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Topological Classification of $\mathbf{Z}_p^m$ Actions on Surfaces

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1. Introduction

Let G be a group isomorphic to $\mathbf{Z}_p^m$, where p is a prime integer. Abelian group actions on surfaces constitutes a classical subject (see [E; J1; J2; Na1; N; S; Z]). In [E; J1; J2; Z], a connection is established between the topological equivalence classes of actions and the second homology of the group that is acting. But some attempts to use these results for the classification of abelian actions give wrong results in some cases (cf. [E, Rem. 4.5] with Corollary 12 in our Section 4). The full classification has been found in the cyclic case by Nielsen in [N] and for $\mathbf{Z}_2^m$ in [Na1]. In this paper we present a direct and complete way to deal with the topological classification of $\mathbf{Z}_p^m$ actions, where p is a prime integer ($\mathbf{Z}_p^m = \mathbf{Z}_p \oplus \overset{m}{\cdots} \oplus \mathbf{Z}_p$ and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$). The main idea of our work is the fact that a fixed point–free action of $\mathbf{Z}_p^m$ provides an alternating bilinear form on $\mathbf{Z}_p^m$.

We give the full description of strong equivalence classes, in particular. In the case of fixed point–free actions, every action of G on a surface define an alternating bilinear form $(\cdot,\cdot)\colon G^* \times G^* \to \mathbf{Z}_p$, where G^* is the group of forms of G on $\mathbf{Z}_p$ (see Definition 7). Two actions of G are strongly equivalent if and only if the actions define the same bilinear form (Theorem 8). All possible such actions are described in Theorem 9. The case of actions having elements with fixed points is considered in Theorems 13 and 14.

Since G is a finite group, it is possible—given an action $(\tilde{S}, f)$ of G—to construct an analytic structure on $\tilde{S}$ such that $f(G)$ is a group of automorphisms of $\tilde{S}$. Hence all the actions considered in this paper appear as automorphism group actions of complex algebraic curves.

A motivation for our study is the description of the set of connected components in the moduli space $M^{p,m}$ of pairs (C, G), where C is a complex algebraic curve and $G \cong \mathbf{Z}_p^m$ is a group of automorphisms of C. According to [Na2], the description of connected components of $M^{p,m}$ is reduced to the description of topological classes of pairs $(\tilde{S}, K)$, where K is a group of autohomeomorphisms of $\tilde{S}$ and where K is isomorphic to $\mathbf{Z}_p^m$. We consider $(\tilde{S}, K)$ and $(\tilde{S}', K')$ to be equivalent if there exists a homeomorphism $\varphi\colon \tilde{S} \to \tilde{S}'$ such that $K' = \varphi \circ K \circ \varphi^{-1}$. These equivalence classes are in one-to-one correspondence with classes of weak

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equivalence (in the terminology of Edmonds [E]). In Theorem 16 we describe the weak equivalence classes of actions of G on surfaces.

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2. Algebraic Preliminaries

Let us consider the standard lattice $\mathbf{Z}^{2g} = \mathbf{Z} \oplus \overset{2g}{\cdots} \oplus \mathbf{Z}$ with the standard basis $(e_i) = ((0, \dots, 1^{(i)}, \dots, 0))$. We define the alternating bilinear form $(\cdot,\cdot)\colon \mathbf{Z}^{2g} \times \mathbf{Z}^{2g} \to \mathbf{Z}$ by $(e_i, e_j) = \delta_{i+j,2g+1}$ for $i < j$.

We consider also the group $\mathbf{Z}_p^{2g} = \mathbf{Z}_p \oplus \overset{2g}{\cdots} \oplus \mathbf{Z}_p$, where p is a prime and $\mathbf{Z}_p = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{p-1}\}$. Let $\varphi\colon \mathbf{Z}^{2g} \to \mathbf{Z}_p^{2g}$ be the natural projection defined by $\varphi(e_i) = \bar{e_i}$, where $\bar{e_i} = (0, \dots, \bar{1}^{(i)}, \dots, 0)$. Then we have an alternating bilinear form $(\cdot,\cdot)_p\colon \mathbf{Z}_p^{2g} \times \mathbf{Z}_p^{2g} \to \mathbf{Z}_p$ defined by $(\bar{e_i}, \bar{e_j}) = (\varphi(e_i), \varphi(e_j))_p = (e_i, e_j) \bmod p$.

Let $\mathrm{Sp}(2g, \mathbf{Z})$ and $\mathrm{Sp}_p(2g, \mathbf{Z}_p)$ be the subgroups of the automorphism groups of $\mathbf{Z}^{2g}$ and $\mathbf{Z}_p^{2g}$ that preserve the bilinear forms $(\cdot,\cdot)$ and $(\cdot,\cdot)_p$, respectively. The natural projection $\varphi\colon \mathbf{Z}^{2g} \to \mathbf{Z}_p^{2g}$ induces a homomorphism $\varphi_*\colon \mathrm{Sp}(2g, \mathbf{Z}) \to \mathrm{Sp}_p(2g, \mathbf{Z}_p)$ such that $\varphi_*(f) \circ \varphi = \varphi \circ f$ for all $f \in \mathrm{Sp}(2g, \mathbf{Z})$.

