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Geodesics on Quotient Manifolds and Their Corresponding Limit Points

Torbjörn Lundh

1. Introduction

Consider a discrete group and its quotient manifold. We can choose to view a point ξ on the unit circle or its corresponding geodesics on the manifold. It is well known that if ξ is in the conical limit set then the geodesic will return infinitely often. What is the general relation of the tangency level of the accumulating orbit points at ξ and the "escape rate" of the geodesic?

What is the relation between the geodesic escape rate and the *thinness* of the *archipelago* of the group seen from the limit point ξ?

Let us now give some more background and details on these two main questions.

1.1. Discrete Groups and Geodesics

Let Γ be a discrete group of Möbius mappings preserving the unit ball (or the unit disc when $n = 2$). Since Γ is discrete, we know that all the cluster points of the orbit $\bigcup_{\gamma\in\Gamma}\gamma(0)$ are situated on the unit sphere. This set of accumulation points is called the limit set, Λ. An important subset of Λ is the conical limit set, Λ_c, which consists of those limit points that have orbit points accumulating in a non-tangential way.

A way to study how the orbit points accumulate toward the boundary is to see at what "rate" they approach the limit set by using the *Poincaré series, $h(s)$*, of the group Γ; this series can be defined as

$$h(s) = \sum_{\gamma\in\Gamma}(1 - |\gamma(0)|)^s.$$

By Sullivan's dichotomy, the conical limit set has either full or empty Lebesgue measure, depending on whether the series $h(n-1)$ diverges or converges; see [17].

Nicholls [14] studied a family $\mathcal{L}(\alpha)$ of generalized limit sets (see Definition 2.3 in the next section), where for example $\alpha = 1$ gives the usual conical limit set and $\alpha = \frac{1}{2}$ gives a limit set for which parabolic tangency of the clustering orbit points is allowed. From [14, Thm. 2.1.1] we see that if $h(\alpha(n-1))$ converges then $|\mathcal{L}(\alpha)| = 0$.

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The convergence of the Poincaré series is quantified by the *critical exponent* of Γ, defined as $\delta(\Gamma) = \inf(s : h(s) < \infty)$. Bishop and Jones showed in [6, Thm. 1.1] that the Hausdorff dimension of the conical limit set of a nonelementary Kleinian group ($n = 3$) equals the critical exponent.

A point in the conical limit set can be represented by a geodesic on the quotient manifold (see Section 2) that returns infinitely often to a compact neighborhood of the starting point, so the main result of Bishop and Jones was also formulated in terms of studying the set of initial directions of geodesics that remain bounded. There is quite a history to this problem formulation; see for example [7; 11; 15; 16]. More recently, Fernández and Melián [9] have shown a three-way classification for the sets of escaping geodesics on Riemann manifolds—that is, $\lim_{t\to\infty} \varphi(t) = \infty$, depending on the cases if the manifold has finite area or if a Brownian motion on it is recurrent or transient. In [5], Bishop presents a result on the Hausdorff dimension on the limit set using the sets of geodesics that remain bounded as well as the "linear escape limit set". See Remark 4.7 for the connection with the limit sets $\mathcal{L}(\alpha)$.

Bishop suggested the following problem to the author: Let a point particle trace with unit speed along a given geodesic on a quotient manifold and let $\varphi(t)$ be the distance to the starting point from the particle at time t. We know that if $\liminf_{t\to\infty} \varphi(t) < \infty$ then the geodesic corresponds to a conical limit point. *What can be said in general about the limit point ξ if the function $\varphi(t)$ is known?*

In Theorem 4.1 we have the following result connecting φ with the limit sets $\mathcal{L}(\alpha)$. Suppose that ξ is on the unit sphere. Then $\xi \in \mathcal{L}(\alpha)$, for $\frac{1}{2} < \alpha \leq 1$, if and only if

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) < \infty.$$

1.2. *A Potential Theoretic Connection*

By studying different limit sets $\mathcal{L}(\alpha)$, we take into consideration the kind of approach with respect to the rate of tangency of orbit points clustering toward a limit point on the boundary.

In [13] the author studied an alternative measure of the orbit that was linked to potential theory. This was done by asking if the "fattened" orbit (i.e., the archipelago; see Section 8) is "thin", in various meanings, when viewed from the boundary point in question.

In Proposition 9.2 we give a result connecting the function $\varphi(t)$ and its related point ξ on the unit sphere with the thinness of the archipelago of Γ in the following sense. The archipelago is not β-thin at ξ (see Definition 9.1) if

$$\sum_i e^{-(n-\beta)\varphi_i} = \infty,$$

where φ_i is the ith (generalized) local minimum of $\varphi(t)$.

From various results in [13, p. 310], one sees that the conical limit set and the set on the boundary where the archipelago is not minimally thin (see Definition 8.1) are quite close. The question has been raised of whether they do in fact coincide there; that question is now settled by an example (see Section 10) of a Fuchsian group with a limit point outside the conical limit set but where the archipelago is not minimally thin.

In Remarks 8.2, 8.4, and 10.1 we give an intuitive probabilistic motivation for why the counterexample works by implicating recurrence of a special stochastic process on the Riemann surface depicted in Figure 6.

2. The Setup

Let B be the unit ball in $\mathbb{R}^n$ (or the unit disk if $n = 2$) and let Γ be a discrete group of Möbius transformations that preserves B. We will denote the elements in Γ by γ_i. Let $S = B/\Gamma$ be the (Riemann) quotient manifold obtained from B by identification of Γ-equivalent points. (If $n \leq 3$ then S is a manifold. For higher dimensions it may be an orbifold rather than a manifold, but we will adopt the notion from [1, p. 79] and call S a quotient manifold nevertheless.)

Furthermore, let x_0 be the base point on S corresponding to the origin in B, and let $g(t)$ be a parameterized geodesic on S such that $g(0) = x_0$ and such that the arc length of $g(t)$ for t from 0 to τ is τ. Let $\varphi(t)$ be the distance $d(g(t), x_0)$ on the manifold. Thus we have that $\varphi(t) \leq t$. The geodesic from x_0 is viewed in B as a straight line from the origin to a boundary point ξ and thus corresponds to a geodesic, $g(t)$, on S.

We can think of S as the result of taking the Dirichlet domain in B around 0 and gluing together corresponding sides according to the generators of Γ. The "seams" on S will then correspond to the set on S where the graph of φ "has a corner"—that is, there are at least two different geodesics from x_0 to a seam point; see Figure 1.

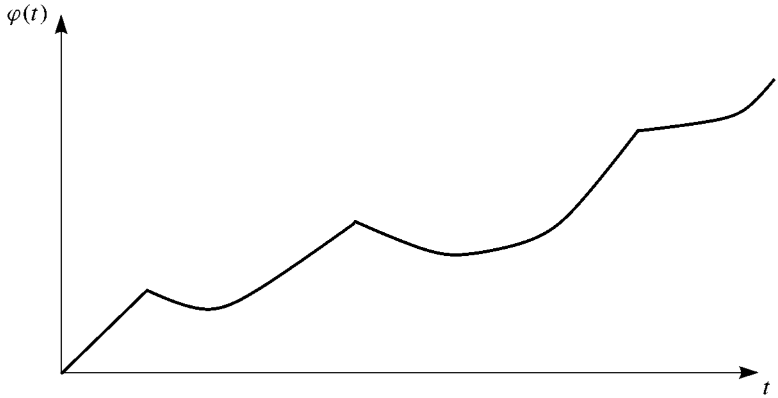

Figure 1 Example of a graph of φ

We will now give some definitions taken from [13] and [14].

DEFINITION 2.1 [14, p. 5]. Let $a \in B$ and $k, \alpha > 0$. We define

$$I(a : k, \alpha) = \left\{ x \in \partial B : \left| x - \frac{a}{|a|} \right| < k(1 - |a|)^{\alpha} \right\}.$$

DEFINITION 2.2 [14, p. 23]. Let γ_i be the elements of the discrete group Γ and let z be the base point of the orbit. Then

$$L(z : k, \alpha) = \bigcap_{m=1}^{\infty} \bigcup_{i>m}^{\infty} I(\gamma_i(z) : k, \alpha).$$

In the next definition we take the base point to be the origin.

DEFINITION 2.3 [13, Def. 3.14]. Denote the α-limit set by

$$\mathcal{L}(\alpha) = \bigcup_{k>0} L(0 : k, \alpha).$$

REMARK 2.4. The special case when $\alpha = 1$ gives us the conical limit set (also called the nontangential limit set); that is, $\mathcal{L}(1) = \Lambda_c$. See for example [13, Lemma 3.13] for a more detailed comparison.

In [13, Def. 5.2], a subset of the limit set $\mathcal{L}(\alpha)$ was introduced by taking the intersection instead of the union in the following manner.

DEFINITION 2.5. We define the *strong α-limit set* to be

$$\mathcal{L}_s(\alpha) = \bigcap_{k>0} L(0 : k, \alpha).$$

For any strictly positive α, we have

$$\partial B \supset \mathcal{L}(\alpha) \supset \mathcal{L}_s(\alpha) \supset \mathcal{L}(\alpha + \varepsilon) \quad \text{for all } \varepsilon > 0.$$

It is well known that the conical limit set $\mathcal{L}(1)$ is independent of the choice of base point (see e.g. [10, p. 29]). We will show that the same holds when $\alpha \in (0, 1)$, telling us that our restriction in Definitions 2.3 and 2.5 to fix the base point to the origin is not that essential.

