On Fixed Points and Determining Sets
for Holomorphic Automorphisms

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0. Introduction

It is a result of classical function theory (see [FiF; Les; Mas; PeL; S]) that if
$f : U \to U$ is a conformal self-mapping of a plane domain that fixes three distinct
points then $f(\zeta) \equiv \zeta$. The purpose of the present paper is to put this result into a
geometrically natural context and to extend it to higher-dimensional domains and
manifolds. For an examination of fixed point questions from a slightly different
point of view, we refer the reader to the work of Vigué (see e.g. [V1; V2]).

The third-named author thanks Robert Burckel for early discussions of this topic
and for basic references.

1. Spanning Cartan–Hadamard Subsets

In this section, we let M be a connected, complete Riemannian manifold.

1.1. Cut Points and Cut Loci

Let $x \in M$. A point $y \in M$ is called a *cut point* of x if there are two or more
length-minimizing geodesics from x to y in M. We also use the following basic
terminology and facts from Riemannian geometry. A geodesic $\gamma : [a, b] \to M$
is called a *length-minimizing geodesic* (or, alternatively, a *minimal geodesic* or a
minimal connector) from x to y if $\gamma(a) = x$, $\gamma(b) = y$, and $\mathrm{dis}(x, y) =$ arc
length of γ. Any two points in a complete Riemannian manifold can be connected
by a minimizing geodesic by the Hopf–Rinow theorem. If there is a smooth family
of minimizing geodesics from x to y, then these two points are said to be *conjugate*. Conjugate points are cut points. The collection of cut points of x in M is
called the *cut locus* of x, which we denote by C_x in this paper. It is known that
C_x is nowhere dense in M (see e.g. [GKM; K]).

1.2. Spanning Cartan–Hadamard Sets

A subset X of M is a *Cartan–Hadamard set* if there exists an $x_0 \in X$ such that $X \subset
M \setminus C_{x_0}$. We will call x_0 a *pole* of X. A pole of a set is in no way unique. But, for

convenience, we will commonly use the notation (X, x_0) for a Cartan–Hadamard subset paired with a pole.

We now fix notation. For $y \in M \setminus C_x$, we denote by $\gamma_{xy} : [0, \ell] \to M$ the unit-speed length-minimizing geodesic with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(\ell) = y$.

Now we say that a Cartan–Hadamard subset (X, x_0) is *spanning* if the pole x_0 has the property that the set $\{\gamma'_{x_0 y}(0) \mid y \in X \setminus \{x_0\}\}$ spans (in the sense of linear algebra) the tangent space $T_{x_0} M$.

1.3. Determining Sets for Isometries

We now can discuss the existence of finite subsets that may determine isometries. We begin with the following lemma.

LEMMA 1.1.   *If (X, x_0) is a spanning Cartan–Hadamard subset of M, and if $f : M \to M$ is an isometry with $f(x) = x$ for every $x \in X$, then f coincides with the identity map.*

Proof. Since isometries preserve geodesics and arc-lengths of curves, it follows that df_{x_0} must fix each $\gamma'_{x_0 y}(0)$ for every $y \in X \setminus \{x_0\}$. Owing to the spanning property of these vectors, df_{x_0} thus coincides with the identity map of $T_{x_0} M$. As a result, f must fix every point in a geodesic polar coordinate neighborhood of x_0 or, equivalently, every point of $M \setminus C_{x_0}$. Since C_{x_0} is nowhere dense, we see that $f = \mathrm{id}_M$. □

In the case that $\dim_{\mathbb{R}} M = d \geq 2$, we see that one has great freedom in choosing a spanning Cartan–Hadamard subset consisting of $d + 1$ points. This can be done in general as follows.