The following result is known.

THEOREM 1 [Ne, Thms. VII.20, VII.21]. *The following equality holds*:

$$\varphi_*(\mathrm{Sp}(2g, \mathbf{Z})) = \mathrm{Sp}_p(2g, \mathbf{Z}_p).$$

Proof (Sketch). We say that an element of $\mathbf{Z}^{2g}$ is *primitive* if $e \neq nf$ for all $n \in \mathbf{Z} - \{\pm 1\}$ and $f \in \mathbf{Z}^{2g}$. For every $a_p \in \mathbf{Z}_p^{2g}$ there is a primitive $a \in \mathbf{Z}^{2g}$ such that $\varphi(a) = a_p$.

The proof makes use of the following two claims.

(1) Assume that $\langle \tilde{a}_1, \tilde{a}_2 \rangle$ is a subgroup of $\mathbf{Z}_p^{2g}$ such that $\langle \tilde{a}_1, \tilde{a}_2 \rangle \cong \mathbf{Z}_p^2$ and the bilinear form restricted to $\langle \tilde{a}_1, \tilde{a}_2 \rangle$ is not trivial. Then there exist primitive elements $a_1, a_2 \in \mathbf{Z}^{2g}$ such that $(a_1, a_2) = 1$ and $\varphi(a_1), \varphi(a_2) \in \langle \tilde{a}_1, \tilde{a}_2 \rangle$.

(2) Let $a, b \in \mathbf{Z}^2$, $\varphi(a) \neq 0$, $\varphi(b) \neq 0$, and $(a, b) = mp$ with $m \in \mathbf{Z}$. Then $\langle \varphi(a) \rangle = \langle \varphi(b) \rangle$.

Using induction together with claims (1) and (2), it is easy to prove the following. Let G be a subgroup of $\mathbf{Z}_p^{2g}$. Then, for $i = 1, \dots, r$ and $j = 1, \dots, k \leq s$ (where k may be 0), there exists a $\Delta = (a_i, b_j) \subset \mathbf{Z}^{2g}$ such that $\varphi(\Delta)$ generates G, Δ is linear independent, $(a_i, a_j) = (b_i, b_j) = 0$, and $(a_i, b_j) = \delta_{ij}$.

From this fact and induction, the theorem follows. □

We shall also need the following results of symplectic geometry.

LEMMA 2 (see [A, Thm. 3.8]). *Let $H \cong \mathbf{Z}_p^m$ and $(\cdot,\cdot)\colon H \times H \to \mathbf{Z}_p$ be an alternating bilinear form. Let $\Delta = (a_i\ (i = 1, \dots, r), b_j\ (j = 1, \dots, k)) \subset H$ be a set* *of linear independent elements* (k *may be* 0) *such that* $(a_i, a_j) = (b_i, b_j) = 0$ *and* $(a_i, b_j) = \delta_{ij}$. *Then there is a basis of* H, $(\tilde{a}_i\ (i = 1, \dots, t),\ \tilde{b}_j\ (j = 1, \dots, s))$, *such that* $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$, $\tilde{a}_i = a_i\ (i = 1, \dots, r)$, *and* $\tilde{b}_i = b_i\ (i = 1, \dots, k)$.

Proof (Sketch). Let us consider all systems $\Delta' = (a_i'\ (i = 1, \dots, t),\ b_j'\ (j = 1, \dots, s))$ such that $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$, $a_i' = a_i\ (i = 1, \dots, r)$, and $b_i' = b_i\ (i = 1, \dots, k)$. Between them we choose a system Λ with maximal $t + s$. Then Λ is a basis with the conditions that we need. □

THEOREM 3 (Witt's Theorem; see [A, 3.9]). *Let* G, G' *be subgroups of* $\mathbf{Z}_p^{2g}$ *and let* $\psi\colon G \to G'$ *be an isomorphism such that* $(\psi(a), \psi(b))_p = (a, b)_p$ *for all* $a, b \in G$. *Then there is an automorphism* $\tilde{\psi} \in \mathrm{Sp}_p(2g, \mathbf{Z}_p)$ *such that* $\tilde{\psi}$ *restricted to* G *is* ψ.

3. Strong Classification of Fixed Point–Free Orientation-Preserving Actions of $\mathbf{Z}_p^m$ on Surfaces

In this section we shall consider (orientation-preserving) fixed point–free actions. Let $\tilde{S}$ be a closed (compact without boundary) oriented surface with genus g and let G be a group isomorphic to $\mathbf{Z}_p^m$, where p is a prime integer.

DEFINITION 4 (*Strong Equivalence*). Two actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are called *strongly equivalent* if there is a homeomorphism, $\tilde{\psi}\colon \tilde{S} \to \tilde{S}'$, sending the orientation of $\tilde{S}$ to the orientation of $\tilde{S}'$ and such that $f'(h) = \tilde{\psi} \circ f(h) \circ \tilde{\psi}^{-1}$ for all $h \in G$.