LEMMA 2.6. *For any point z in B,*

$$\bigcup_{k>0} L(z : k, \alpha) = \bigcup_{k>0} L(0 : k, \alpha)$$

if $\alpha \leq 1$. That is, $\mathcal{L}(\alpha)$ is independent of the base point of the orbit if $\alpha \leq 1$. Similarly, $\mathcal{L}_s(\alpha)$ is independent of the base point of the orbit if $\alpha < 1$.

Proof. Given an $\alpha \leq 1$ and a $z \in B$, suppose $x \in \bigcup_{k>0} L(z : k, \alpha)$; then $x \in L(z : k, \alpha)$ for some $k > 0$. We want to show that there is a K such that $x \in L(0 : K, \alpha)$, where K is dependent on α, z, and k.

Denote by δ the hyperbolic distance from 0 to z, $\delta = d(0, z)$. Since a Möbius mapping acts as an isometry, it follows that $\delta = d(\gamma_i(0), \gamma_i(z))$.

Since $x \in L(z : k, \alpha)$, we have $x \in I(\gamma_i(z), k, \alpha)$ for infinitely many indices i. Call that set of indices J. Let $\varepsilon < 1/2e^{\delta}$ and define J_ε to be the (infinite) subset of J such that

$$J_\varepsilon = \{i \in J \mid 1 - |\gamma_i(z)| < \varepsilon\}.$$

Thus, if $i \in J_\varepsilon$ then

$$\left| x - \frac{\gamma_i(z)}{|\gamma_i(z)|} \right| < k\varepsilon^\alpha. \tag{1}$$

We know that $\gamma_i(0)$ lies on the hyperbolic sphere

$$C = \{\zeta \in B \mid d(\zeta, \gamma_i(z)) = \delta\}.$$

Let us now make an Euclidean estimate of how far $\gamma_i(0)$ can be from $\gamma_i(z)$ by computing the two extremal distances to C from $\gamma_i(z)$. Let a be the distance from $\gamma_i(z)$ to ν, the point closest to the origin in C, and let b be the distance to the point β furthest away from the origin in C. See Figure 2.

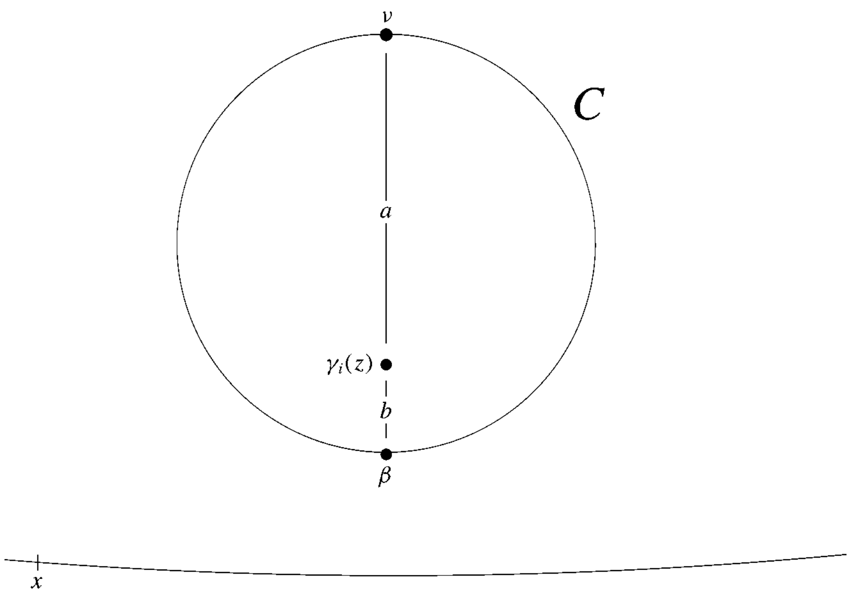

Figure 2 $\gamma_i(0)$ lies on the hyperbolic sphere C centered at $y = \gamma_i(z)$ with ν, β, and y lying in the unit ball on the ray from the origin through y, where ν is the point on the sphere that is closest to the origin and β is the point farthest away; a is the Euclidean distance from y to ν, and similarly b is $|\beta - y|$

We have that

$$\delta = d(\gamma_i(z), \nu) = d(0, \gamma_i(z)) - d(0, \nu) = \log\left(\frac{(1 - |\nu|)}{1 - |\gamma_i(z)|} \frac{1 + |\gamma_i(z)|}{(1 + |\nu|)} \right)$$

and thus may derive the following rough estimate:

$$\tfrac{1}{2} e^\delta (1 - |\gamma_i(z)|) < 1 - |\nu| < 2e^\delta (1 - |\gamma_i(z)|).$$

We can then estimate a as

$$a = |\gamma_i(z)| - |\nu| = (1 - |\nu|) - (1 - |\gamma_i(z)|)$$
$$< 2e^{\delta}(1 - |\gamma_i(z)|) - (1 - |\gamma_i(z)|) < \varepsilon(2e^{\delta} - 1). \tag{2}$$

Similarly, for b we have

$$b = |\beta| - |\gamma_i(z)| < (1 - |\gamma_i(z)|) - \frac{1}{2e^{\delta}}(1 - |\gamma_i(z)|) < \varepsilon\left(1 - \frac{1}{2e^{\delta}}\right). \tag{3}$$

Defining θ to be $\arctan(a/|\gamma_i(z)|)$, we estimate

$$\left| x - \frac{\gamma_i(0)}{|\gamma_i(0)|} \right| < \left| x - \frac{\gamma_i(z)}{|\gamma_i(z)|} \right| + \theta.$$

But from (2) and since we have chosen $\varepsilon < 1/2e^{\delta}$, we can estimate θ as follows:

$$\theta = \arctan \frac{a}{|\gamma_i(z)|} \leq \frac{a}{|\gamma_i(z)|} < \frac{a}{1 - \varepsilon} < \frac{\varepsilon(2e^{\delta} - 1)}{1 - \varepsilon} < \varepsilon 2e^{\delta}.$$

Using inequality (1), it follows that

$$\left| x - \frac{\gamma_i(0)}{|\gamma_i(0)|} \right| < k\varepsilon^{\alpha} + \varepsilon 2e^{\delta} \quad \text{for } i \in J_{\varepsilon}. \tag{4}$$

Now (3) yields

$$1 - |\gamma_i(0)| > \varepsilon - b > \varepsilon - \varepsilon\left(1 - \frac{1}{2e^{\delta}}\right) = \frac{\varepsilon}{2e^{\delta}}. \tag{5}$$

We aim to find a K such that $x \in L(0 : K, \alpha)$, that is,

$$K > \left| x - \frac{\gamma_i(0)}{|\gamma_i(0)|} \right| (1 - |\gamma_i(0)|)^{-\alpha}. \tag{6}$$

Let us therefore study the right-hand side of this expression. From (4) and (5) we obtain

$$\left| x - \frac{\gamma_i(0)}{|\gamma_i(0)|} \right| (1 - |\gamma_i(0)|)^{-\alpha} < (k\varepsilon^{\alpha} + \varepsilon 2e^{\delta})\left(\frac{\varepsilon}{2e^{\delta}}\right)^{-\alpha}$$
$$= (2e^{\delta})^{\alpha}(k + 2e^{\delta}\varepsilon^{1-\alpha}). \tag{7}$$

Since $\alpha \leq 1$ we have that $\varepsilon^{1-\alpha} \leq 1$. Hence, by picking $K = (2e^{\delta})^{\alpha}(k + 2e^{\delta})$, inequality (6) will be satisfied and so $x \in L(0 : K, \alpha)$. This proves that $\mathcal{L}(\alpha)$ is independent of the base point of the orbit if $\alpha \leq 1$, which ends the first part of the proof.

To prove the statement about the strong α-limit set, let us suppose that

$$x \in \bigcap_{k>0} L(z : k, \alpha), \quad \alpha < 1. \tag{8}$$

We aim to show that $x \in \bigcap_{k>0} L(0 : k, \alpha)$. It is therefore enough to show that $x \in L(0 : \kappa, \alpha)$ for any given $\kappa > 0$.

