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# Propagation of Regularity and Global Hypoellipticity

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## 1. Introduction

If $X = \{X_1, \dots, X_m\}$ is a collection of real $C^\infty$ vector fields on a $C^\infty$ manifold $\mathcal{M}$, then the formulation of necessary and sufficient conditions for the global (or local) hypoellipticity of their *sub-Laplacian* $\Delta_X \doteq -(X_1^2 + \dots + X_m^2)$ is an open problem. We recall that an operator $P$ is said to be globally hypoelliptic if, for any distribution $u$ in $\mathcal{M}$ such that $Pu$ is in $C^\infty(\mathcal{M})$, we have that $u$ is in $C^\infty(\mathcal{M})$. An operator $P$ is said to be locally hypoelliptic if the last condition holds in any open subset of the manifold. Global and local analytic hypoellipticity are defined similarly. Also, we recall that a point in $\mathcal{M}$ is said to be of finite type (or satisfies the bracket condition) if the Lie algebra generated by the vector fields $X_1, \dots, X_m$ spans the tangent space of $\mathcal{M}$ at the given point. Otherwise, it is said to be of infinite type. By the celebrated theorem of Hörmander [Hö] (see also Kohn [K], Oleinik and Radkevic [OR], and Rothschild and Stein [RS]), the finite-type condition is sufficient for the local hypoellipticity of $\Delta_X$ and hence for its global hypoellipticity. In the analytic category, Derridj [D] proved that the finite-type condition is also necessary for hypoellipticity. Baouendi and Goulaouic [BG] proved that the finite-type condition is not sufficient for the analytic hypoellipticity of $\Delta_X$. We shall not discuss here the problem of analytic hypoellipticity, for which we refer the reader to Bernadi, Bove, and Tartakoff [BBT], Christ [C2], Grigis and Sjöstrand [GS], Hanges and Himonas [HH2], Helffer [Hel], Metivier [M], Tartakoff [Ta], Treves [Tr], and the references therein.

Our first result here is about semi-local propagation of regularity for an operator that is the sum of a sub-Laplacian and lower-order terms: $P = \Delta_X + X_0 + ib(t)$.

Theorem 1. *On the torus $\mathbb{T}^{(n+1)+m}$ with variables $(t, x)$ let $P$ be the operator*

$$P = -\Delta_t - \sum_{j=1}^{n} X_j^2 + X_0 + ib(t), \tag{1.1}$$

*where $X_j = \partial_{t_j} + \sum_{k=1}^m a_{jk}(t)\partial_{x_k}$ for $j = 0, \dots, n$ and with $a_{jk}(t)$ and $b(t)$ real-valued functions in $C^\infty(\mathbb{T}^{n+1})$. If $u \in D'(\mathbb{T}^{n+1+m})$, $Pu \in C^\infty(\mathbb{T}^{n+1+m})$, and $u \in C^\infty(U \times \mathbb{T}^m)$ for some open set $U \subset \mathbb{T}^{n+1}$, then $u \in C^\infty(\mathbb{T}^{n+1+m})$.*

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In general, the operator $P$ in (1.1) is not globally hypoelliptic, since if all $a_{jk}(t)$ and $b(t)$ are identically equal to zero then any function $u = u(x)$ will be a solution to $Pu = 0$. However, Theorem 1 implies the following result.

COROLLARY 1. *Let $P$ be as in* (1.1). *If there exists a point $(t^0, x^0) \in \mathbb{T}^{n+1+m}$ of finite type for the vector fields $\partial_{t_0}, \partial t_1, \dots, \partial_{t_n}$ and $X_0, X_1, \dots, X_n$, then $P$ is globally hypoelliptic in $\mathbb{T}^{n+1+m}$.*

In fact, since the finite type is an open condition, there exists an open set $U \subset \mathbb{T}^{n+1}$ such that $t^0 \in U$ and all points of the set $U \times \mathbb{T}^m$ are of finite type. Thus, by Hörmander's theorem [Hö], the operator $P$ is hypoelliptic in $U \times \mathbb{T}^m$. Therefore, if $u \in D'(\mathbb{T}^{n+1+m})$ is such that $Pu \in C^\infty(\mathbb{T}^{n+1+m})$, then Theorem 1 implies that $u \in C^\infty(\mathbb{T}^{n+1+m})$ and hence $P$ is globally hypoelliptic in $\mathbb{T}^{n+1+m}$.