We are interested in finding all strong equivalence classes of actions of $\mathbf{Z}_p^m$.

We denote by $S = \tilde{S}/f(G)$ and by $\varphi = \varphi(f)\colon \tilde{S} \to S$ the natural projection. We shall consider first the case when $f(h)$ has no fixed points for any $h \in G - \{\mathrm{id}\}$, that is, when the action of $(\tilde{S}, f)$ is fixed point–free. The general case will be considered in Section 5. Then the projection $\varphi(f)\colon \tilde{S} \to S$ is an unbranched covering with deck transformation group $f(G)$.

Let us consider $\pi_1(S)$ as the group of deck transformations of the universal covering of S. Then we have

$$\omega(\tilde{S}, f)\colon \pi_1(S) \to \pi_1(S)/\pi_1(\tilde{S}) = f(G) \xrightarrow{f^{-1}} G.$$

The resulting epimorphism $\omega(\tilde{S}, f)\colon \pi_1(S) \to G \cong \mathbf{Z}_p^m$ is the monodromy epimorphism of the covering $\varphi(f)\colon \tilde{S} \to S$. The epimorphism $\omega(\tilde{S}, f)\colon \pi_1(S) \to G$ induces the epimorphism $\theta_p(\tilde{S}, f)\colon H_1(S, \mathbf{Z}_p) \to G$, since G is abelian.

Conversely, given an epimorphism $\theta_p\colon H_1(S, \mathbf{Z}_p) \to G$, there is an action $(\tilde{S}, f)$ such that $\theta_p = \theta_p(\tilde{S}, f)$. To obtain $\tilde{S}$ it is enough to consider the monodromy $\omega\colon \pi_1(S) \to H_1(S, \mathbf{Z}_p) \xrightarrow{\theta_p} G$ and then $\tilde{S} = U/\ker\omega$, where U is the universal covering of S and the action of G is given by $G = \pi_1(S)/\ker\omega$.

DEFINITION 5. Let S and S' be two surfaces. Then two epimorphisms $\theta\colon H_1(S, \mathbf{Z}_p) \to G$ and $\theta'\colon H_1(S', \mathbf{Z}_p) \to G$ are called *strongly equivalent* if there is an orientation-preserving homeomorphism $\psi: S \to S'$ inducing an isomorphism $\psi_p: H_1(S, \mathbf{Z}_p) \to H_1(S', \mathbf{Z}_p)$ such that $\theta = \theta' \circ \psi_p$.

THEOREM 6 [S]. *Two actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are strongly equivalent if and only if the epimorphisms $\theta_p(\tilde{S}, f)$ and $\theta_p(\tilde{S}', f')$ are strongly equivalent.*

DEFINITION 7. Let $(\tilde{S}, f)$ be an action of G, let $S = \tilde{S}/f(G)$, and let $\theta = \theta_p(\tilde{S}, f): H_1(S, \mathbf{Z}_p) \to G$ be the epimorphism defined by the action $(\tilde{S}, f)$. Now consider the spaces of homomorphisms $G^* = \{e: G \to \mathbf{Z}_p\}$ and $H^1(S, \mathbf{Z}_p) = \{e: H_1(S, \mathbf{Z}_p) \to \mathbf{Z}_p\}$. Then θ generates a monomorphism $\theta^* = \theta^*(\tilde{S}, f)$: $G^* \to H^1(S, \mathbf{Z}_p)$. The intersection form $(\cdot, \cdot)_p = (\cdot, \cdot)_p^S$ on $H_1(S, \mathbf{Z}_p)$ induces an isomorphism $i: H^1(S, \mathbf{Z}_p) \to H_1(S, \mathbf{Z}_p)$ defined by $(a, \cdot) \to a$ and a form $(\cdot, \cdot)_{(\tilde{S}, f)}: G^* \times G^* \to \mathbf{Z}_p$ such that $(a, b)_{(\tilde{S}, f)} = (i \circ \theta^*(a), i \circ \theta^*(b))_p$.

THEOREM 8. *Two actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ of the group G are strongly equivalent if and only if $\tilde{S}$ and $\tilde{S}'$ have the same genus and $(\cdot, \cdot)_{(\tilde{S}, f)} = (\cdot, \cdot)_{(\tilde{S}', f')}$.*