Note from (8) that $x \in L(z, k, \alpha)$ holds trivially with the special choice of $k = \kappa/2^{\alpha+1}e^{\alpha\delta}$, where (as before) $\delta = d(0, z)$. Thus, $x \in I(\gamma_i(z) : k, \alpha)$ for infinitely many indices i. Let us denote this set of indices by J. Now pick

$$\varepsilon = \left(\frac{k}{2e^{\delta}}\right)^{1/(1-\alpha)}.$$

Since

$$L(0 : c_1, \alpha) \subset L(0 : c_2, \alpha) \quad \text{if } c_1 < c_2,$$

we can without loss of generality assume that $\kappa < 2(2e^{\delta})^{2\alpha}$. That gives us that $k < (2e^{\delta})^{\alpha}$ and $\varepsilon < 1/2e^{\delta}$, so we can use the estimate in (7). Let $i \in J_{\varepsilon}$; then

$$\left| x - \frac{\gamma_i(0)}{|\gamma_i(0)|} \right| (1 - |\gamma_i(0)|)^{-\alpha} < (2e^{\delta})^{\alpha}(k + 2e^{\delta}\varepsilon^{1-\alpha}) = 2k(2e^{\delta})^{\alpha} = \kappa.$$

Thus as in (6) we have that, for every $i \in J_{\varepsilon}$ (which are infinitely many), $x \in I(\gamma_i(0) : \kappa, \alpha)$. Hence $x \in L(0 : \kappa, \alpha)$, which ends the proof. □

REMARK 2.7. It is easy to see that $\mathcal{L}_s(1)$ is not independent of the base point. For example, let Γ be a Fuchsian group generated by a single hyperbolic generator in the unit ball. Then the limit set consists only of the two fixed points that are in Λ_c; however, the points are in $\mathcal{L}_s(1)$ if and only if the base point is taken to be the origin.

3. The Case $0 < \alpha \leq \frac{1}{2}$

Suppose ξ is a parabolic fixed point that can be reached from the origin along a ray inside the Dirichlet domain. The geodesic will then run straight out on a parabolic cusp on the quotient manifold. Hence $\varphi(t) = t$, which is the maximal escape rate. That is, $\varphi(t) \leq 1$ will always be true, even for $\xi \notin \Lambda$.

Thus the φ function will not be of any help for classifying points in $\mathcal{L}(\alpha)$ when $0 < \alpha \leq \frac{1}{2}$. Let us quickly turn to the next case.

4. The Case $\frac{1}{2} \leq \alpha < 1$

THEOREM 4.1. *Suppose that ξ is on the unit sphere. Then we have the following two equivalences*:

(i) $\xi \in \mathcal{L}(\alpha)$, *for* $\frac{1}{2} < \alpha \leq 1$, *if and only if*

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) < \infty;$$

(ii) $\xi \in \mathcal{L}_s(\alpha)$, *for* $\frac{1}{2} \leq \alpha < 1$, *if and only if*

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) = -\infty.$$

COROLLARY 4.2. *Let* $\frac{1}{2} < \alpha < 1$. *Then*:

$$\xi \in \mathcal{L}(\alpha) \implies \liminf_{t\to\infty} \frac{\varphi(t)}{t} \leq \frac{1-\alpha}{\alpha};$$

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} < \frac{1-\alpha}{\alpha} \implies \xi \in \mathcal{L}_s(\alpha).$$

Before we plunge into the proofs, let us turn our attention to an auxiliary sequence related to the function φ.

4.1. The Sequence of Generalized Local Minima for $\varphi(t)$

Let the sequence $\{\varphi(t_i)\}$ be the sequence of *generalized* local minima for the function φ in the following sense. Let us follow the geodesic mapped to the unit ball, where it will be the ray from the origin to the boundary point ξ, and let D_i be the ith fundamental domain that we visit on our way from 0 to ξ. Observe that D_i is a copy of the Dirichlet domain around the origin mapped by γ_i. Let now φ_i be the (hyperbolic) distance to the ray from the single orbit point $\gamma_i(0)$ in D_i, and let t_i be the distance from the origin to the point on the ray that is closest to $\gamma_i(0)$; see Figure 3.

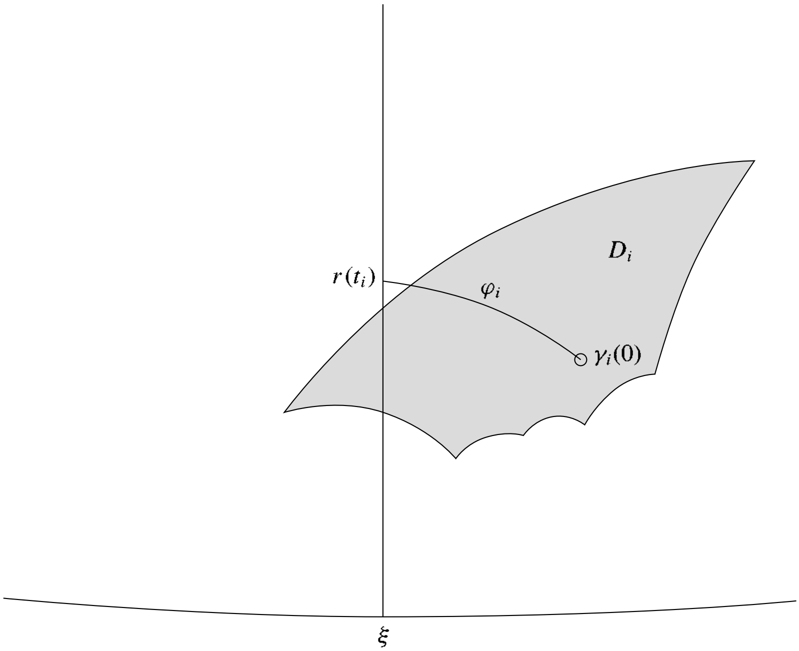

Figure 3 Depiction of case where the closest point (on the ray, $(t, r(t))$, toward ξ) to $\gamma_i(0)$ lies outside the fundamental domain D_i; compare to φ_3 in Figure 4

If the closest point on the ray lies inside D_i then $\varphi_i = \varphi(t_i)$ will be a local minimum, since we travel with unit speed along the ray. However, in the case shown schematically in Figure 3, we will get a *generalized* local minimum point outside the graph of $\varphi(t)$; see Figure 4.

REMARK 4.3. Note that such generalized local minima φ_i will never be smaller than $\varphi(t_i)$ because the closest orbit point is $\gamma_{i-1}(0)$ for every point in D_{i-1}.

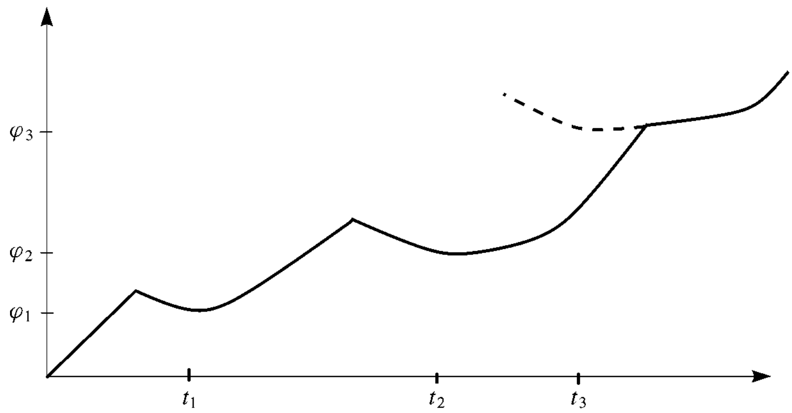

Figure 4 Example of the $\varphi(t)$ graph and the sequence of generalized local minima at $\varphi_i = \varphi(t_i)$; note that $\varphi_1 = \varphi(t_1)$ and $\varphi_2 = \varphi(t_2)$ but $\varphi_3 > \varphi(t_3)$

We will repeatedly use this auxiliary sequence $\{t_i, \varphi_i\}$ of generalized local minima.

LEMMA 4.4. *If*

$$0 < \liminf_{t\to\infty} \frac{\varphi(t)}{t} < 1,$$

then

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

Proof. We can assume that $\{\varphi_i\}$ is an infinite sequence. If not, then there exists an integer I such that φ_I is the last generalized local minimum. That is, the ray from the origin in the unit ball toward the point ξ would never leave the fundamental domain D_I. Let $r(t)$ be the ray toward $\xi \in \partial B$ such that $d(r(t), 0) = t$. Let $t > t_I$ and consider the hyperbolic triangle in D_I with corners in $r(t)$, $r(t_I)$, and $\gamma_I(0)$. The side lengths are $t - t_I$, φ_I, and $\varphi(t)$. Note that the angle at $r(t_I)$ is $\pi/2$. From the triangle inequality we have that

$$t - t_I \le \varphi(t) \le t - t_I + \varphi_I.$$

Hence,

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = 1,$$

which is not allowed. Thus we have that $\{\varphi_i\}$ is an infinite sequence, and then

$$\liminf_{i\to\infty} \frac{\varphi_i}{t_i}$$

exists.

By Remark 4.3 we have immediately that

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \le \liminf_{i\to\infty} \frac{\varphi_i}{t_i}. \tag{9}$$

Let us first assume $\varphi(t_i) = \varphi_i$ for all i; that is, we assume the generalized local minima are true local minima for $\varphi(\cdot)$. (At the very end of this proof we will treat the general case.) In this situation we have that, for a given i, there is a δ_i such that $t_i + \delta_i$ is a local minimum of the function $\varphi(\cdot)/\cdot$. Note that the line from the origin to the point $(t_i + \delta_i, \varphi(t_i + \delta_i))$ will be a tangent to the graph of φ. As before, we examine a right-angled triangle with corners in $r(t_i + \delta_i)$, $r(t_i)$, and $\varphi_i(0)$. The side lengths are then δ_i, φ_i, and $\psi_i := \varphi(t_i + \delta_i)$.