In Section 3 we state a necessary and sufficient condition for the global hypoellipticity of the operator (1.1) (when $n = 1$, $X_0 = 0$, and $b = 0$) using Diophantine approximations (see Theorem 5). Here we state a result concerning semi-local propagation of regularity for our second family of operators.

THEOREM 2. *On $\mathbb{T}^{n+1}$ with variables $(t_1, \dots, t_n, x)$, let $P$ be the operator defined by*

$$P = -(\partial_{t_1}^2 + \cdots + \partial_{t_{n-1}}^2) - (\partial_{t_n} + a(t_1, \dots, t_n)\partial_x)^2, \tag{1.2}$$

*where $a(t_1, \dots, t_n)$ is a real-valued function in $C^\infty(\mathbb{T}^n)$. If $u \in D'(\mathbb{T}^{n+1})$, $Pu \in C^\infty(\mathbb{T}^{n+1})$, and $u \in C^\infty(U \times \mathbb{T}^2)$ for some open set $U \subset \mathbb{T}^{n-1}$, then $u \in C^\infty(\mathbb{T}^{n+1})$.*

Operator (1.2) is globally hypoelliptic when the finite-type condition holds on a "2-dimensional torus" set. More precisely, we have the following result.

THEOREM 3. *If there exists a point $(t_1^0, \dots, t_{n-1}^0) \in \mathbb{T}^{n-1}$ such that all points in the set $\{(t_1^0, \dots, t_{n-1}^0)\} \times \mathbb{T}^2$ are of finite type for the vector fields $X_j = \partial_{t_j}$ ($j = 1, \dots, n-1$) and $X_n = \partial_{t_n} + a(t_1, \dots, t_n)\partial_x$, then the operator $P$ defined by* (1.2) *is globally hypoelliptic in $\mathbb{T}^{n+1}$.*

If $n = 2$ then the operator (1.2) takes the familiar form

$$\Delta_X = -\partial_{t_1}^2 - [\partial_{t_2} + a(t_1, t_2)\partial_x]^2. \tag{1.3}$$

The analytic hypoellipticity of this operator has been considered by several authors (see [C1; HH1; PR]). If $a$ is an analytic function, then $\Delta_X$ is globally analytic hypoelliptic if the bracket condition holds [CH]. If $a = a(t_1)$ and is analytic near the origin, then $\Delta_X$ is not locally analytic hypoelliptic if $a(0) = a'(0) = 0$ [C1]. If $a = a(t_1)$ and is in $C^\infty(\mathbb{T})$, then $\Delta_X$ is globally hypoelliptic if and only if the range of $a$ contains a non-Liouville number [H]. As a consequence of Theorem 3 it follows that, if there exists a point $t_1^0 \in \mathbb{T}$ such that all points in the set $\{t_1^0\} \times \mathbb{T}^2$ are of finite type, then the operator $\Delta_X$ is globally hypoelliptic in $\mathbb{T}^3$. Moreover, if every point in $\mathbb{T}^3$ is of infinite type, then it is globally hypoelliptic if and only if the average of the function $a$ is a non-Liouville number (see Theorem 4 in Section 3).

For more results on local and global hypoellipticity, we refer the reader to [A; BM; FO; GPY; GW; F; HP1; HP2; KS; T] and the references therein.

## 2. Proofs of Theorems 1–3

*Proof of Theorem 1.* Let $u \in D'(\mathbb{T}^{n+1+m})$ be such that

$$Pu = f, \quad f \in C^\infty(\mathbb{T}^{n+1+m}), \tag{2.1}$$

and let $u \in C^\infty(U \times \mathbb{T}^m)$ for some open set $U \subset \mathbb{T}^{n+1}$.