Let $x_0 \in M$ be chosen arbitrarily. Let W_0 be the largest connected open subset, containing x_0, of $M \setminus C_{x_0}$. Then we may find a connected open subset U of $T_{x_0} M$ that is star-shaped at the origin and such that the exponential map $\exp_{x_0} : U \to W_0$ is a diffeomorphism. We let

$$W_1 \equiv W_0 \setminus \{x_0\}$$

and choose x_1 to be an arbitrary point of W_1. Then let

$$W_2 = W_1 \setminus \left(\exp_{x_0}(\mathrm{Span}\{\gamma'_{x_0 x_1}(0)\}) \cap W_0 \right)$$

and let x_2 be an arbitrary point of W_2. Now W_{k+1} will be chosen inductively to be

$$W_{k+1} \equiv W_0 \setminus \left(\exp_{x_0}(\mathrm{Span}\{\gamma'_{x_0 x_1}(0), \dots, \gamma'_{x_0 x_k}(0)\}) \cap W_0 \right)$$

for $k = 2, \dots$. Then, of course, x_{k+1} is chosen from W_{k+1} without any further restrictions. Because each W_k constructed in this way is nowhere dense in M as long as $k \leq d$, we may always find $d + 1$ points in this fashion. Moreover, it is now clear that such a $(d + 1)$-point set is a spanning Cartan–Hadamard subset of M, and that spanning Cartan–Hadamard subsets are generic. We may summarize this discussion in the following proposition.

PROPOSITION 1.2. *Let M be a d-dimensional, connected, complete Riemannian manifold. There exists an open dense subset W of the product manifold $M \times \cdots \times M$ of $d + 1$ copies of M with the following property: If f is an isometry of M with $f(x_j) = x_j$ for every $j = 0, \ldots, d$ and if $(x_0, \ldots, x_d) \in W$, then $f = \mathrm{id}_M$.*

2. Biholomorphisms and Determining Subsets

If $\Omega \subset \mathbb{C}^n$ is a domain (connected open set) or M is a complex manifold, then $\mathrm{Aut}(\Omega)$ (resp. $\mathrm{Aut}(M)$) denotes the group, under composition, of biholomorphic self-maps of Ω (resp. M). We call such mappings *automorphisms* of Ω (resp. M).

At this point, we remark that the study of determining sets is meaningful. It is indeed known that most domains (or manifolds) are *rigid*; that is, they have automorphism group consisting of just the identity mapping. This assertion means that the collection of rigid, smoothly bounded, strongly pseudoconvex domains is dense in the collection of all smoothly bounded, strongly pseudoconvex domains in the C^∞ topology (see [GrKr]). A complementary fact, however, is that the collection of bounded domains with nontrivial automorphism group is dense in the collection of all domains in the topology induced by the Hausdorff distance (see [FrP, Thm. 2.1]). Moreover, every compact Lie group occurs as the automorphism group of a bounded strongly pseudoconvex domain (see [BD]).

DEFINITION. Let K be a subset of a complex manifold M. The set K is said to be a *determining subset* of M if each automorphism g of M satisfying the condition $g(x) = x$ for all $x \in K$ is the identity map of M.

As mentioned previously, a self-map of a domain in $\mathbb{C}$ that fixes three points is necessarily the identity (see e.g. [PeL]). Hence any 3-point set is a determining set for plane domains. Note that no "general position" hypothesis need be mandated on the points of the determining set. (However, a certain general position hypothesis is essential even in dimension 1 if one considers nonplanar Riemann surfaces; we will clarify this point in a later section.)

In an attempt to extend this result to higher dimensions, one can ask the following question. For $n \geq 2$, does there exist a positive integer k such that, if S is a set of k points in "general" position in $\mathbb{C}^n$ and if $D \subset \mathbb{C}^n$ is a domain containing S, then each automorphism of D fixing S is necessarily the identity? The answer to that question is negative: no such "general" position can be defined to obtain a positive answer, as shown by the following theorem.