We will use the hyperbolic version of Pythagoras's theorem (cf. [4, p. 146]):

$$\cosh\psi_i = \cosh\delta_i \cosh\varphi_i. \tag{10}$$

From our previous assumption and (9), we have that

$$0 < \liminf_{t\to\infty} \frac{\varphi(t)}{t} \le \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

Thus $\varphi_i \to \infty$. Hence

$$\cosh\varphi_i \approx \frac{e^{\varphi_i}}{2}$$

if i is large.

Since $\psi_i > \varphi_i$, we have the following approximation of (10) for large index i:

$$e^{\psi_i} \approx \cosh\delta_i e^{\varphi_i}.$$

Therefore,

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\psi_i}{t_i+\delta_i} = \liminf_{i\to\infty} \frac{\varphi_i + \log(\cosh\delta_i)}{t_i+\delta_i}. \tag{11}$$

Let us now separately study two cases.

Case 1: $\limsup_{i\to\infty}\delta_i < \infty$. In this case we see from (11) that

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\varphi_i/t_i + \log(\cosh\delta_i))/t_i}{1+\delta_i/t_i} = \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

Case 2: $\limsup_{i\to\infty}\delta_i = \infty$. In this case there are infinitely many indices i such that

$$\cosh\delta_i \approx \frac{e^{\delta_i}}{2}.$$

From (11) we have that

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} = \liminf_{i\to\infty} \frac{\varphi_i + \delta_i - \log 2}{t_i+\delta_i} = \liminf_{i\to\infty} \frac{\varphi_i/t_i + \delta_i/t_i}{1+\delta_i/t_i}. \tag{12}$$

To simplify the notation, let

$$k = \liminf_{i\to\infty} \frac{\varphi_i}{t_i} \quad \text{and} \quad f(x) = \frac{k+x}{1+x}.$$

We have then that

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \ge \liminf_{i\to\infty} f\left(\frac{\delta_i}{t_i}\right).$$

We note also that $0 < k \le 1$ and that

$$f'(x) > 0 \iff k < 1.$$

If $k = 1$ then $f(x) \equiv 1$ and thus

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \geq 1,$$

which is not allowed. Hence we have that $k < 1$, which yields

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \geq \liminf_{i\to\infty} f\left(\frac{\delta_i}{t_i}\right) \geq f(0) = k = \liminf_{i\to\infty} \frac{\varphi_i}{t_i}.$$

We are now done under the assumption that $\varphi(t_i) = \varphi_i$ for all i.

Finally, let us treat the case where $\varphi(t_i) < \varphi_i$, schematically depicted in Figure 4 for $i = 3$. We first concentrate on the graph consisting of the dotted-arc continuation when t is such that $r(t) \in D_i$. In other words, we examine the quotients $d(r(t), \gamma_i(0))/t$. As in the preceding arguments (starting at the point where we assumed $\varphi(t_i) = \varphi_i$), we conclude that

$$\liminf_{i\to\infty} \frac{\varphi_i}{t_i} = \liminf_{i\to\infty} \inf_t \frac{d(r(t), \gamma_i(0))}{t}.$$

On the other hand,

$$\inf_t \frac{d(r(t), \gamma_i(0))}{t} \leq \inf_{r(t)\in D_i} \frac{d(r(t), \gamma_i(0))}{t} = \inf_{r(t)\in D_i} \frac{\varphi(t)}{t}.$$

Thus

$$\liminf_{i\to\infty} \frac{\varphi_i}{t_i} \leq \liminf_{i\to\infty} \inf_{r(t)\in D_i} \frac{\varphi(t)}{t} = \liminf_{t\to\infty} \frac{\varphi(t)}{t}. \qquad \square$$

We will also need the following variant of Lemma 4.4.

LEMMA 4.5. *Let* $\frac{1}{2} < \alpha < 1$. *If* $\liminf_{t\to\infty} \varphi(t) = \infty$ *and if*

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) < \infty,$$

then

$$\liminf_{t\to\infty} (\alpha(\varphi(t) + t) - t) = \liminf_{i\to\infty} (\alpha(\varphi_i + t_i) - t_i).$$

Proof. Since the proof is completely analogous to the proof of Lemma 4.4, we give only a short outline.

For simplicity we define $\Phi(t) := \alpha(\varphi(t) + t) - t$. With the same triangle argument as before, we see that if $\{\varphi_i\}$ is finite then

$$\alpha(2t - t_I) - t \leq \Phi(t) \leq \alpha(2t - t_I + \varphi_I) - t,$$

and since $\alpha > \frac{1}{2}$ we have that

$$\liminf_{t\to\infty} \Phi(t) = \infty.$$

Hence we can safely assume that the sequence $\{\varphi_i\}$ is infinite.

Next we assume that $\Phi(t)$ has a local minimum at $t_i + \delta_i$. Similarly to (11), we have

$$\liminf_{t\to\infty} \Phi(t) = \liminf_{i\to\infty} \alpha(\varphi_i + \log(\cosh \delta_i) + t_i + \delta_i) - t_i - \delta_i.$$

Now we treat the two cases.

Case 1: $\limsup \delta_i < \infty$. Then

$$\liminf_{t\to\infty} \Phi(t) = \liminf_{i\to\infty} \alpha(\varphi_i + t_i) - t_i.$$

Case 2: $\limsup \delta_i = \infty$. Then

$$\liminf_{t\to\infty} \Phi(t) = \liminf_{i\to\infty} \alpha(\varphi_i + t_i) - t_i + \delta_i(2\alpha - 1) \geq \liminf_{i\to\infty} \alpha(\varphi_i + t_i) - t_i,$$

since $\alpha > \frac{1}{2}$. □

4.2. Proof of Theorem 4.1

For part (i), note that we already know that $\xi \in \mathcal{L}(1) = \Lambda_c$ if and only if $\liminf_{t\to\infty} \varphi(t) < \infty$. This takes care of the case $\alpha = 1$. From now on in the proof, we will assume that $\alpha < 1$ and that $\liminf_{t\to\infty} \varphi(t) = \infty$.

We have that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a $k > 0$ such that $\xi \in L(0 : k, \alpha)$. This is equivalent to saying that, for infinitely many γ_i in Γ,

$$\left|\xi - \frac{\gamma_i}{|\gamma_i|}\right| < k(1 - |\gamma_i|)^\alpha.$$

This can be expressed using the notation in Figure 5 as $R_i < kh_i^\alpha$ or as

$$r_i \sin(\theta_i) < k(r_i \cos(\theta_i))^\alpha \tag{13}$$

for infinitely many i.

We note from Figure 5 that φ_i is small if and only if the angle θ_i is small. Therefore $\xi \in \mathcal{L}_s(\alpha)$ if and only if (13) holds for infinitely many such local minimum points in $\{t_i\}$.

Let us now give estimates for the components in (13). From a standard calculation we have that

$$t_i = \log\left(\frac{2 - r_i}{r_i}\right).$$

Since $r_i \ll 1$, we have the estimate

$$r_i \approx 2\exp(-t_i). \tag{14}$$

See [4, p. 162] for the following relations between φ_i and the angle θ_i in Figure 5:

$$\sin(\theta_i) = \tanh(\varphi_i); \tag{15}$$

$$\cos(\theta_i) = \frac{1}{\cosh(\varphi_i)}; \tag{16}$$

$$\tan(\theta_i) = \sinh(\varphi_i). \tag{17}$$

Since we assumed that ξ is not a conical limit point, we know that $\liminf_{i\to\infty} \varphi_i = \infty$ and thus we can make two estimates of (15) and (16) as follows:

$$\sin(\theta_i) \approx 1; \tag{18}$$

$$\cos(\theta_i) \approx \frac{2}{\exp(\varphi_i)}. \tag{19}$$

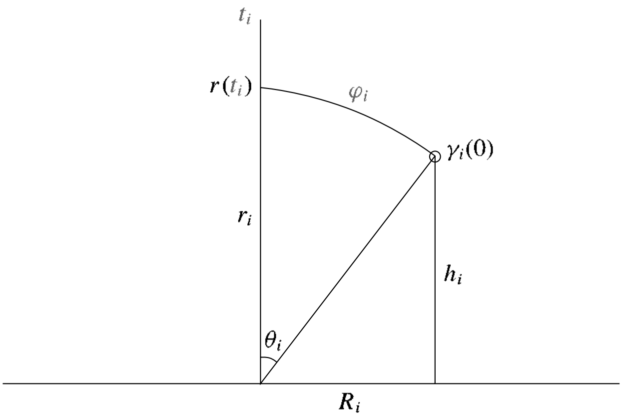

Figure 5 Geometric relations between θ_i, φ_i, and t_i (the latter two shaded to indicate they are the only hyperbolic distances in the figure) in the upper half-space, where φ_i is the shortest hyperbolic distance from the orbit point to the vertical line (i.e., the hyperbolic distance along the circular arc from the orbit point to $r(t_i)$ on the vertical line); the hyperbolic distance from $r(t_i)$ to the image of the origin in the upper half plane is t_i

Using the estimates (14), (18), and (19) in condition (13), we obtain the relation

$$\xi \in \mathcal{L}(\alpha) \iff (2\exp(-t_i))^{1-\alpha} < K(2\exp(-\varphi_i))^{\alpha}$$

for infinitely many indices i and for some constant K. Let

$$C = \log(K) + (2\alpha - 1)\log 2.$$

We can now write the preceding inequality as

$$-t_i(1-\alpha) < -\alpha\varphi_i + C.$$

Hence $\xi \in \mathcal{L}(\alpha)$ if and only if there exists a C such that

$$\alpha(\varphi_i + t_i) - t_i < C$$

for infinitely many indices i. Thus

$$\xi \in \mathcal{L}(\alpha) \iff \liminf_{i\to\infty}(\alpha(\varphi_i + t_i) - t_i) < \infty, \tag{20}$$

which (thanks to Lemma 4.5) is equivalent to the statement

$$\xi \in \mathcal{L}(\alpha) \iff \liminf_{t\to\infty}(\alpha(\varphi(t) + t) - t) < \infty.$$

(Recall that we could make the assumption that $\liminf_{t\to\infty}\varphi(t) = \infty$ after treating the case $\alpha = 1$ separately in the beginning; hence all the assumptions in Lemma 4.5 are fulfilled.) We are done with the proof of the first statement.