If, in (2.1), we take the partial Fourier transform with respect to $x \in \mathbb{T}^m$, then

$$\left[-\Delta_t - \sum_{j=1}^{n} Y_j^2 + Y_0 + ib(t)\right]\hat{u}(t,\xi) = \hat{f}(t,\xi) \quad \text{for all } \xi \in \mathbb{Z}^m, \tag{2.2}$$

where

$$Y_j = \partial_{t_j} + i\sum_{k=1}^{m} a_{jk}(t)\xi_k, \quad j = 0, \dots, n. \tag{2.3}$$

For any fixed $\xi \in \mathbb{Z}^m$, we have that $\hat{u}(t,\xi)$ is in $C^\infty(\mathbb{T}^{n+1})$ because (2.2) is elliptic in $t$. Therefore, if we multiply (2.2) with $\bar{\hat{u}}$, integrate by parts with respect to $t \in \mathbb{T}^{n+1}$, and use (2.3), then

$$\begin{aligned} &\sum_{j=0}^{n} \|\hat{u}_{t_j}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} + \sum_{j=1}^{n} \|Y_j\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} \\ &\qquad + i\left[\operatorname{Im}\int_{\mathbb{T}^{n+1}} (\partial_{t_0}\hat{u}(t,\xi))\bar{\hat{u}}\,dt + \int_{\mathbb{T}^{n+1}} \sum_{k=1}^{m} a_{0k}(t)\xi_k|\hat{u}(t,\xi)|^2\,dt \right.\\ &\qquad\qquad \left. + \int_{\mathbb{T}^{n+1}} b(t)|\hat{u}(t,\xi)|^2\,dt\right] \\ &\quad = \int_{\mathbb{T}^{n+1}} \hat{f}(t,\xi)\bar{\hat{u}}(t,\xi)\,dt. \end{aligned}$$

Taking the real part in the last relation, we obtain

$$\begin{aligned} &\sum_{j=0}^{n} \|\hat{u}_{t_j}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} + \sum_{j=1}^{n} \|Y_j\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} \\ &\qquad\qquad = \operatorname{Re}\int_{\mathbb{T}^{n+1}} \hat{f}(t,\xi)\bar{\hat{u}}(t,\xi)\,dt. \end{aligned} \tag{2.4}$$

Using the Cauchy–Schwarz inequality, relation (2.4) gives

$$\sum_{j=0}^{n} \|\hat{u}_{t_j}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} \le \|\hat{f}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})}\|\hat{u}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})}. \tag{2.5}$$

Furthermore, using the fundamental theorem of calculus yields

$$\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} \le C\bigg(\int_V |\hat{u}(s,\xi)|^2\,ds + \sum_{j=0}^{n} \|\hat{u}_{t_j}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})}\bigg), \tag{2.6}$$

where $V \subset \bar{V} \subset U$ and $\bar{V}$ is a compact set.

From now on we shall use the letter $C$ to represent a constant, which may change a finite number of times. Since $u \in C^\infty(U \times \mathbb{T}^m)$ for a given $N \in \mathbb{N}$, there exists a $C_N > 0$ such that

$$|\hat{u}(s,\xi)| \le C_N|\xi|^{-2N} \quad \forall s \in V \text{ and } \forall \xi \in \mathbb{Z}^m - \{0\}. \tag{2.7}$$

By (2.5)–(2.7) it then follows that, for a given $N \in \mathbb{N}$, there are $C_N > 0$ and $C > 0$ such that

$$\begin{aligned}
\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} &\le C\bigg(\int_V |\hat{u}(s,\xi)|^2\,ds + \sum_{j=0}^{n} \|\hat{u}_{t_j}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})}\bigg) \\
&\le C\int_V |\hat{u}(s,\xi)|^2\,ds + C\|\hat{f}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})}\|\hat{u}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})} \\
&\le C_N\int_V |\xi|^{-2N}\,ds + C\|\hat{f}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})}\|\hat{u}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})} \\
&\le C_N|\xi|^{-2N} + C\|\hat{f}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})}\|\hat{u}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})} \\
&\le C_N|\xi|^{-2N} + C\bigg[\frac{1}{2\varepsilon^2}\|\hat{f}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} + \frac{\varepsilon^2}{2}\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})}\bigg].
\end{aligned}$$