Proof. Let S and S' denote $\tilde{S}/f(G)$ and $\tilde{S}'/f'(G)$, respectively. Assume that $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are strongly equivalent. Then, according to Theorem 6, there exists a homeomorphism $\psi: S \to S'$, which induces an isomorphism $\psi_p: H_1(S, \mathbf{Z}_p) \to H_1(S', \mathbf{Z}_p)$ such that $\theta = \theta' \circ \psi_p$. Because ψ_p is induced by a homeomorphism, it follows that ψ_p preserves the intersection form and induces an isomorphism $\psi^*: H^1(S', \mathbf{Z}_p) \to H^1(S, \mathbf{Z}_p)$ such that $(a, b)_p^{S'} = (\psi^*(a), \psi^*(b))_p^S$ and $\theta^*(\tilde{S}, f) = \psi^* \circ \theta^*(\tilde{S}', f')$. Hence, for $a, b \in G^*$,

$$\begin{aligned}(a, b)_{(\tilde{S}, f)} &= (i \circ \theta^*(\tilde{S}, f)(a),\ i \circ \theta^*(\tilde{S}, f)(b))_p^S \\ &= (i \circ \psi^* \circ \theta^*(\tilde{S}', f')(a),\ i \circ \psi^* \circ \theta^*(\tilde{S}', f')(b))_p^S \\ &= (i' \circ \theta^*(\tilde{S}', f')(a),\ i' \circ \theta^*(\tilde{S}', f')(b))_p^{S'} = (a, b)_{(\tilde{S}', f')}.\end{aligned}$$

Assume now that $(\cdot, \cdot)_{(\tilde{S}, f)} = (\cdot, \cdot)_{(\tilde{S}', f')}$, and consider the isomorphisms

$$Q: H^1(S, \mathbf{Z}_p) \to (\mathbf{Z}_p^{2g}, (\cdot, \cdot)), \qquad Q': H^1(S', \mathbf{Z}_p) \to (\mathbf{Z}_p^{2g}, (\cdot, \cdot)),$$

such that $(Q(a), Q(b))_p = (a, b)_p^S$ and $(Q'(a'), Q'(b'))_p = (a', b')_p^S$ for any $a, b \in H^1(S, \mathbf{Z}_p)$ and $a', b' \in H^1(S', \mathbf{Z}_p)$.

We note $\tilde{G} = Q \circ \theta^*(\tilde{S}, f)(G^*) \subset \mathbf{Z}_p^{2g}$ and $\tilde{G}' = Q' \circ \theta^*(\tilde{S}', f')(G^*) \subset \mathbf{Z}_p^{2g}$. Let $\psi: \tilde{G} \to \tilde{G}'$ be the isomorphism given by $\psi = Q' \circ Q^{-1}$. Then, for every $a, b \in \tilde{G}$, we have $(\psi(a), \psi(b))_p = (a, b)_p$. From Theorem 3 it follows that there is a $\tilde{\psi} \in \mathrm{Sp}_p(2g, \mathbf{Z})$ such that $\tilde{\psi}$ restricted to $\tilde{G}$ is ψ. Consider now $\Psi = Q^{-1} \circ \tilde{\psi} \circ Q': H^1(S', \mathbf{Z}_p) \to H^1(S, \mathbf{Z}_p)$. Since $\tilde{\psi} \in \mathrm{Sp}_p(2g, \mathbf{Z})$, we see that Ψ comes from an isomorphism $\psi_*: H_1(S, \mathbf{Z}) \to H_1(S', \mathbf{Z})$ sending the intersection form of $H_1(S, \mathbf{Z})$ to the intersection form of $H_1(S', \mathbf{Z})$ (Theorem 1). Then, by a classical result of H. Burkardt in 1890 (see [MKS, p. 178]), there exists a homeomorphism $\psi: S \to S'$ inducing ψ_* and Ψ; by construction, $\theta^*(\tilde{S}, f) = \Psi \circ \theta^*(\tilde{S}', f')$. Then, by Theorem 6, the actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are strongly equivalent. □

THEOREM 9. *Let $G \cong \mathbf{Z}_p^m$, let $(\cdot,\cdot)\colon G^* \times G^* \to \mathbf{Z}_p$ be an alternating bilinear form, and let $k = \dim\{h \in G^* \colon (h, G^*) = 0\}$. Then there exists an action $(\tilde{S}, f)$ such that $(\cdot,\cdot) = (\cdot,\cdot)_{(\tilde{S},f)}$ and $g = g(\tilde{S}/f(G))$ if and only if $g \geq \frac{1}{2}(m+k)$ for $k = m \bmod 2$ and $k \leq m$.*

Proof. First we construct the action from the form and the numerical conditions. Applying Lemma 2 yields a basis of G^*, $(a_i^*\ (i = 1, \ldots, r),\ b_j^*\ (j = 1, \ldots, s))$, $0 \leq s \leq r$, such that $(a_i^*, a_j^*) = (b_i^*, b_j^*) = 0$, $(a_i^*, b_j^*) = \delta_{ij}$, and $s - r = k$. Let $(a_i\ (i = 1, \ldots, r),\ b_j\ (j = 1, \ldots, s))$ be the dual basis of the one just described. Now consider a surface S of genus g and a basis of $H_1(S, \mathbf{Z}_p)$, $(\alpha_i\ (i = 1, \ldots, g),\ \beta_i\ (i = 1, \ldots, g))$. Then we construct the epimorphism $\theta\colon H_1(S, \mathbf{Z}_p) \to G^*$, which is defined by

$$\theta(\alpha_i) = \begin{cases} a_i & \text{if } i \leq r, \\ 0 & \text{if } i > r, \end{cases}$$

and by

$$\theta(\beta_i) = \begin{cases} b_i & \text{if } i \leq s, \\ 0 & \text{if } i > s. \end{cases}$$

Then the epimorphism θ defines a regular covering $\tilde{S} \to S$ with automorphism group G, and the action of G on $\tilde{S}$ satisfies $(\cdot,\cdot)_{(\tilde{S},f)} = (\cdot,\cdot)$.