In proving part (ii) of Theorem 4.1, we shall treat the case $\alpha = \frac{1}{2}$ separately at the end.

We note that $\xi \in \mathcal{L}_s(\alpha)$ if and only if $\xi \in L(0 : k, \alpha)$ for *all* $k > 0$. Using the same arguments as before, we will in this case have the analogue to condition (13) as $\xi \in \mathcal{L}_s(\alpha)$ if and only if

$$r_i \sin(\theta_i) < k(r_i \cos(\theta_i))^\alpha \tag{21}$$

for infinitely many i and for all $k > 0$. Using (14), (18), and (19), we see that (21) is equivalent to

$$-t_i(1-\alpha) < -\alpha\varphi_i - C$$

for infinitely many indices i and for all $C < \infty$. Thus $\xi \in \mathcal{L}_s(\alpha)$ if and only if

$$\liminf_{i\to\infty}(\alpha(\varphi_i + t_i) - t_i) = -\infty, \tag{22}$$

which by Lemma 4.5 is equivalent to

$$\xi \in \mathcal{L}(\alpha) \iff \liminf_{t\to\infty}(\alpha(\varphi(t) + t) - t) = -\infty.$$

For the case $\alpha = \frac{1}{2}$, we note that the assumption $\alpha > \frac{1}{2}$ in Lemma 4.5 is used only to make sure that we have an infinite sequence $\{\varphi_i\}$.

Suppose now that $\alpha = \frac{1}{2}$ and that $\xi \in \mathcal{L}_s\left(\frac{1}{2}\right)$. We will show that in this case $\{\varphi_i\}$ must be infinite. The idea of that argument is taken from the proof of [14, Thm. 2.4.10], which states that a conical limit point cannot appear on the boundary on a Dirichlet domain.

Let us study the unit ball tessellated by images of the Dirichlet domain around the origin. We know that the ray to ξ visits such a domain only once. Since the number of local minima is finite, we conclude that there is a domain, F_i, where the ray finally enters and then never leaves on its way to ξ; that is, there is a C such that $c\xi \in F_i$ for every $c > C$. Recall that every point in this open domain F_i has the property that it is closer to the orbit point $\gamma_i(0)$ in it than to any other orbit point. Let us for simplicity map the whole picture by the mapping γ_i^{-1}, so that $F_0 = \gamma_i^{-1}(F_i)$ is a Dirichlet domain centered at the origin. Let us now study the ray from the origin to $\xi_0 = \gamma_i^{-1}(\xi)$. We see that this ray is in F_0 and parameterize it by $c\xi_0$ for $c \in (0, 1)$. Let us now construct an open hyperbolic ball centered at $c\xi_0$ with radius $d(c\xi, \gamma_i(0))$. That is, let

$$B_c = \{z : d(z, c\xi_0) < d(c\xi_0, 0)\}.$$

We note that B_c does not contain any orbit points, and the same is true for the union

$$\hat{B} = \bigcup_{c\in(0,1)} B_c.$$

We note that $\hat{B}$ is a horoball $\left(\text{with Euclidean radius } \frac{1}{2}\right)$ that is tangent to the unit ball at ξ_0.

Finally, let us map $\hat{B}$ back to $\gamma_i(\hat{B})$, which is tangent to ξ, has a radius greater than or equal to $(1 - |\gamma_i(0)|)/2$, and contains no orbit points. We conclude then from Definition 2.5 that $\xi \notin \mathcal{L}_s\left(\frac{1}{2}\right)$. We thus conclude that the sequence $\{\varphi_i\}$ is infinite.

Furthermore, since $\xi \in \mathcal{L}_s\left(\frac{1}{2}\right)$ we have

$$\liminf_{i\to\infty}(\varphi_i - t_i) = -\infty < \infty$$

by the reasoning that led to equation (22), which is still valid. Because $\{\varphi_i\}$ is infinite, we can now use the proof for Lemma 4.5 to obtain that

$$\liminf_{t\to\infty}(\varphi(t) - t) = -\infty.$$

On the other hand, assume that

$$\liminf_{t\to\infty}(\varphi(t) - t) = -\infty.$$

If $\{\varphi_i\}$ is finite then, from the beginning of the proof of Lemma 4.5, we have $\left(\text{since } \alpha = \frac{1}{2}\right)$ that $\frac{1}{2}(\varphi(t) - t) = \Phi(t) \geq -\frac{1}{2}t_I$, which contradicts our assumption. Hence we conclude that $\{\varphi_i\}$ is infinite. As before, we can use the proof of Lemma 4.5 to see that

$$\liminf_{i\to\infty}(\varphi_i - t_i) = -\infty,$$

which is equivalent to $\xi \in \mathcal{L}_s\left(\frac{1}{2}\right)$. □

4.3. Proof of Corollary 4.2

Suppose $\xi \in \mathcal{L}(\alpha)$. Then, from Theorem 4.1(i), it follows that

$$\liminf_{t\to\infty}(\alpha(\varphi(t) + t) - t) < \infty.$$

Hence there is a $K < \infty$ such that

$$\liminf_{t\to\infty}\left(\varphi(t) - \frac{1-\alpha}{\alpha}t\right) = K.$$

Thus

$$\liminf_{t\to\infty}\frac{\varphi(t)}{t} \leq \frac{1-\alpha}{\alpha}.$$

For the second implication, suppose that

$$a := \liminf_{t\to\infty}\frac{\varphi(t)}{t} < \frac{1-\alpha}{\alpha}.$$

By Lemma 4.4, $a = \varphi_i/t_i$. Let $\delta := (1-\alpha)/\alpha - a$.

We now have that for, every $\varepsilon > 0$, there is an infinite set of indices $J = \{j\}$ such that

$$\frac{\varphi(t_j)}{t_j} - a < \varepsilon \quad \text{for all} \quad j \in J.$$

This will especially be valid for our choice of $\varepsilon = \delta/2$ (note that $\delta > 0$ by our previous assumption). Therefore,

$$\varphi(t_j) < t_j(a + \varepsilon) \quad \text{for all} \quad j \in J$$

and so

$$\varphi(t_j) - \frac{1-\alpha}{\alpha}t_j < t_j\left(a + \varepsilon - \frac{1-\alpha}{\alpha}\right) = t_j(\varepsilon - \delta) = -\frac{\delta}{2}t_j.$$

We conclude, using Lemma 4.5, that

$$\liminf_{t\to\infty} \alpha(\varphi(t)+t)-t) = -\infty,$$

which by Theorem 4.1(ii) is equivalent to $\xi \in \mathcal{L}_s(\alpha)$. □

4.4. A Global Result

Using Theorem 4.1 together with a Borel–Cantelli type result from [14] yields a global result for the limit sets $\mathcal{L}_s(\alpha)$ and their corresponding Poincaré series.

COROLLARY 4.6. *Let γ_θ be a geodesic on $\mathcal{B}/\Gamma$ starting at the reference point x_0 in the θ direction, where θ is on the unit sphere, and let φ_θ be the φ distance function for γ_θ. Let $|\cdot|$ be the $(n-1)$-dimensional Lebesgue measure on the unit sphere, and let $\frac{1}{2} < \alpha \le 1$. If*

$$\left|\left\{\theta : \liminf_{t\to\infty}(\alpha(\varphi_\theta(t)+t)-t) < \infty\right\}\right| > 0$$

then the Poincaré type series

$$\sum_{\gamma_i\in\Gamma}(1-|\gamma_i(0)|)^{(n-1)\alpha} = \infty.$$

Proof. If $\liminf_{t\to\infty}(\alpha(\varphi_\theta(t)+t)-t) < \infty$ then by Theorem 4.1 we have that, when γ_θ is transformed into the unit ball, it ends at a point in $\mathcal{L}(\alpha)$. Hence if

$$\left|\left\{\theta : \liminf_{t\to\infty}(\alpha(\varphi_\theta(t)+t)-t) < \infty\right\}\right| > 0$$

then $|\mathcal{L}(\alpha)| > 0$. Now [14, Thm. 2.1.1] tells us that if $|\mathcal{L}(\alpha)| > 0$ then

$$\sum_{\gamma_i\in\Gamma}(1-|\gamma_i(0)|)^{(n-1)\alpha} = \infty.$$ □

REMARK 4.7. Compare this corollary with the forthcoming [5, Lemma 2]. From the definition of Λ_α in [5] and Corollary 4.2 here, it follows that $\Lambda\setminus\Lambda_\alpha = \mathcal{L}\big(\frac{1}{1+\alpha}\big)$.