If we choose $\varepsilon > 0$ such that $1 - c\varepsilon^2/2 > 1/2$, then

$$\frac{1}{2}\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})} \le C_N|\xi|^{-2N} + \frac{C}{2\varepsilon^2}\|\hat{f}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n+1})},$$

which gives

$$\|\hat{u}(\cdot,\xi)\|_{L^2(\mathbb{T}^{n+1})} \le C_N|\xi|^{-N} \quad \forall \xi \in \mathbb{Z}^m - \{0\}, \tag{2.8}$$

since $f \in C^\infty(\mathbb{T}^{n+1+m})$. Finally, using (2.8) and a standard microlocal analysis argument (see [H]), we prove that $u \in C^\infty(\mathbb{T}^{n+1+m})$. □

*Proof of Theorem 2.* The proof of Theorem 2 is similar to that of Theorem 1, if one replaces inequality (2.6) with

$$\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n})} \le C\bigg(\int_{-\pi}^{\pi}\int_I |\hat{u}(s,t_n,\xi)|^2\,ds\,dt_n + \sum_{j=1}^{n-1} \|\hat{u}_{t_j}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^{n})}\bigg), \tag{2.9}$$

where $I \subset [-\pi,\pi]^{n-1}$ and $C$ is a constant independent of $\xi$. To verify inequality (2.9), let $\phi(t) = \hat{u}(\cdot,\xi)$, $s \in I$, and $t \in [-\pi,\pi]^n$. Then, by the fundamental theorem of calculus, we have

$$\phi(t) = \phi(s,t_n) + \sum_{j=1}^{n-1}\int_{s_j}^{t_j} \phi_{y_j}(s_1,\dots,s_{j-1},y_j,t_{j+1},\dots,t_n)\,dy_j.$$

Using the Cauchy–Schwarz inequality gives

$$|\phi(t)|^2 \le C\left(|\phi(s,t_n)|^2 + \sum_{j=1}^{n-1}\int_{-\pi}^{\pi}|\phi_{y_j}(s_1,\dots,s_{j-1},y_j,t_{j+1},\dots,t_n)|^2\,dy_j\right).$$

Finally, integrating this inequality for $s \in I$ and $t \in [-\pi,\pi]^n$ yields (2.9). □

*Proof of Theorem 3.* For simplicity we may assume that $(t_1^0,\dots,t_{n-1}^0)$ is the origin in $\mathbb{T}^{n-1}$. We will show that there exist $\delta$ $(0 < \delta \le \pi)$, functions $c_\ell(t) \in C^\infty([-\delta,\delta]^{n-1}\times\mathbb{T})$ for $\ell = 1,\dots,M$, and $J_1,\dots,J_M \in \mathcal{J}$ with $|J| \ge 2$ such that

$$\partial_x = \sum_{\ell=1}^{M} c_\ell(t) X_{J_\ell} \ \text{ on } [-\delta,\delta]^{n-1}\times\mathbb{T}, \tag{2.10}$$

where for $J = (j_1,\dots,j_p) \in \mathcal{J} = \bigcup_{\gamma=1}^{\infty}\{1,\dots,n\}^{\gamma}$ we define

$$X_J = [X_{j_1},[X_{j_2},[X_{j_3},\dots,X_{j_p}]]].$$

Also, we define $|J| = p$. By the finite-type assumption, if $(0,t_n,x) \in \mathbb{T}^{n+1}$ then there are $J_1,\dots,J_{n+1} \in \mathcal{J}$ such that $X_{J_1},\dots,X_{J_{n+1}}$ span the tangent space of $\mathbb{T}^{n+1}$ at $(0,t_n,x)$. Since either $X_J = 0$ or $X_J = C_J(t)\partial_x$ for all $J \in \mathcal{J}$, where $C_J(t) = \partial_t^\alpha a(t)$ for some $\alpha \in \mathbb{N}^n$, it follows that the list $X_{J_1},\dots,X_{J_{n+1}}$ just displayed necessarily must contain the vector fields $X_1,\dots,X_n$. Now, using the assumption that all points in the set $\{0\}\times\mathbb{T}^2$ are of finite type, for each point $t_n \in \mathbb{T}$ there exist an open set $V_{t_n}$ containing 0 and an open interval $U_{t_n}$ containing $t_n$ such that, for some $|J| \ge 2$,

$$\partial_x = C_J^{-1}(t)X_J, \quad C_J^{-1}(t) \in C^\infty(V_{t_n}\times U_{t_n}).$$

