *Then* $f^{-1}(B_R)$ *is* L*-regular.*

Proof. By [Ze, Prop. 3.14], L-regularity of a compact set E at $x \in E$ is equivalent to $h^*_{E,\Omega}(x) = 0$, where $\Omega \supset E$ is an open set in $\mathbb{CP}^2 \setminus A$ and $h_{E,\Omega} = \sup\{u \in \mathrm{PSH}(\Omega) : u \leq 1, u|_E \leq 0\}$. Let $R' > R$ and $\Omega = \{g < R'\}$. By the second part of Theorem 3.6 in [Ze], $h_{B_R,\Omega} = (g - R)^+/(R' - R)$ in Ω. In analogy to [K1, Prop. 4.5.14] it can be proven that $h_{B_R,\Omega} \circ f = h_{f^{-1}(B_R), f^{-1}(\Omega)}$. For $x \in f^{-1}(B_R)$ this gives $0 = h_{f^{-1}(B_R), f^{-1}(\Omega)}(x) = G^*_{f^{-1}(B_R)}$. □

4. A Plurisubharmonic Lyapunov Function

From now on we assume that the variety $A = \{zw - t^2 = 0\}$ is invariant under a holomorphic map $f : \mathbb{CP}^2 \mapsto \mathbb{CP}^2$ and that A is contained in the critical set for f. (It would be enough to assume that A is a nonsingular hypersurface in $\mathbb{CP}^2$, invariant under a holomorphic endomorphism f and contained in the critical set of f, but no examples are available with A of degree at least 3.) We will need the following estimate for $\mathrm{dist}(f(x), A)$, which is sharper than that provided by Theorem 1.

THEOREM 4. *If* f *and* A *satisfy the previous assumptions, then for some constant* $M > 0$ *it follows that* $\mathrm{dist}(f(x), A) \leq M(\mathrm{dist}(x, A))^2$.

Proof. We will use the Fermi coordinates (x_1, x_2) around $q \in A$ defined in a neighborhood $V \subset A$ of q, relative to a local coordinate Y in V and a section $\mathbf{u}$ of the restriction of the normal bundle N of A to V. These are defined as follows (cf. [Gr, (2.2) and (2.3)]):

$$x_1(\exp_N(s\mathbf{u}(q'))) = Y(q') \quad \text{and} \quad x_2(\exp_N(s\mathbf{u}(q'))) = s$$

for $q' \in V$, where $\exp_N : N \mapsto \mathbb{CP}^2$ maps a neighborhood of the zero section of N diffeomorphically onto a (tubular) neighborhood $\mathcal{U}$ of $A \subset \mathbb{CP}^2$ and where the complex number s is small enough so that $s\mathbf{u}(q') \in \exp_N^{-1}(\mathcal{U})$.

Now take $p' \in \mathcal{U}$. Then $p' = \exp_N^{-1}(p, t\mathbf{v})$ for some $p \in A$, $t \in \mathbb{C}$, and $\mathbf{v} \perp A$ at p. The Taylor formula for f (in the normal coordinate t around p and the Fermi coordinates around $q = f(p) \in A$) yields $f(t) = f(0) + Df(0) \cdot t + \mathcal{O}(|t|^2)$. Note that $Df(0)$ has rank 1, so the gradient of $P(X, Y)$ is orthogonal to $tDf(0)$ at q (where $P(X, Y) = 0$ defines A near q). Using the Taylor formula for the local coordinate Y centered at q, we can replace $f(0) + Df(0) \cdot t$ by $q' - \mathcal{O}(|t|^2)$ with $q' = Y(t) = x_1(f(p))$, from which the estimate follows. □

By Theorem 1 (or Theorem 4), there is a neighborhood W of A such that $f(W) \subset W$. In fact, we can take W equal to the complement of some B_R, as follows.

PROPOSITION 6. *There is an* $R > 0$ *such that* $W = \{x : g(x) \geq R\}$ *satisfies* $f(W) \subset \operatorname{int} W$.

Proof. Instead of the standard Fubini–Study distance in $\mathbb{CP}^2$, we can work with the pullback to $\mathbb{CP}^2$ of the Fubini–Study distance in $\mathbb{CP}^5$ by the Veronese embedding Φ. In the chart $\phi_6 = 1$ we have $\operatorname{dist}(p, L_\infty) = \mathcal{O}(|\phi_3|)$ for points $p = (\phi_1, \dots, \phi_5)$ near the hyperplane $L_\infty = \{\phi_3 = 0\}$ ([KoM, Thm. 3.10.2]; the argument for other coordinate charts is the same). Consider an $R > 0$ such that the level set $\{g = R\} \subset \mathbb{CP}^2$ is contained in some open neighborhood W' of A with $f(W') \subset W'$ (this is possible since g is a plurisubharmonic exhaustion). For large values of R we have $|\phi_3| = \mathcal{O}(e^{-R})$ for points $x \in \{g = R\}$. Hence for such x, $\operatorname{dist}(x, A) \leq Me^{-R}$ and $\operatorname{dist}(f(x), A) \leq M'e^{-2R}$ (cf. Theorem 4). Also, $|\phi_3(f(x))| \leq M''e^{-2R}$, which gives $g(f(x)) > R$. □

Recall that, by [Ta], $-\log \operatorname{dist}(\cdot, A)$ is a plurisubharmonic exhaustion on $\mathbb{CP}^2 \setminus A$. Hence the class $\mathcal{L}$ defined by means of the parabolic potential g can be also characterized as $\{u \in \operatorname{PSH}(\mathbb{CP}^2 \setminus A) : u(x) \leq c - \log \operatorname{dist}(x, A)\}$ (this is how $\mathcal{L}$ is defined in [BT3]). This characterization allows us to prove the following.