5. A Ladder-like Example

We now study a Riemann surface that looks like a ladder or a "one-dimensional jungle gym". Our surface is an infinitely long body with evenly distributed "holes"; see Figure 6.

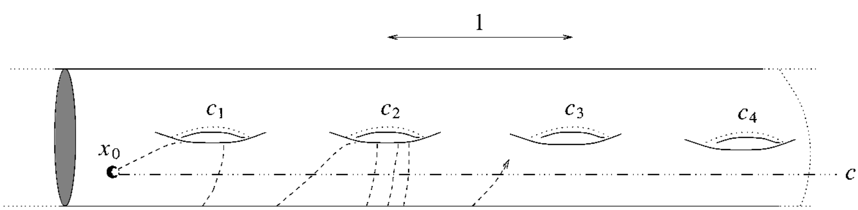

Figure 6 A one-dimensional jungle gym

For simplicity let us assume that the distance between the centers of two consecutive holes is 1, and denote the shortest curve circumscribing hole j by c_j. Furthermore, let c be a geodesic from x_0 not intersecting any c_j and such that $\varphi(t) = t$ along c. We will also assume that the shortest closed arc crossing both c_j and c has length 1.

We will study a geodesic $g(t)$ started in x_0 and winding through the holes in a consecutive way. That is, the geodesic will alternate between crossing a c_j and c, and if $g(s) \in c_j$ then $g(t) \notin c_k$ for $t > s$ and $k < j$. Let $N(j)$ be the number of intersections of $g(t)$ with c_j. Note that $g(t)$, and hence the corresponding limit point $\xi \in \partial B$, will be completely determined by the sequence $\{N(j)\}_{j=1}^{\infty}$.

This jungle-gym construction, together with Corollary 4.2, will be used to give simple examples of $\xi \in \mathcal{L}(\alpha) \setminus \mathcal{L}(\alpha')$ where $\frac{1}{2} < \alpha < \alpha' < 1$.

COROLLARY 5.1. *Let Γ be given by the jungle-gym construction just described and ξ by the geodesic making $N(j)$ turns in hole j twisting out in a consecutive way. Assume that $N(j) \geq 1$ and that the limit average of the number of turns is bounded, that is,*

$$\bar{N} := \limsup_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} N(j) < \infty.$$

Then

$$\xi \in \mathcal{L}_s\left(\frac{\bar{N}}{\bar{N}+1}\right) \Big\backslash \mathcal{L}\left(\frac{\bar{N}+1}{\bar{N}+2}\right).$$

Proof. We have that $\xi \notin \mathcal{L}(1)$ because the geodesic does not return at all.

Now we estimate the local minimum of $\varphi(t)$ about where the geodesic has made $N(i)$ turns in hole i. Let us denote that "ending" local minimum by $\varphi_{i_e} = \varphi(t_{i_e})$. Using the "little ordo" $o(\cdot)$ function immediately yields the estimate

$$i_e - o(i_e) < \varphi_{i_e} < i_e + o(i_e).$$

We will use the following estimate:

$$i_e + \sum_{j=1}^{i}(N(j) - 1) - o(i_e) < t_{i_e} < i_e + \sum_{j=1}^{i} N(j) + o(i_e).$$

We see from the construction that

$$\liminf_{i\to\infty} \frac{\varphi_i}{t_i} = \liminf_{i\to\infty} \frac{\varphi_{i_e}}{t_{i_e}}.$$

Hence the previous estimates give

$$\liminf_{i\to\infty} \frac{1}{1 + \frac{1}{i}\sum_{j=1}^{i} N(j)} < \liminf_{i\to\infty} \frac{\varphi_i}{t_i} < \liminf_{i\to\infty} \frac{1}{\frac{1}{i}\sum_{j=1}^{i} N(j)}.$$

Thus we see that

$$\frac{1}{1+\bar{N}} < \liminf_{i\to\infty} \frac{\varphi_i}{t_i} < \frac{1}{\bar{N}}.$$

Now, if $x = (1-\alpha)/\alpha$ then $\alpha = 1/(x+1)$. We can then use Corollary 4.2 and Lemma 4.4 to conclude that

$$\xi \in \mathcal{L}_s\left(\frac{\bar{N}}{\bar{N}+1}\right) \Big\backslash \mathcal{L}\left(\frac{\bar{N}+1}{\bar{N}+2}\right),$$

which is the desired expression. □

6. The Case $\alpha = 1$

It is well known that if the geodesic $g(t)$ returns infinitely often to a compact neighborhood of x_0 then the limit point ξ is in the nontangentially limit set Λ_c. Let us try to be a little more precise about this.

PROPOSITION 6.1. *Let the set $L(\cdot : \cdot, \cdot)$ be as in Definition 2.2. Then*

$$\xi \in L(0 : \sinh(K), 1) \iff \liminf_{t\to\infty} \varphi(t) < K.$$

Furthermore,

$$\xi \in \mathcal{L}_s(1) \iff \liminf_{t\to\infty} \varphi(t) = 0.$$

Proof. Suppose that $\xi \in L(0 : \sinh(K), 1)$. Using the notation from Figure 5, we know that there are infinitely many orbit points $\gamma_i(0)$ such that $R_i < \sinh(K) h_i$. We can reformulate this as: there are infinitely many $\gamma_i(0)$ such that

$$\tan(\theta_i) < \sinh(K).$$

Using equation (17) gives us that, for infinitely many indices i, $\sinh(\varphi_i) < \sinh(K)$ and thus $\varphi_i < K$ for infinitely many i. Hence

$$\xi \in L(0 : \sinh(K), 1) \iff \liminf_{i\to\infty} \varphi_i < K,$$

which by Lemma 6.2 yields the first equivalence:

$$\xi \in L(0 : \sinh(K), 1) \iff \liminf_{t\to\infty} \varphi(t) < K.$$

(Note that we can use Lemma 6.2 because we can safely assume that $\{\varphi_i\}$ is infinite.)

For the second statement, we argue as before and conclude that $\xi \in \mathcal{L}_s(1)$ if and only if, for every $K > 0$, there are infinitely many indices i such that $\varphi_i < K$. Hence, using Lemma 6.2, we obtain

$$\xi \in \mathcal{L}_s(1) \iff \liminf_{t\to\infty} \varphi(t) = 0.$$ □

LEMMA 6.2. *If the sequence $\{\varphi_i\}$ is infinite then*

$$\liminf_{i\to\infty} \varphi_i = \liminf_{t\to\infty} \varphi(t).$$

Proof. We have from Remark 4.3 that

$$\liminf_{i\to\infty} \varphi_i \geq \liminf_{i\to\infty} \varphi(t_i) \geq \liminf_{t\to\infty} \varphi(t).$$

On the other hand, we can use an argument similar to that used in (the latter part of) the proof of Lemma 4.4 to obtain the following inequality:

$$\varphi_i = \inf_t d(r(t), \gamma_i(0)) \le \inf_{r(t)\in D_i} d(r(t), \gamma_i(0)) = \inf_{r(t)\in D_i} \varphi(t).$$

Therefore,

$$\liminf_{i\to\infty} \varphi_i \le \liminf_{i\to\infty} \inf_{r(t)\in D_i} \varphi(t) = \liminf_{t\to\infty} \varphi(t). \qquad \square$$

7. The Case $\alpha > 1$

When $\xi \in \mathcal{L}(\alpha)$ for $\alpha > 1$, we have immediately that $\xi \in \Lambda_c$ and hence that there exists a bounded subsequence of $\{\varphi_i\}$. But we can say more than this.