Since the family of the intervals $\{U_{t_n}\}_{t_n\in\mathbb{T}}$ cover $\mathbb{T}$, by the compactness of $\mathbb{T}$ there exist finitely many intervals $U_1,\dots,U_M$ covering $\mathbb{T}$. If we define $V$ to be the intersection of the corresponding sets $V_1,\dots,V_M$, then

$$\partial_x = C_\ell^{-1}(t)X_{J_\ell}, \quad C_\ell^{-1}(t) \in C^\infty(V\times U_\ell),\ |J_\ell| \ge 2,\ \ell = 1,\dots,M.$$

If we choose $\delta > 0$ such that $[-\delta,\delta]^{n-1} \subset V$, then the open sets $V \times U_\ell$ cover the compact set $[-\delta,\delta]^{n-1}\times\mathbb{T}$. Now, taking a partition of unity $\{\psi_\ell\}$ subordinate to this covering and letting $c_\ell(t) = \psi_\ell(t)C_\ell^{-1}(t)$, we obtain the desired relation (2.10).

Applying Hörmander's theorem [Hö], we find that the operator $P$ is hypoelliptic in $U\times\mathbb{T}^2$, where $U \subset [-\delta,\delta]^{n-1}$ is an open set. Therefore, if $u \in D'(\mathbb{T}^{n+1})$ is such that $Pu \in C^\infty(\mathbb{T}^{n+1})$, then $u \in C^\infty(U\times\mathbb{T}^2)$. Using Theorem 2, we conclude that $u \in C^\infty(\mathbb{T}^{n+1})$ and hence $P$ is globally hypoelliptic in $\mathbb{T}^{n+1}$. □

## 3. Global Hypoellipticity and Diophantine Approximations

Finding necessary and sufficient conditions for the global hypoellipticity of a sub-Laplacian is a difficult open problem. One of the main obstacles is the appearance of Diophantine phenomena (see e.g. [FO; GPY; GW; H; HP1; HP2]). Such is the case in our next result for the operator (1.2), when the finite-type condition fails everywhere.

THEOREM 4. *Let $X_1, \dots, X_n$ be as in Theorem 3, and let $P$ be as in* (1.2). *If every point in $\mathbb{T}^{n+1}$ is of infinite type for the vector fields $X_1, \dots, X_n$, then the operator $P$ is globally hypoelliptic in $\mathbb{T}^{n+1}$ if and only if the average of the function $a$ is a non-Liouville number.*

*Proof*. Suppose that every point in $\mathbb{T}^{n+1}$ is of infinite type for the vector fields $X_1, \dots, X_n$. Then we must have $\partial_{t_j} a(t) = 0$ for all $t \in \mathbb{T}^n$ and for all $j = 1, \dots, n-1$. This means that $a(t) = a(t_n)$. Thus, the average of the function $a$ is given by

$$a_0 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} a(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(t_n)\, dt_n.$$

If we now change the variables $t_1, \dots, t_n$ and $x$ to the new variables $s_1, \dots, s_n$ and $y$, where $s_j = t_j$ $(j = 1, \dots, n)$ and

$$y = x - \int_{-\pi}^{t_n} a(r)\, dr + a_0(t_n + \pi),$$

then the operator $P$ becomes

$$Q = -(\partial_{s_1}^2 + \cdots + \partial_{s_{n-1}}^2) - (\partial_{s_n} + a_0 \partial_y)^2.$$

Thus, $P$ is globally hypoelliptic in $\mathbb{T}^{n+1}$ if and only if $Q$ is globally hypoelliptic in $\mathbb{T}^{n+1}$. It follows from [H, Thm. 1.2] that $Q$ is globally hypoelliptic in $\mathbb{T}^{n+1}$ if and only if $a_0$ is a non-Liouville number. This completes the proof of the theorem. □

Although Theorems 3 and 4 provide significant information about the global hypoellipticity of the operator (1.2), we still do not understand the full picture. On the other hand, for the operator (1.1) with $n = 1$, $X_0 = 0$, and $b = 0$, we have the following complete result using Diophantine approximations.