Conversely, if there is an action $(\tilde{S}, f)$ such that $(\cdot,\cdot) = (\cdot,\cdot)_{(\tilde{S},f)}$, then it is obvious that $g = g(\tilde{S}/f(G)) \geq \frac{1}{2}(m+k)$ for $k = m \bmod 2$ and $k \leq m$. □

4. Weak Classification of Fixed Point–Free Orientation-Preserving Actions of $\mathbf{Z}_p^m$ on Surfaces

In this section we shall continue to consider only fixed point–free actions.

DEFINITION 10 (*Weak Equivalence*). Let $(\tilde{S}, f)$ and $(\tilde{S}', f')$ be two actions of a group $G \cong \mathbf{Z}_p^m$. We shall say that $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are *weakly equivalent* if there is a homeomorphism $\tilde{\psi}\colon \tilde{S} \to \tilde{S}'$ and an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $f' \circ \alpha(h) = \tilde{\psi} \circ f(h) \circ \tilde{\psi}^{-1}$, $h \in G$.

The next theorem solves the problem of weak classification of actions of $\mathbf{Z}_p^m$ on surfaces.

THEOREM 11. *Let $(\tilde{S}, f)$ and $(\tilde{S}', f')$ be two actions of a group $G \cong \mathbf{Z}_p^m$. Let $(\cdot,\cdot)_{(\tilde{S},f)}$ and $(\cdot,\cdot)_{(\tilde{S}',f')}$ be the alternating bilinear forms induced by the two actions, $k(\tilde{S}, f) = \dim\{h \in G^* : (h, G^*)_{(\tilde{S},f)} = 0\}$ and $k(\tilde{S}', f') = \dim\{h \in G^* : (h, G^*)_{(\tilde{S}',f')} = 0\}$. Then the actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are weakly equivalent if and only if $g(\tilde{S}/f(G)) = g(\tilde{S}'/f'(G))$ and $k(\tilde{S}, f) = k(\tilde{S}', f')$.*

Proof. Let us put $S = \tilde{S}/f(G)$ and $S' = \tilde{S}'/f'(G)$, and let $g = g(S)$ and $g' = g(S')$. Let $\theta^*(\tilde{S}, f)$ and $\theta^*(\tilde{S}', f')$ be the epimorphisms defined by the two actions, let $\tilde{G}$ be the image of G^* in $H_1(S, \mathbf{Z}_p)$ by $\theta^*(\tilde{S}, f)$, and let $\tilde{G}'$ be the image of G^* in $H_1(S', \mathbf{Z}_p)$ by $\theta^*(\tilde{S}', f')$.

Assume that $g(S) = g(S')$ and $k(\tilde{S}, f) = k(\tilde{S}', f')$. Since $k(\tilde{S}, f) = k(\tilde{S}', f')$, there exists an isomorphism $\psi\colon \tilde{G}' \to \tilde{G}$ such that

$$(\psi(a), \psi(b))_{(\tilde{S}', f')} = (a, b)_{(\tilde{S}, f)}.$$

Then, using Theorem 3 and $g(S) = g(S')$, there is an isomorphism $\tilde{\psi}\colon H^1(S', \mathbf{Z}_p) \to H^1(S, \mathbf{Z}_p)$ giving by restriction ψ and sending the intersection form of $H^1(S', \mathbf{Z}_p)$ to the intersection form of $H^1(S, \mathbf{Z}_p)$. By [MKS, p. 178], there exists a homeomorphism $\varphi\colon S \to S'$ inducing $\tilde{\psi}$ on cohomology. Then, by Theorem 6, the actions $(\tilde{S}, f)$ and $(\tilde{S}, \varphi^{-1} \circ f' \circ \varphi)$ are strongly equivalent. The isomorphism ψ defines an automorphism of G, giving the weak equivalence between $(\tilde{S}, f)$ and $(\tilde{S}', f')$. □

The following corollary is one of the main results of this paper.