Proposition 7.1. *Suppose that $\alpha > 1$. Then we have the following two equivalences*:

(i) $\xi \in \mathcal{L}(\alpha)$ *if and only if there exists a $K < \infty$ such that*

$$\liminf_{t\to\infty} \varphi(t) e^{(\alpha-1)t} < K;$$

(ii) $\xi \in \mathcal{L}_s(\alpha)$ *if and only if*

$$\liminf_{t\to\infty} \varphi(t) e^{(\alpha-1)t} = 0.$$

Proof. We have that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a $k < \infty$ such that ξ is in infinitely many $L(0 : k, \alpha)$. With the notation from Figure 5, this translates into $R_i < kh_i^\alpha$ for infinitely many i, or

$$r_i \sin(\theta_i) < k(r_i \cos(\theta_i))^\alpha \quad \text{for infinitely many } i.$$

Now, using (14), (15), and (16) yields that $\xi \in \mathcal{L}(\alpha)$ if and only if there exists a k such that, for infinitely many i,

$$2e^{-t_i} \tanh(\varphi_i) < k\left(2e^{-t_i} \frac{1}{\cosh(\varphi_i)}\right)^\alpha. \tag{23}$$

We know that $\alpha > 1$, so it follows that every cone with vertex at ξ has infinitely many orbit points inside even if the opening angle is very small. Thus we have

$$\liminf_{i\to\infty} \theta_i \to 0$$

and hence, using equation (15),

$$0 = \liminf_{i\to\infty} \sin(\theta_i) = \liminf_{i\to\infty} \tanh(\varphi_i).$$

Therefore,

$$\liminf_{i\to\infty} \varphi_i = 0.$$

Using this fact in (23) gives us the following asymptotic relation: $\xi \in \mathcal{L}(\alpha)$ if and only if there is a k such that

$$\liminf_{i\to\infty} \varphi_i < k 2^{\alpha-1} e^{-t_i(\alpha-1)}.$$

We conclude that $\xi \in \mathcal{L}(\alpha)$ if and only if there is a $K < \infty$ such that

$$\liminf_{i\to\infty} \varphi_i e^{t_i(\alpha-1)} < K. \tag{24}$$

We will now show that inequality (24) is equivalent to

$$\liminf_{t\to\infty} \varphi(t) e^{t(\alpha-1)} < K' \tag{25}$$

for some K'.

In order to simplify the notation, let

$$f(t) = \varphi(t)e^{t(\alpha-1)}, \quad g_i(t) = d(r(t), \gamma_i(0)), \quad f_i(t) = g_i(t)e^{t(\alpha-1)}.$$

Note that

$$f_i(t) = f(t) \quad \text{when } r(t) \in D_i \tag{26}$$

and that $g_i(t_i) = \varphi_i$ is a local minimum for g_i.

On the other hand, f_i has a local minimum not at t_i but rather at $t_i - \delta_i$ for some positive δ_i. Hence, after differentiation, we get that

$$g_i'(t_i - \delta_i) = -(\alpha - 1)g_i(t_i - \delta_i). \tag{27}$$

So again the Pythagorean theorem is used to obtain

$$\cosh(g_i(t_i - \delta_i)) = \cosh(\delta_i)\cosh(\varphi_i),$$

and by differentiation with respect to δ_i we have

$$-\sinh(g_i(t_i - \delta_i))g_i'(t_i - \delta_i) = \sinh(\delta_i)\cosh(\varphi_i). \tag{28}$$

Combining (27) and (28) then yields

$$\sinh(g_i(t_i - \delta_i))g_i(t_i - \delta_i)(\alpha - 1) = \sinh(\delta_i)\cosh(\varphi_i). \tag{29}$$

From (28) we see that $\frac{d}{dx}g(t_i - x)$ decreases from 0 to $-\cosh(\varphi_i)$ as x goes from 0 to ∞. Thus,

$$-gi'(t_i - \delta_i) \le \cosh(\varphi_i). \tag{30}$$

This estimate, together with (27), gives us that

$$g_i(t_i - \delta_i) \le \frac{\cosh(\varphi_i)}{\alpha - 1}. \tag{31}$$

By (28) we see that

$$\delta_i \le \sinh(\delta_i) = \frac{-\sinh(g_i(t_i - \delta_i))g_i'(t_i - \delta_i)}{\cosh(\varphi_i)} \le \sinh(g_i(t_i - \delta_i));$$

using (31), we obtain

$$\delta_i \le \sinh\left(\frac{\cosh(\varphi_i)}{\alpha - 1}\right). \tag{32}$$

Since

$$f_i(t_i - \delta_i) = g_i(t_i - \delta_i)e^{(t_i-\delta_i)(\alpha-1)} = \frac{g_i(t_i - \delta_i)}{\varphi_i} f_i(t_i)e^{-\delta_i(\alpha-1)},$$

we have the following estimate for the function f_i:

$$f_i(t_i - \delta_i) \le f_i(t_i)e^{-\delta_i(\alpha-1)}. \tag{33}$$

We now have that

$$\liminf_{t\to\infty} f(t) \le \liminf_{i\to\infty} f_i(t_i),$$

and the following inverse inequality will be used to prove the equivalence of (24) and (25):

$$\begin{aligned}\liminf_{t\to\infty} f(t) &= \liminf_{i\to\infty} \inf_{r(t)\in D_i} f(t) = (26) = \liminf_{i\to\infty} \inf_{r(t)\in D_i} f_i(t)\\ &\ge \liminf_{i\to\infty} f_i(t_i-\delta_i) \ge (33) \ge \liminf_{i\to\infty} f_i(t_i)\exp(-\delta_i(\alpha-1))\\ &\ge (32) \ge \liminf_{i\to\infty} f_i(t_i)\exp\left(-\sinh\left(\frac{\cosh(\varphi_i)}{\alpha-1}\right)(\alpha-1)\right)\\ &\ge \left(\liminf_{i\to\infty} f_i(t_i)\right)\exp\left(-\sinh\left(\frac{\cosh(\liminf_{i\to\infty}\varphi_i)}{\alpha-1}\right)(\alpha-1)\right).\end{aligned}$$

We can assume that $\liminf_{i\to\infty}\varphi_i = 0$ since otherwise it would follow, by Proposition 6.1 and Lemma 6.2, that

$$\xi \notin \mathcal{L}_s(1) \supset \mathcal{L}(\alpha) \quad \text{for all } \alpha > 1.$$

Hence

$$\liminf_{t\to\infty} f(t) \ge \liminf_{i\to\infty} f_i(t_i)\exp\left(-\sinh\left(\frac{1}{\alpha-1}\right)(\alpha-1)\right)$$

and so, if

$$K' = K\exp\left(-\sinh\left(\frac{1}{\alpha-1}\right)(\alpha-1)\right),$$

then (24) and (25) are equivalent. This concludes our proof of part (i).

To prove part (ii) we need only observe that $\xi \in \mathcal{L}(\alpha)$ if and only if, for every $k>0$, ξ is in infinitely many $L(0:k,\alpha)$. Following the same arguments as before, we obtain that $\xi\in\mathcal{L}(\alpha)$ if and only if

$$\liminf_{i\to\infty} \varphi_i e^{t_i(\alpha-1)} = 0,$$

where this expression is equivalent, by our previous reasoning (with $K=0$), to

$$\liminf_{t\to\infty} \varphi(t) e^{t(\alpha-1)} = 0. \qquad \square$$

Point- and Line-Transitive Sets

Note from Remark 2.7 that the results in Proposition 7.1 depend on the choice of base point $x_0 \in B/\Gamma$. We now allow the base point to vary, letting $\varphi_a(t)$ be as $\varphi(t)$ above except that we replace x_0, the image of the origin, by x_a, the image of $a\in B$. Then it is easy to see, using Figure 5 and the definitions in [14, pp. 26, 27], that ξ is a point transitive limit point ($\xi\in T_p$) if and only if

$$\liminf_{t\to\infty} \varphi_a(t) = 0 \quad \text{for all } a\in B$$

and that ξ is a line transitive limit point (T_l) if and only if

$$\liminf_{t\to\infty} (\varphi_a(t)+\varphi_b(t)) = 0 \quad \text{for all pairs } a, b\in B.$$

REMARK 7.2. We have trivially that $T_l \subset T_p$. Furthermore, $T_l \neq \emptyset$ if Γ is of the first kind, and $T_p = \emptyset$ if Γ is of the second kind (see e.g. [14, Thms. 2.2.2, 2.3.3]).

8. A Question about the Archipelago of Γ

The *archipelago* of a discrete group Γ is defined in [13, p. 300]. Let $B_j := \{z \in B \mid d(z, \gamma_j(0)) < r_\Gamma,\ \gamma_j \in \Gamma \setminus \{I\}\}$. Since Γ is discrete, it is possible to find an $r_\Gamma > 0$ such that the balls B_j do not intersect each other. Let us fix such an r_Γ and let $E := \bigcup_j B_j$. That is, E is the "fattened" orbit of Γ, and we call it the *archipelago* of Γ.

Minimal Thinness

For the convenience of the reader, let us here include a short background and a definition of minimal thinness (essentially taken from [13, Sec. 4]).

We denote the class of nonnegative superharmonic functions in the unit ball by $\mathrm{SH}(B)$ and the Poisson kernel at $\tau \in \partial B$, $(1 - |z|^2)/|z - \tau|^n$, by P_τ. The Poisson kernel is a harmonic function. It is minimal in the sense that, if h is a positive harmonic function such that $h(z) \leq P_\tau(z)$ for all $z \in B$, then $h(z) \equiv 0$ or $h(z) = cP_\tau(z)$ for a constant c.

Let us now make a variant of this. Let $u \in \mathrm{SH}(B)$ be such that $u(z) \geq P_\tau(z)$ holds on a subset E of the unit ball. How strong is this condition? Can there be such a function u and a point z in $B \setminus E$ such that $u(z) < P_\tau(z)$? The answer depends on how "big" E is close to the pole τ. The concept of *minimal thinness* was introduced when studying similar questions in [12]. Let us now turn to the definition.

The *reduced function* of h with respect to a subset E of B is defined as

$$R_h^E(w) = \inf\{u(w) : u \in \mathrm{SH}(B) \text{ and } u \geq h \text{ on } E\}.$$

We can make this function lower semicontinuous by regularizing it—that is, obtaining the *regularized reduced function* $\hat{R}_h^E(z) = \liminf_{w \to z} R_h^E(w)$.