THEOREM 5. *Let $P$ be the differential operator defined by*

$$P = -\partial_t^2 - \left(\partial_t + \sum_{j=1}^{m} a_j(t) \partial_{x_j}\right)^2, \tag{3.1}$$

*where $(t, x) \in \mathbb{T}^{1+m}$ and $a_j$ $(j = 1, \dots, m)$ are real-valued functions in $C^\infty(\mathbb{T})$. Then $P$ is globally hypoelliptic in $\mathbb{T}^{1+m}$ if and only if, after a possible renaming of the variables $x_1, \dots, x_m$ and the corresponding coefficients $a_1, \dots, a_m$, the following Diophantine condition* $(\mathrm{DC})_j$ *is satisfied for some $j \in \{0, 1, \dots, m-1\}$:*

$(\mathrm{DC})_j$ *$a_1, \dots, a_{m-j}$ are $\mathbb{R}$-independent and*
$(a_{m-j+1}, \dots, a_n) \in (SA)^c(a_1, \dots, a_{m-j})$.

We recall the following definitions from [HP2]. A collection of vectors $v_1, \dots, v_\ell$ in $\mathbb{R}^d$ is said to be not simultaneously approximable if there exist a $C > 0$ and a $K > 0$ such that, for any $\eta = (\eta_1, \dots, \eta_\ell) \in \mathbb{Z}^\ell$ and $\xi \in \mathbb{Z}^d - \{0\}$, we have

$$|\eta_j - v_j \cdot \xi| \geq \frac{C}{|\xi|^K} \quad \text{for some } \; j = 1, \ldots, \ell.$$

When $\ell = 1$, this is the definition of a non-Liouville vector (see [Her] and [HP1]). When $d = 1$, this is the definition of a collection of real numbers $v_1, \ldots, v_\ell$ that are not simultaneously approximable (see [HP1]). If $\ell = 1$ and $d = 1$, then this is the well-known definition of a non-Liouville number.

A vector $(f_1(t), \ldots, f_d(t))$ of real-valued functions that are linearly independent over $\mathbb{R}$ is said to belong to $(SA)^c(b_1, \ldots, b_\ell)$ if the following conditions hold:

(1) $\{f_1, \ldots, f_d\}$ is contained in the linear span of $\{b_1, \ldots, b_\ell\}$; and
(2) the $\ell$ column vectors of the matrix $(\lambda_{jk})$ in the expression

$$(f_1, \ldots, f_d)^t = (\lambda_{jk})(b_1, \ldots, b_\ell)^t$$

are not simultaneously approximable vectors in $\mathbb{R}^d$.

REMARK. In [HP2] it was shown that condition $(\mathrm{DC})_j$ is necessary and sufficient for the global hypoellipticity of the operator

$$Q = -\partial_t^2 - \left( \sum_{j=1}^{m} a_j(t) \partial_{x_j} \right)^2. \tag{3.2}$$

Therefore, with respect to global hypoellipticity, the operators (3.1) and (3.2) are equivalent.

*Proof of Theorem 5.*

*Necessity.* Let $j_0$ be the number of functions among $a_1(t), \ldots, a_m(t)$ that are linearly independent over $\mathbb{R}$. Thus $0 \leq j_0 \leq m$. If condition $(\mathrm{DC})_j$ does not hold then it implies that, after a possible renaming of the variables $x_1, \ldots, x_m$ and the corresponding coefficients $a_1, \ldots, a_m$, either $a_1 \equiv 0, \ldots, a_m \equiv 0$ or the following condition holds:

$(\widetilde{\mathrm{DC}})_{j_0}$ $1 \leq j_0 \leq n - 1$ and $\{a_{j_0+1}, \ldots, a_m\} \in (SA)(a_1, \ldots, a_{j_0})$.