COROLLARY 12. *Weak equivalence classes of $\mathbf{Z}_p^m$ actions are in bijection with the set of pairs of positive integers (k, g) such that $k \le m$, $k = m \bmod 2$, and $g \ge \frac{1}{2}(m + k)$.*

Proof. Theorem 11 tells us that each pair (k, g) determines a weak equivalence class. By Theorem 9, each pair of numbers (k, g) satisfying the conditions in the corollary defines a nonempty weak equivalence class. □

EXAMPLE. By Corollary 12, there are four weak equivalence classes of fixed point–free $\mathbf{Z}_3^6$ actions on surfaces of genus 3649 (cf. [E, Rem. 4.5]); in this case, by the Riemann–Hurwitz formula, the number g in the corollary is 6. We shall construct a representative for each action. We take a Fuchsian surface group Γ of genus 6 acting conformally on the complex disc D. The group Γ has a canonical presentation

$$\left\langle A_i,\ B_i,\ i = 1, \dots, 6;\ \prod_{i=1}^{6} [A_i, B_i] = 1 \right\rangle.$$

We consider the epimorphisms $\theta_j\colon \Gamma \to \mathbf{Z}_3^6 = \bigoplus_{i=1}^{6} \langle g_i : g_i^3 = 1 \rangle$, $j = 0, 2, 4, 6$, defined by:

$$\theta_j(A_i) = g_i, \quad \theta_j(B_i) = 0, \quad i = 1, \dots, j,$$
$$\theta_j(A_i) = g_{2i-j-1}, \quad \theta_j(B_i) = g_{2i-j}, \quad j < i \le \tfrac{j+6}{2},$$
$$\theta_j(A_i) = \theta_j(B_i) = 0, \quad \tfrac{j+6}{2} < i \le 6.$$

The Riemann surfaces $D/\ker\theta_j$ admit automorphism groups representing each weak equivalence class of fixed point–free $\mathbf{Z}_3^6$ actions.

5. Classification of Orientation-Preserving Actions of $\mathbf{Z}_p^m$ with Elements Having Fixed Points

Let G be a group isomorphic to $\mathbf{Z}_p^m$, and let $(\tilde{S}, f)$ be an action of G on an oriented closed surface $\tilde{S}$. By G_{fix} we denote the subgroup of G generated by the elements of $f(G)$ having fixed points.

The projection $\varphi = \varphi(f)\colon \tilde{S} \to S = \tilde{S}/f(G)$ is a covering branched on a finite set of points $\mathcal{B} = \{b_1, \ldots, b_r\}$. The covering φ is now determined by an epimorphism $\theta_p(\tilde{S}, f)\colon H_1(S - \mathcal{B}, \mathbf{Z}_p) \to G$.

Let X_i $(i = 1, \ldots, r)$ denote the element of $H_1(S - \mathcal{B}, \mathbf{Z}_p)$ represented by the boundary of a small disc in S around the branched point b_i, and with the orientation given by the orientation of S. Then the set $\{\theta_p(\tilde{S}, f)(X_i)\}$ is a topological invariant for the action $(\tilde{S}, f)$. We have

$$\langle \theta_p(\tilde{S}, f)(X_i),\ i = 1, \ldots, r\rangle = G_{\text{fix}}.$$

Then we have an epimorphism $\vartheta\colon H_1(S, \mathbf{Z}_p) \to G_{\text{free}}$, defined by

$$H_1(S, \mathbf{Z}_p) \to H_1(S - \mathcal{B}, \mathbf{Z}_p)/\langle X_i,\ i = 1, \ldots, r\rangle \to G/G_{\text{fix}} = G_{\text{free}}.$$

In fact, the epimorphism ϑ is the epimorphism defined by the fixed point–free action defined by the unbranched covering $\tilde{S}/f(G_{\text{fix}}) \to S$. If $G_{\text{free}} = G/G_{\text{fix}}$ then ϑ defines, as in Section 2, a bilinear form $(\cdot,\cdot)_{(\tilde{S},f)}\colon G^*_{\text{free}} \times G^*_{\text{free}} \to \mathbf{Z}_p$.

THEOREM 13. *Two actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ of the group $G \cong \mathbf{Z}_p^m$ are strongly equivalent if and only if the following statements hold.*

(1) *$\tilde{S}$ and $\tilde{S}'$ have the same genus.*

(2) *The number of branched points $(r = \#\mathcal{B})$ of the covering $\tilde{S} \to S = \tilde{S}/f(G)$ is the same as the number of branched points $(r' = \#\mathcal{B}')$ of the covering $\tilde{S}' \to S' = \tilde{S}'/f'(G)$.*

(3) $$[\theta_p(\tilde{S}, f)(X_1), \theta_p(\tilde{S}, f)(X_2), \ldots, \theta_p(\tilde{S}, f)(X_r)] \\ = [\theta_p(\tilde{S}, f)(X_1), \theta_p(\tilde{S}, f)(X_2), \ldots, \theta_p(\tilde{S}, f)(X_r)],$$

where $[\cdot, \ldots, \cdot]$ is used to denote unordered r-tuples of elements of $G - \{\text{id}\}$. As a consequence, $G^f_{\text{free}} = G^{f'}_{\text{free}} = G_{\text{free}}$.