THEOREM 2.1. *For each finite set $K = \{p_1, \ldots, p_k\} \subset \mathbb{C}^n$ ($n > 1$), there exist a bounded domain D containing K and a subgroup $H \subset \mathrm{Aut}(D)$ isomorphic to $U(n - 1)$ (the complex unitary group of $\mathbb{C}^{n-1}$) such that each element of H fixes each point of K.*

Proof. Let $p_j = (u_j, v_j)$ with $u_j \in \mathbb{C}$ and $v_j \in \mathbb{C}^{n-1}$. Without any loss of generality, we assume that the u_j are all distinct and that $|u_j| < 1$. Consider the polynomial transformation

$$F : w_1 = z_1, \ w' = z' + f(z_1),$$

where $f : \mathbb{C} \to \mathbb{C}^{n-1}$ is the Lagrange interpolation polynomial map satisfying $f(u_j) = v_j$. Then $F(u_j, 0) = p_j$ for $j = 1, \ldots, k$. Let $D = F(B)$, where B is the unit ball in $\mathbb{C}^n$. Let U_{n-1} be the unitary group acting on B in the last $n-1$ coordinates, and let $H = F \circ U_{n-1} \circ F^{-1}$. Now the assertions of the theorem can be verified directly.   □

Although no given finite set in "general position" can be a determining subset for all bounded domains containing the set, we will establish in the sequel that, for each given bounded pseudoconvex domain in $\mathbb{C}^n$, "almost any" subset of $n + 1$ points is a determining subset.

Consider the group of biholomorphic automorphisms, $\mathrm{Aut}(M)$, of a complex manifold M. For the next theorem, we assume that

(A) M is a connected, complete Hermitian manifold such that each automorphism in $\mathrm{Aut}(M)$ is an isometry.

We would like to point out that these restrictions are rather mild in the sense that we have a broad collection of examples. Every bounded pseudoconvex domain in $\mathbb{C}^m$ admits a complete Kähler–Einstein metric ([MY]; see also [O]). Then there is an ample collection of compact complex manifolds that admit complete Bergman or Kähler–Einstein metrics; see [GrW; Ko; Y] and further references therein.

The discussion in Section 1 naturally yields the following theorem.

THEOREM 2.2.   *For a complex manifold M satisfying* (A) *and of dimension $m = \dim_{\mathbb{C}} M \geq 1$, there exists an open dense subset W of the $(m + 1)$-fold product $M \times \cdots \times M$ such that any automorphism f fixing $p_0, \ldots, p_m$ coincides with the identity map of M whenever $(p_0, \ldots, p_m) \in W$.*

Proof. If one follows the proof of Proposition 1.2 line by line, using the invariant Hermitian metric, the only difference one encounters is in the number of points and their choices. We therefore replace the exponentiation of the real span of vectors by the exponentiation of the complex span of vectors given by the minimal geodesics emanating from the pole point. We now exploit the fact that automorphisms are isometries that preserve the complex holomorphic tangent subspaces. Then all the arguments simply go through.   □

Suppose that K is a determining subset of a bounded domain D in $\mathbb{C}^n$. We next prove a "stability" theorem: K is also a determining subset for a small perturbation $\tilde{D}$ of D.

THEOREM 2.3.   *If D is a bounded domain in $\mathbb{C}^n$ and if K is a nonempty determining subset of D, then each domain $\tilde{D}$, containing K and for which $\partial \tilde{D}$ is sufficiently close to ∂D in the Hausdorff metric, also has K as a determining subset.*

Proof. Seeking a contradiction, we assume that there exists a sequence $\{D_j\}$ of domains converging to D such that, for each j, D_j contains K, some $g_j \in \mathrm{Aut}(D_j)$

satisfies $g_j \neq$ id, and g_j fixes each point of K. Choose $z \in K$ and an $r > 0$ so that the closure $Q = \bar{B}(z, r)$ of the ball with center z and radius r is contained in D and in all D_j. Let

$$H_j = \{g \in \text{Aut}(D_j) : g \text{ fixes each point of } K\}.$$

By assumption, $H_j \neq \{\text{id}\}$. It is clear that H_j is a compact Lie subgroup of $\text{Aut}(D_j)$. By [Ma, Thm. 2.4], for each j there exists a point $x_j \in Q$ and an $h_j \in H_j$ such that $|h_j(x_j) - x_j| \geq r/2$. Passing to a subsequence if necessary, we can assume that $x_j \to x$ and $h_j(x_j) \to y$. Using a normal families argument (again passing to a subsequence if necessary) and the fact that $h_j(z) = z$, one can show that the sequence h_j converges in the compact-open topology to an $h \in \text{Aut}(D)$. It is clear that $h(x) = y \neq x$ and h fixes each point of K, contradicting the hypothesis that K is a determining subset of D. □