The condition $(\widetilde{\mathrm{DC}})_{j_0}$ means that $a_1, \ldots, a_{j_0}$ are linearly independent over $\mathbb{R}$, $\{a_{j_0+1}, \ldots, a_m\}$ is contained in the linear span of $\{a_1, \ldots, a_{j_0}\}$, and the $j_0$ column vectors of the matrix $(\lambda_{lk})$ in the expression

$$(a_{j_0+1}, \ldots, a_m)^t = (\lambda_{lk})(a_1, \ldots, a_{j_0})^t$$

are simultaneously approximable vectors in $\mathbb{R}^{m-j_0}$.

*Case 1.* Assume that $a_1 \equiv \cdots \equiv a_m \equiv 0$. Then, for any function $u \in C^0(\mathbb{T}_x) - C^\infty(\mathbb{T}_x)$, we have $Pu = 0$. Therefore, $P$ is not globally hypoelliptic in $\mathbb{T}^{1+m}$.

*Case 2.* Assume that condition $(\widetilde{\mathrm{DC}})_{j_0}$ holds. Then

$$a_p = \sum_{k=1}^{j_0} \lambda_k^p a_k, \quad p = j_0 + 1, \ldots, m,$$

where the vectors $(\lambda_k^{j_0+1}, \dots, \lambda_k^m)$, $k = 1, \dots, j_0$, are simultaneously approximable. Thus the operator $P$ takes the form

$$P = -\partial_t^2 - \left(\partial_t + \sum_{k=1}^{j_0} a_k(t)\left(\partial_{x_k} + \sum_{p=j_0+1}^{m} \lambda_k^p \partial_{x_p}\right)\right)^2. \tag{3.3}$$

Since the $j_0$ vectors $(\lambda_k^{j_0+1}, \dots, \lambda_k^m)$, $k = 1, \dots, j_0$, are simultaneously approximable, there exist sequences $\{\xi_\ell\} = \{(\xi_{j_0+1,\ell}, \dots, \xi_{m,\ell})\}$ for $\xi_\ell \in \mathbb{Z}^{m-j_0} - \{0\}$ and $\{\eta_\ell\} = \{(\eta_{1,\ell}, \dots, \eta_{j_0,\ell})\}$ for $\eta_\ell \in \mathbb{Z}^{j_0}$ such that

$$\left|\eta_{k,\ell} - \sum_{p=j_0+1}^{m} \lambda_k^p \xi_{p,\ell}\right| < |\xi_\ell|^{-\ell}, \quad \ell = 1, 2, \dots, \tag{3.4}$$

for any $k = 1, \dots, j_0$.

We now define $u \in D'(\mathbb{T}^{1+n}) - C^\infty(\mathbb{T}^{1+n})$ by

$$u(t, x) = \sum_{\ell=1}^{\infty} e^{i(\eta_\ell \cdot x' - \xi_\ell \cdot x'')},$$

where $x' = (x_1, \dots, x_{j_0})$ and $x'' = (x_{j_0+1}, \dots, x_m)$. Then

$$\begin{aligned} Pu = &\sum_{\ell=1}^{\infty} \left\{ \sum_{k=1}^{j_0} \partial_t a_k(t)\left(\eta_{k,\ell} - \sum_{p=j_0+1}^{m} \lambda_k^p \xi_{p,\ell}\right)\right\} e^{i(\eta_\ell \cdot x' - \xi_\ell \cdot x'')} \\ &+ \sum_{\ell=1}^{\infty} \left\{ \left[ \sum_{k=1}^{j_0} a_k(t)\left(\eta_{k,\ell} - \sum_{p=j_0+1}^{m} \lambda_k^p \xi_{p,\ell}\right)\right]^2 \right\} e^{i(\eta_\ell \cdot x' - \xi_\ell \cdot x'')}. \end{aligned}$$

It follows from this and (3.4) that $Pu \in C^\infty(\mathbb{T}^{1+n})$. Hence $P$ is not globally hypoelliptic in $\mathbb{T}^{1+n}$. This completes the proof of the necessity.