(4) *The intersection forms on G_{free} induced by f and f' are the same, $(\cdot,\cdot)_{(\tilde{S},f)} = (\cdot,\cdot)_{(\tilde{S}',f')}$.*

Proof. Using Dehn twists along curves around the branched points (see [C, p. 151, move (6)]), it is possible to obtain a basis $(A_i\ (i = 1, \ldots, g),\ B_i\ (i = 1, \ldots, g),\ X_i\ (i = 1, \ldots, r))$ of $H_1(S - \mathcal{B}, \mathbf{Z}_p)$ such that

$$\theta_p(\tilde{S}, f)(A_i) \in G_{\text{free}}, \quad \theta_p(\tilde{S}, f)(B_i) \in G_{\text{free}}, \quad i = 1, \ldots, g;$$
$$(A_i, A_j) = 0, \quad (B_i, B_j) = 0, \quad (A_i, B_j) = \delta_{ij}.$$

In the same way, we can construct a basis $(A'_i\ (i = 1, \ldots, g),\ B'_i\ (i = 1, \ldots, g),\ X'_i\ (i = 1, \ldots, r))$ of $H_1(S' - \mathcal{B}', \mathbf{Z}_p)$ such that

$$\theta_p(\tilde{S}', f')(A'_i) \in G_{\text{free}}, \quad \theta_p(\tilde{S}', f')(B'_i) \in G_{\text{free}}, \quad i = 1, \ldots, g;$$
$$(A'_i, A'_j) = 0, \quad (B'_i, B'_j) = 0, \quad (A'_i, B'_j) = \delta_{ij}.$$

Note that, by conditions (1) and (2), $g = g'$ and $r = r'$.

By condition (4) and Theorem 8, it follows that the fixed point–free action of G_{free} on $\tilde{S}/f(G_{\text{fix}})$ given by f and the fixed point–free action of G_{free} on

$\tilde{S}'/f'(G_{\text{fix}})$ given by f' are strongly equivalent. Then there exists an orientation-preserving homeomorphism $\varphi\colon S \to S'$ inducing on homology an isomorphism $\psi\colon H_1(S, \mathbf{Z}_p) \to H_1(S', \mathbf{Z}_p)$ and, by the proof of Theorem 8, we can construct φ such that $\psi(A_i) = A'_i$ and $\psi(B_i) = B'_i$. We now consider a disc D on S containing $\mathcal{B}$ and a disc D' on S' containing $\mathcal{B}'$. Then we can modify φ by composing with an isotopy in S' in order that $\varphi(D) = D'$ and $\varphi(b_i) = b_{\sigma(i)}$, where σ is a permutation of $\{1, \dots, r\}$ such that

$$(\theta_p(\tilde{S}, f)(X_1), \theta_p(\tilde{S}, f)(X_2), \dots, \theta_p(\tilde{S}, f)(X_r)) \\ = (\theta_p(\tilde{S}, f)(X_{\sigma(1)}), \theta_p(\tilde{S}, f)(X_{\sigma(2)}), \dots, \theta_p(\tilde{S}, f)(X_{\sigma(r)})).$$

Now φ defines an isomorphism $\tilde{\psi}\colon H_1(S - \mathcal{B}, \mathbf{Z}_p) \to H_1(S' - \mathcal{B}', \mathbf{Z}_p)$ such that $\tilde{\psi}(A_i) = A'_i$, $\tilde{\psi}(B_i) = B'_i$, and $\tilde{\psi}(X_i) = X'_i$, so $\theta_p(\tilde{S}, f) = \theta_p(\tilde{S}', f') \circ \varphi$. Hence the actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ are strongly equivalent. □

As a consequence of Theorem 13 and Theorem 9, we have the following.

THEOREM 14. *Let $G \cong \mathbf{Z}_p^m$, and let $H \cong \mathbf{Z}_p^n$ be a subgroup of G. Assume that $[C_1, \dots, C_r]$, $r \geq n$, is an unordered element of $(H - \{0\})^r$, where $\{C_1, \dots, C_r\}$ generates H and $\sum_1^r C_i = 0$. Let $(\cdot, \cdot)$ be an alternating bilinear form on G/H and let $k = \dim\{h \in (G/H)^* : (h, (G/H)^*) = 0\}$. Then, for $g \geq \frac{1}{2}(m - n + k)$ and only for such g, there is an action $(\tilde{S}, f)$ with $g = g(\tilde{S}/f(G))$ and $(\cdot, \cdot) = (\cdot, \cdot)_{(\tilde{S}, f)}$, where $(\cdot, \cdot)_{(\tilde{S}, f)}$ is the bilinear form induced by the fixed point–free action on G/H, the elements acting with fixed points generate H, and*

$$[\theta_p(\tilde{S}, f)(X_1), \theta_p(\tilde{S}, f)(X_2), \dots, \theta_p(\tilde{S}, f)(X_r)] = [C_1, C_2, \dots, C_r].$$

REMARK. The unordered elements of H^r are in one-to-one correspondence with the functions $F\colon H \to (\mathbf{Z}_+)^{p-1}$. From $[C_1, C_2, \dots, C_r]$ we define $F(h) = (k_1, k_2, \dots, k_{p-1})$ if the element h^i appears k_i times in $[C_1, C_2, \dots, C_r]$. The function F gives the topological type of the action of H.