DEFINITION 8.1. A set E is *minimally thin* at $\tau \in \partial B$ if there is a z in the unit ball such that $\hat{R}_{P_\tau}^E(z) < P_\tau(z)$.

REMARK 8.2. There is an interesting probabilistic interpretation of minimal thinness; see for example [3, p. 102] or [8, (b1), p. 208] combined with [8, (7.3sm), p. 686]. Let B_t be a Brownian motion in the unit ball B that is conditioned by the Doob's h-condition, where h is the Poisson kernel with pole at ξ. This process will then be conditioned to exit B at $\xi \in \partial B$. Hence, E is minimally thin at ξ if and only if there exists a point $x \in B$ such that

$$\Pr_x[B_t \text{ avoids } E] > 0.$$

Let us now combine the discrete group and the concept of minimal thinness, with help from the following set.

Definition 8.3 [13, Def. 5.1]. We define the set $\mathfrak{N}$ to be

$$\mathfrak{N} = \{x \in \partial B : \text{the archipelago is not minimally thin at } x\}.$$

Remark 8.4. Using the probabilistic interpretation of minimal thinness in Remark 8.2, we can view the set $\mathfrak{N}$ in the following heuristic way. The Brownian motion in B is conditioned using the Poisson kernel and so we see that, at each point $x \in B$, there is a drift perpendicular to the level sets of the Poisson kernel with pole in ξ. Since these level sets are horospheres, it follows that the drifts are directed along hyperbolic geodesics from x toward ξ. Lifting this conditioned process in the unit ball to the quotient-manifold results in a stochastic process that is conditioned to "eventually follow" the geodesic $g(t)$ on the manifold. This is a recurrent process if and only if $\xi \in \mathfrak{N}$.

By [13, Secs. 5, 6],

$$\Lambda_c \subset \mathfrak{N} \subset \mathcal{L}_s(\alpha)$$

when $\alpha < 1$. We also have $\mathcal{L}_s(1) \subseteq \mathcal{L}(1) = \Lambda_c$. Furthermore, $\mathfrak{N}$ and Λ_c have the same Hausdorff dimension and, in the case where Γ is geometrically finite, $\mathfrak{N} = \Lambda_c$ (see [13, Thm. 5.4, Cor. 6.1]).

The following question was raised in [13, Sec. 5, p. 310]: *Is in fact* $\mathfrak{N} = \Lambda_c$? We will answer this question negatively in Section 10.

9. A Generalized Version of Minimal Thinness

We now give a relationship between minimal thinness and the function $\varphi(t)$. The result holds for a generalization of minimal thinness given in the next definition.

Definition 9.1. The set E is β-thin at y if there is a measure μ such that

$$\liminf_{x \to y,\, x \in E} k_\beta * \mu(x) > k_\beta * \mu(y),$$

where $k_\beta(x)$ is the Riesz kernel $|x|^{\beta - n}$.

Note that we here used β instead of α as the parameter so as to avoid confusion. To find out more about this type of thinness, see for example [2, pp. 155–158]. We have that 0-thinness is the same as minimal thinness (cf. [2, Cors. 7.4.3(iv), 7.4.7(iv)]).

Let E be the archipelago of Γ. What can be said about the β-thinness of E if the sequence $\{\varphi_i\}$ is known? We have immediately that, if there is a bounded subsequence of $\{\varphi_i\}$, then $\xi \in \Lambda_c$ (and thus, by [13, Prop. 4.14], $\xi \in \mathfrak{N}$), which then would imply that E is not β-thin at ξ for any $\beta \geq 0$ (cf. again [2, Cor. 7.4.3(iv)]). The following result gives a more precise statement.

Proposition 9.2. *Let $\{\varphi_i\}$ and ξ be as before and let $\beta \in [0, 1)$. Then the archipelago of Γ is not β-thin at ξ if*

$$\sum_i e^{-(n-\beta)\varphi_i} = \infty.$$

Proof. Let E be the archipelago of Γ, and let $\{Q_k\}$ be a Whitney decomposition of the unit ball. Using the estimates in [13, Lemma 4.11; 2, Cor. 7.4.3, p. 155], we obtain that E is β-thin at ξ if and only if

$$\bigcup_{Q_k \cap E \neq \emptyset} Q_k \text{ is } \beta\text{-thin at } \xi.$$

By [2, Cor. 7.4.3(iv)] it then follows that E is β-thin at ξ if and only if

$$\sum_{Q_k \cap E \neq \emptyset} \left(\frac{\operatorname{diam}(Q_k)}{\operatorname{dist}(Q_k, \xi)} \right)^{n-\beta} < \infty.$$

Thus, by Lemma 4.11 in [13] and (7) and (8) in its proof, we have

$$\sum_{Q_k \cap E \neq \emptyset} \left(\frac{\operatorname{diam}(Q_k)}{\operatorname{dist}(Q_k, \xi)} \right)^{n-\beta} \geq C \sum_{\{\gamma_j \in \Gamma\}} \left(\frac{|1 - |\gamma_j(0)||}{|\xi - \gamma_j(0)|} \right)^{n-\beta} \geq C \sum_{\{\varphi_i\}} \left(\frac{h_i}{R_i} \right)^{n-\beta},$$

with the notation from Figure 5. Since

$$\frac{h_i}{R_i} = \cos(\theta_i) = \frac{1}{\cosh(\varphi_i)},$$

we conclude that E is not β-thin at ξ if

$$\sum_i e^{-(n-\beta)\varphi_i} = \infty. \qquad \square$$

COROLLARY 9.3. *If* $\sum_i e^{-n\varphi_i} = \infty$ *then* $\xi \in \mathfrak{N}$.

The corollary follows immediately from the preceding proposition because 0-thinness is minimal thinness. We will use Corollary 9.3 to give a concrete example in Section 10 of a Fuchsian group with a limit point $\xi \in \mathfrak{N} \setminus \Lambda_c$.

Rarefiedness

Beside minimal thinness, let us study another potential theoretic set measure called *rarefiedness.* To give a definition of rarefiedness, we recall that the *Riesz mass* of a positive superharmonic function u is a measure μ such that, by the Riesz representation theorem, $u(x) = G\mu + h$, where $G\mu$ is the Green potential of the measure μ and h is a harmonic function.

DEFINITION 9.4 (cf. [2, Def. 12.4, p. 74). A subset E of the unit ball B is *rarefied* at $\xi \in \partial B$ if there exists a positive function u in the upper half-space $\mathbb{H} = \{x = (x_1, \dots, x_n); 0 < x_n\}$ with no Riesz mass at infinity such that

$$u(x) \geq |x|, \quad x \in E',$$

where E' is the image of E under the Möbius mapping that maps B to $\mathbb{H}$ and ξ to ∞.

What can we say about rarefiedness of the archipelago at ξ if we know $\varphi(t)$? "Nothing in general" is the negative answer, as seen in the following example.

Consider a Fuchsian group Γ with a parabolic element for which ξ is a fixed point. The corresponding geodesic on the Riemann surface will be going out on a parabolic cusp with maximal rate (i.e., $\varphi(t) = t$), and we cannot tell just by looking at $\varphi(t)$ that we are heading toward a limit point at all. On the other hand, [13, Lemma 6.3] tells us that the archipelago of Γ is not rarefied at ξ.

10. A Counterexample

Again we use the jungle-gym construction of Section 5. Recall that given a starting point x_0 we can completely determine the geodesic, and thus the related limit point ξ on the unit sphere for the underlying discrete group Γ, by the number of turns $N(j)$ that the geodesic makes in each hole. Because we suppose that the holes are visited in strict order, going to the "right" for example as in Figure 6, it follows that φ_j is increasing. We will show that if $N(j)$, the number of turns in the jth hole, is chosen to be the upper integer part of $\exp(2j)/j$ then ξ will be in $\mathfrak{N}$ but not in Λ_c.

Note that in this setup $\varphi_j \approx j$ and thus $\varphi_j \to \infty$, hence $\xi \notin \Lambda_c$. From Corollary 9.3 it is sufficient to show that, with the choice of $N(j)$ as before, $\sum_{\{\varphi_i\}} e^{-2\varphi_i} = \infty$ and

$$\sum_{\{\varphi_i\}} e^{-2\varphi_i} \geq \sum_{\{\text{hole}_j\}} N(j)e^{-2\varphi_j} \approx \sum_{\{\text{hole}_j\}} N(j)e^{-2j}$$
$$\geq \sum_{\{\text{hole}_j\}} \frac{e^{2j}}{j} e^{-2j} = \sum_{\{\text{hole}_j\}} \frac{1}{j} = \infty.$$

Hence we conclude that $\Lambda_c \neq \mathfrak{N}$.

REMARK 10.1. Recalling the conditioned stochastic process described in the heuristic Remark 8.4, we can argue that, by making more and more turns, the resulting outward drift on the jungle gym itself will be relatively small—so small that the process will be recurrent.

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Mathematical Sciences
Chalmers University of Technology
SE–412 96 Göteborg
Sweden

torbjrn@math.chalmers.se