*Sufficiency*. We will prove that, if condition $(\mathrm{DC})_j$ holds for some $j \in \{0, 1, \dots, m-1\}$, then $P$ is globally hypoelliptic. For this, let $u \in D'(\mathbb{T}^{1+n})$ be such that

$$Pu = f, \quad f \in C^\infty(\mathbb{T}^{1+n}). \tag{3.5}$$

If, in (3.5), we take the partial Fourier transform with respect to $x \in \mathbb{T}^m$, then

$$\left[-\partial_t^2 - \left(\partial_t + i\sum_{j=1}^{m} a_j(t)\xi_j\right)^2\right]\hat{u}(t, \xi) = \hat{f}(t, \xi) \quad \text{for all } \xi \in \mathbb{Z}^m. \tag{3.6}$$

For any fixed $\xi$, we have that $\hat{u}(t, \xi)$ is in $C^\infty(\mathbb{T})$ because (3.6) is elliptic in $t$. Therefore, if we multiply (3.6) with $\hat{u}$ and integrate by parts with respect to $t \in \mathbb{T}$, then

$$\|\hat{u}_t(\cdot, \xi)\|_{L^2(\mathbb{T})}^2 + \int_{\mathbb{T}} |\partial_t \hat{u}(t, \xi) + ib(t, \xi)\hat{u}(t, \xi)|^2 \, dt = \int_{\mathbb{T}} \hat{f}(t, \xi)\bar{\hat{u}}(t, \xi) \, dt, \tag{3.7}$$

where

$$b(t, \xi) = \sum_{j=1}^{m} a_j(t)\xi_j. \tag{3.8}$$

First we have the following inequality:

$$\|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}} b^2(t,\xi)|\hat{u}(t,\xi)|^2\,dt \le 3\|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + 3\int_{\mathbb{T}} |\partial_t\hat{u}(t,\xi) + ib(t,\xi)\hat{u}(t,\xi)|^2\,dt. \quad (3.9)$$

In fact,

$$\begin{aligned}
&\|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}} b^2(t,\xi)|\hat{u}(t,\xi)|^2\,dt \\
&\quad = \|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}} |ib(t,\xi)\hat{u}(t,\xi)|^2\,dt \\
&\quad = \|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}} |\partial_t\hat{u}(t,\xi) + ib(t,\xi)\hat{u}(t,\xi) - \partial_t\hat{u}(t,\xi)|^2\,dt \\
&\quad \le \|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + 2\int_{\mathbb{T}} |\partial_t\hat{u}(t,\xi) + ib(t,\xi)\hat{u}(t,\xi)|^2\,dt + 2\int_{\mathbb{T}} |\partial_t\hat{u}(t,\xi)|^2\,dt.
\end{aligned}$$

Now, since condition $(\mathrm{DC})_j$ holds for some $j \in \{0, 1, \ldots, m-1\}$, it follows from [HP2, (2.13)] with $\varphi(t) = \hat{u}(t,\xi)$ that

$$\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T})} \le C|\xi|^K \left( \|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}} b^2(t,\xi)|\hat{u}(t,\xi)|^2\,dt \right). \quad (3.10)$$

Using (3.7), (3.9), and (3.10), we have

$$\begin{aligned}
\|\hat{u}(\cdot,\xi)\|^2_{L^2(\mathbb{T})} &\le C|\xi|^K \left( 3\|\hat{u}_t(\cdot,\xi)\|^2_{L^2(\mathbb{T})} + 3\int_{\mathbb{T}} |\partial_t\hat{u}(t,\xi) + ib(t,\xi)\hat{u}(t,\xi)|^2\,dt \right) \\
&= C|\xi|^K \int_{\mathbb{T}} \hat{f}(t,\xi)\overline{\hat{u}}(t,\xi)\,dt. \quad (3.11)
\end{aligned}$$

This and the Cauchy–Schwarz inequality imply that

$$\|\hat{u}(\cdot,\xi)\|_{L^2(\mathbb{T})} \le C|\xi|^K \|\hat{f}(\cdot,\xi)\|_{L^2(\mathbb{T})}. \quad (3.12)$$

Finally, using a standard microlocal analysis (see [H]), one can prove that $P$ is globally hypoelliptic. □

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