3. Automorphisms, Isometries, Fixed Points, and Cut Loci

We would now like to address the fact that if the fixed points of isometries actually lie in a cut locus then the number of fixed points can be arbitrarily large, making it impossible to relate them to the complex dimension of the manifold.

If the dimension is ≥ 2, this claim was exhibited in Theorem 2.1. The next two examples show the validity of our claim in dimension 1.

EXAMPLE 3.1.   Consider the complex, 1-dimensional torus T generated from the lattice $\{1, i\}$. Let $\pi : \mathbb{C} \to T$ be the standard covering map. Then $z \to -z$ on the complex plane generates an automorphism, say τ, on T. Now τ has *four* fixed points, which are

$$\pi(0), \ \pi(1/2 + i/2), \ \pi(1/2), \ \pi(i/2).$$

Yet τ does not fix $\pi\left(\frac{1}{4}\right)$, so it is not the identity map.

EXAMPLE 3.2.   We now consider a 2-holed torus. This manifold can be generated by a regular octagon centered at the origin of the Poincaré disc together with its reflections. Again, $z \to -z$ generates a nontrivial automorphism of this Riemann surface. The number of fixed points is now *six,* coming from the center (the origin), the vertices, and the corresponding pairs of midpoints of the sides of the octagon.

It is now clear that one can obtain arbitrarily large numbers of fixed points just from among the compact Riemann surfaces. By standard embedding and thickening processes, one can construct examples of this nature for bounded domains as well.

4. The Plane Domain Case

For the sake of completeness of this exposition, we now consider the following well-known theorem that follows from work of Maskit [Mas], Peschl and Lehtinen [PeL], Leschinger [Les], and others.

Theorem 4.1.   *Let Ω be a domain in $\mathbb{C}$. If an automorphism of Ω fixes three distinct points, then it is the identity.*

Here, we would like to give a slightly more geometric rephrasing of the proof of [PeL] in order to demonstrate our geometric methods. Planar domains are rather special among the Riemann surfaces. Indeed, the reason why one does not have to take the cut loci into consideration for planar domains is this topological fact: Every Jordan curve in the plane bounds a cell. Our arguments here concentrate more upon Ω itself and on its geometry, especially emphasizing the role of our topological fact.

First of all, the case of $\Omega = \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ or a topological annulus is simple. Thus, let us assume that Ω is a plane domain that has at least three boundary components. Then, by the uniformization theorem for instance, it admits a complete Hermitian (automatically Kähler) metric with negative constant curvature and for which every holomorphic mapping is an isometry.

Now let f be a holomorphic automorphism of Ω with three distinct fixed points—say a, b, and c. We are to show that f is the identity map.

If b is not a cut point of a, then there is one and only one length-minimizing geodesic joining a and b. In such a case, every point on this geodesic must be fixed by f. Then, by the uniqueness theorem for analytic functions, f is in fact the identity map.

Hence we may now assume that there are at least two length-minimizing geodesics joining any pair of fixed points. At this juncture, we might note that the negativity of the curvature eliminates the possibility of conjugate points, owing to the second variation formula of arc length.

We now suppose that $f \in \mathrm{Aut}(\Omega)$ is not an identity map but does have three distinct fixed points in Ω. To reach a contradiction, let us start with the fixed point a. If the set of fixed points accumulates at a, we are done. Hence we may replace the second fixed point b by the closest (with respect to the Hermitian metric) one to a apart from a itself. This choice may not be unique and so we simply choose one.