THEOREM 15. *Two actions $(\tilde{S}, f)$ and $(\tilde{S}', f')$ of the group $G \cong \mathbf{Z}_p^m$ are weakly equivalent if and only if the following statements hold.*

(1) *$\tilde{S}$ and $\tilde{S}'$ have the same genus.*
(2) *The number of branched points $(r = \#\mathcal{B})$ of the covering $\tilde{S} \to S = \tilde{S}/f(G)$ is the same as the number of branched points $(r' = \#\mathcal{B}')$ of the covering $\tilde{S}' \to S' = \tilde{S}'/f'(G)$.*
(3) $(\theta_p(\tilde{S}, f)(X_1), \theta_p(\tilde{S}, f)(X_2), \dots, \theta_p(\tilde{S}, f)(X_r))$
$$= (\gamma \circ \theta_p(\tilde{S}, f)(X_{\sigma(1)}), \gamma \circ \theta_p(\tilde{S}, f)(X_{\sigma(2)}), \dots, \gamma \circ \theta_p(\tilde{S}, f)(X_{\sigma(r)})),$$
where σ is a permutation of $\{1, \dots, r\}$ and γ is an automorphism of G.
(4) $\dim\{h \in G^*_{\text{free}} : (h, G^*_{\text{free}})_{(\tilde{S}, f)} = 0\} = \dim\{h \in G^*_{\text{free}} : (h, G^*_{\text{free}})_{(\tilde{S}', f')} = 0\}$.

Proof. Similar to the proof of Theorem 13. □

THEOREM 16. *Let $G \cong \mathbf{Z}_p^m$. Then the weak equivalence classes of actions of G, such that there are r points in the surface where the action takes place that are fixed for elements of G, are in bijection with the set of triples $(k, g, \mathrm{Aut}(G)[C_1, \ldots, C_r])$, where*:

(1) *k and g are integers such that, if $r > 0$, then there is an integer n $(n \geq 1)$ such that $k \leq m - n$, $k = (m - n) \bmod 2$, and $g \geq \frac{1}{2}(m - n + k)$ for $r \geq n$; and*

(2) *$[C_1, \ldots, C_r]$ is an unordered r-tuple of nontrivial elements of G such that $\{C_1, \ldots, C_r\}$ generates a group isomorphic to $\mathbf{Z}_p^n$ and $\sum_1^r C_i = 0$.*

Proof. Theorem 15 tells us that each triple determines a weak equivalence class and, by Theorem 14, each triple satisfying the conditions in Theorem 16 defines a nonempty weak equivalence class. □

EXAMPLE. There is only one weak equivalence class of actions of $\mathbf{Z}_p^2$ on surfaces of a fixed genus such that there is exactly one fixed point for some of the elements of $\mathbf{Z}_p^2$. In this situation, where $m = 2$ and $r = 1$, we have $n = 1$ and $k = 1$, and there is only one class of nontrivial elements under the action of $\mathrm{Aut}(G)$. A representative for the weak equivalence class, when $g > 1$, can be constructed in the following way. We take a Fuchsian group Γ with signature $(g, [p])$. The group Γ has a canonical presentation

$$\left\langle A_i,\ B_i,\ i = 1, \ldots, g,\ X;\ X \prod_{i=1}^{g} [A_i, B_i] = 1,\ X^p = 1 \right\rangle.$$

We construct an epimorphism $\theta\colon \Gamma \to \mathbf{Z}_p^2 = \langle g_1 \rangle \oplus \langle g_2 \rangle$ defined by $\theta(A_1) = g_1$, $\theta(A_i) = \theta(B_j) = 0$ for $i \neq 1$, and $\theta(X) = g_2$. The Riemann surface uniformized by $\ker\theta$ has a group of automorphisms representing the weak equivalence class that we are looking for.

Let $M^{p,m}$ be the space of pairs $(\tilde{R}, G)$, where $\tilde{R}$ is a Riemann surface and G is a group of automorphisms of $\tilde{R}$. The covering $\tilde{R} \to \tilde{R}/G$ defines a projection $p\colon M^{p,m} \to M$, where M is the moduli space of Riemann surfaces. The projection $p\colon M^{p,m} \to M$ gives a topology on $M^{p,m}$, the weakest topology where p is continuous.

From Theorem 16 and [Na2, Sec. 6], we have the following.

CONSEQUENCE. *There exists a one-to-one correspondence between the connected components of $M^{p,m}$ with such topology and the triples*

$$(k, g, \mathrm{Aut}(G)[C_1, \ldots, C_r])$$

described in Theorem 16. Each connected component of $M^{p,m}$ is homeomorphic to the quotient $\mathbf{R}^n/\mathrm{Mod}$ of a vector space $\mathbf{R}^n$ by the discontinuous action of a group Mod.

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