PROPOSITION 7. *If* $u \in \operatorname{PSH}(\mathbb{CP}^2 \setminus A)$, *then the formula*

$$\tilde{u}(x) = 2 \max\{u(y) : y \in f^{-1}(x)\}$$

defines a plurisubharmonic function in $\mathbb{CP}^2 \setminus A$. *Moreover, if* u *is in* $\mathcal{L}$, *so is* $\tilde{u}$.

Proof. The first part is essentially the same as [K1, Prop. 2.9.29]. The invariance $f(A) = A$ ensures that the domain of $\tilde{u}$ is indeed $\mathbb{CP}^2 \setminus A$. For the second part, recall that $\operatorname{dist}(f(x), A) \leq M \cdot \operatorname{dist}(x, A)^2$ by Theorem 4 and so the growth condition $u(y) \leq c - \log \operatorname{dist}(y, A)$ gives $\tilde{u} \in \mathcal{L}$, as in Theorem 5.3.1 of [K1]. □

PROPOSITION 8. *For* $E \subset\subset \mathbb{CP}^2 \setminus A$ *we have*

$$2G_{f^{-1}(E)} \leq G_E \circ f \ \text{ in } \mathbb{CP}^2 \setminus f^{-1}(A).$$

Proof. Take a $u \in \mathcal{L}$ such that $u \leq 0$ on $f^{-1}(E)$. Then $\tilde{u} \in \mathcal{L}$ satisfies $\tilde{u} \leq 0$ on E. Hence for any $x \notin f^{-1}(A)$ we have $2u(x) \leq \tilde{u}(f(x)) \leq G_E(f(x))$, which proves the proposition. □

Let K be the repeller dual to the attracting curve A. By Corollary 1, K is not pluripolar, since μ does not charge pluripolar sets.

PROPOSITION 9. G_K *is continuous.*

Proof. Since K is not pluripolar, by Theorem 2 we have $G_K^* \leq c + \log^+|\phi|$. Let $\varepsilon > 0$ and $F_\varepsilon = \{x : G_K(x) \leq \varepsilon\}$. Note that $G_K - \varepsilon \leq G_{F_\varepsilon}$. Let $R > 1$ and $k_0 \in \mathbb{N}$ be such that $f^{-1}(B_R) \subset B_R = \{\log|\phi| \leq R\}$ and $2^{-k_0}(c + \log R) \leq \varepsilon$. Then for $k \geq k_0$ and $x \in f^{-1}(B_R)$ we have

$$G_K(x) \le 2^{-k}(G_K^*(f^k(x)) \le 2^{-k}(c + \log^+|\phi(f^k(x))|) \le 2^{-k_0}(c + \log R) \le \varepsilon.$$

Hence $f^{-k}(B_R) \subset F_\varepsilon$ for $k \ge k_0$. Take now a sequence $\varepsilon_j \searrow 0$ $(j = 1, 2, \dots)$. Then for every j there is a k_j such that $f^k(B_R) \subset F_{\varepsilon_j}$ $(k \ge k_j)$. Moreover, since $K = \bigcap_{k\ge 0} f^{-k}(B_R)$, its Green function G_K satisfies $G_K - \varepsilon_j \le G_{F_{\varepsilon_j}} \le G_{f^{-k}(B_R)} \le G_K$; that is, the functions $G_{f^{-k}(B_R)}$ tend uniformly to G_K in $\mathbb{CP}^2 \setminus A$. Because the sets $f^{-k}(B_R)$ are L-regular, their pluricomplex Green functions are continuous [Ze, Thm. 4.2.3] and so is G_K. □

Combining Propositions 8 and 9 yields our main result, as follows.

THEOREM 5. *G_K is a Lyapunov function for f in $\mathbb{CP}^2 \setminus (A \cup f^{-1}(A))$.*

In topological dynamics, a standard procedure is used to construct a Lyapunov function for a continuous map on a compact metric space (it is a crucial step in the proof of the so-called fundamental theorem of dynamical systems; see [FM]). We will now show that G_K is obtained as a result of a similar procedure in $\mathbb{CP}^2 \setminus A$.

PROPOSITION 10. *Let $v_0 = G_K$, let $v_n(x) = \max_{y\in f^{-n}(x)} v_0(x)$ for $x \in \mathbb{CP}^2 \setminus A$ and $n \ge 1$, and let $v = \sup_{n\ge 0} v_n$. Then $v = G_K$.*

Proof. We only need to show that $v \le G_K$. By Theorem 2, $G_K \in \mathcal{L}$ and so, by the same argument as in Proposition 7 (with Theorem 1 instead of Theorem 4 used to prove the distance estimates), all functions v_n are in the class $\mathcal{L}$. Since K is f^{-1}-invariant and $G_K = 0$ on K, we have $v_n = 0$ on K for every $n \ge 0$. This gives $v \le G_K$. □

REMARK. It is unknown at the moment whether G_K is also a Lyapunov function for the relation C_f or whether it is a maximal function among plurisubharmonic Lyapunov functions for f.

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Department of Mathematics
Purdue University
West Lafayette, IN 47907-2068

mstawisk@math.purdue.edu