As mentioned before, we need only consider the case when b is a cut point (not conjugate) of a. Then there will be several unit-speed minimal connectors (all of which have the same length, of course), say $\gamma_1, \gamma_2, \dots$, joining a to b. First notice that no minimal connector can have a self-intersection. Then the automorphism f maps any one of the minimal connectors to another such, as the endpoints a and b are fixed. Note that $f \circ \gamma_1$ cannot intersect γ_1 except at the endpoints. For if they do intersect at a point other than the endpoints then they must intersect at the same time; otherwise one may find an even shorter connector between a and b than the minimal connector, which is a contradiction. Then the intersection point becomes a fixed point of f closer to a than b, which also is not allowed.

Now, γ_1 and $f \circ \gamma_1$ join to form a piecewise smooth Jordan curve in the plane; thus it bounds a cell, say E, in the plane $\mathbb{C}$. Consider the third fixed point c, which

is distinct from a and b. Notice that we may assume that c is not on any of the minimal connectors for a and b. Suppose that c is inside the cell E. Now join c to a by an arc ξ in $E \cap \Omega$ that does not intersect with either γ_1 or $f \circ \gamma_1$ or, in fact, with any minimal geodesics joining a and b. Notice that the conformality of f at the fixed point a shows that there is a sufficiently small open ball neighborhood U of a on which f must map $U \cap \xi$ to the outside of the cell E. This results in the conclusion that $f \circ \xi$ must cross γ_1 or $f \circ \gamma_1$. But this is impossible, since a point not on any minimal connector from a to b cannot be mapped to a point on a minimal connector from a to b.

If c is outside the cell E then the arguments are similar. Because there are only finitely many minimal connectors between a and b (since a and b are not conjugate to each other and since the quotient from the universal covering space is formed by a properly discontinuous group action), it follows that some iterate f^m of f will move ξ so that its image has points inside E. Then, $f^m \circ \xi$ again crosses one of these minimal geodesics joining a and b, which leads us to another contradiction.

5. Some Examples

We now present several elementary examples that should put our results into perspective.

EXAMPLE 5.1.   Let $A = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. This is an annulus in the plane. The map $\tau(z) = 1/z$ has two fixed points (i.e., 1 and -1), yet τ is not the identity mapping.

EXAMPLE 5.2.   Let $U = \mathbb{C}^2$. Consider a shear of the form $\tau(z, w) = (z, w + \phi(z))$, where ϕ is any entire function on the plane. Then τ is a biholomorphic map of $\mathbb{C}^2$. If ϕ has infinitely many distinct zeros then τ will have infinitely many fixed points, even though τ is not the identity.

By contrast, any biholomorphic (conformal) map of $\mathbb{C}$ that fixes two points must be the identity.

EXAMPLE 5.3.   It can be shown from first principles that a biholomorphic map of the unit ball in $\mathbb{C}^n$ that fixes $n + 1$ points in general position (in the usual sense of topology) must in fact be the identity. We leave the details to the interested reader.

EXAMPLE 5.4.   Consider the domain $U_m \equiv \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^{2m} < 1\}$, any integer $m \geq 2$. Then any automorphism of U_m that fixes two points in general position must be the identity. This result follows because the automorphism group of U_m is well known to consist only of rotations in each variable separately.

Contrast this example with the result from the previous example (for the unit ball in $\mathbb{C}^2$).

EXAMPLE 5.5.   Let U_m be one of the domains from Example 5.4. Let V be *any* rigid domain in $\mathbb{C}^n$ (here *rigid* means that the domain has no automorphisms except the identity). Then, for an adroitly chosen pair of points $z, w \in U_m$ and an arbitrary $x \in V$, any automorphism of $U_m \times V$ that fixes both (z, x) and (w, x) will be the identity. For instance, the points $z = ((1/2, 0), x)$ and $w = ((0, 1/2), x)$ will do.

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