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Intersection Multiplicities and Hilbert Polynomials

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Dedicated to W. Fulton

1. Introduction

In [11], Serre introduced a definition of intersection multiplicity for regular local rings, showed that it satisfied many of the properties which should hold for intersection multiplicities, and stated a number of conjectures. Of these conjectures, the only one that is still open is the positivity conjecture, which states that under certain conditions on dimension (we give a precise statement below), the intersection multiplicity will be positive. Recently, Gabber used a construction of de Jong to prove that these multiplicities are always nonnegative, thus establishing one of the conjectures. In his proof, Gabber constructed a scheme that can be represented by a bigraded ring and reduced the computation of intersection multiplicities to the computation of an Euler characteristic defined by modules over this ring. In this paper we define Hilbert polynomials for bigraded modules over this type of bigraded ring and show that the Euler characteristic can be computed using these Hilbert polynomials. We then use this construction to give a simple proof of a criterion for positivity proven in Kurano and Roberts [7]. Some of these ideas were discussed in Roberts [10]; however, the criterion we prove here was not included in that paper.

The outline of the paper is as follows. In Section 2 we recall the facts we need about the positivity conjecture and Gabber's construction. In Section 3 we prove the existence of Hilbert polynomials in the case we are considering; we then prove (Section 4) a reduction formula for dividing by a homogeneous element. In Section 5 we prove the basic relations between Hilbert polynomials and dimension. Finally, we prove the criterion for positivity in Section 6.

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2. Intersection Multiplicities and Gabber's Construction

Let R be a regular local ring of dimension d with maximal ideal $\mathfrak{m}$, and let $X = \operatorname{Spec}(R)$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\mathfrak{p}+\mathfrak{q}$ is $\mathfrak{m}$-primary or, equivalently, such that $R/\mathfrak{p} \otimes_R R/\mathfrak{q}$ is a module of finite length. Then the intersection multiplicity of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ is defined to be

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$$\chi(R/\mathfrak{p}, R/\mathfrak{q}) = \sum_{i=0}^{d} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q})).$$

For the basic properties of intersection multiplicities we refer to Serre [11]. Serre made several conjectures, of which we state two.

(1) *Nonnegativity:* $\chi(R/\mathfrak{p}, R/\mathfrak{q}) \geq 0$.
(2) *Positivity:* If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$, then $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$.

Serre proved these conjectures in the equicharacteristic case using the method of reduction to the diagonal. This method reduced the problem to the case in which one of the ideals is generated by a regular sequence, and in this case he showed that the intersection multiplicity defined as above could be computed using Samuel multiplicities defined by Hilbert polynomials. For Samuel multiplicities, these properties are easy to verify.

Recently, Gabber proved the nonnegativity conjecture by using a theorem on the existence of regular alterations of de Jong [6]. We next describe this construction briefly; for more details, we refer to Berthelot [1], Hochster [5], and Roberts [10]. In particular, for the theorem of de Jong to apply, R must be essentially of finite type over a field or a ring of Witt vectors; however, the multiplicity conjectures can be reduced to this case, and a description of this reduction can be found in references [1] and [5].

Let R, $\mathfrak{p}$, and $\mathfrak{q}$ be as above. The theorem of de Jong implies that there exists an integer n and a graded prime ideal I of the graded ring $A = R[X_0, \ldots, X_n]$ such that the following conditions hold:

(1) $\operatorname{Proj}(A/I)$ is a regular scheme;
(2) $I \cap R = \mathfrak{p}$; and
(3) the induced map from $\operatorname{Proj}(A/I)$ to $\operatorname{Spec}(R/\mathfrak{p})$ is generically finite.

The third condition means that the extension of fields from the fraction field of $R/\mathfrak{p}$ to the field of rational functions of $\operatorname{Proj}(A/I)$ is finite.

Let $\bar{A}$ denote $A/\mathfrak{q}A$, and let $\bar{I}$ denote the image of I in $\bar{A}$. We consider the associated graded rings defined by I on A and $\bar{I}$ on $\bar{A}$, which we denote by $G(I)$ and $G(\bar{I})$, respectively. Since I and $\bar{I}$ are graded ideals, $G(I)$ and $G(\bar{I})$ are bigraded rings. In this bigrading we assign degree (m, n) to an element of I^n/I^{n+1}, which is represented by an element of I^n that has degree m in A. There is a surjective map from $G(I)$ to $G(\bar{I})$; let K_0 denote its kernel. Then K_0 is a bigraded ideal. We note for future reference that, since $\operatorname{Proj}(A/I)$ is a regular scheme, it follows that I is locally generated by part of a regular system of parameters and that $G(I)$ is locally a polynomial ring over A/I.

Denote the residue field of R by k, and let $C_0 = (A/I) \otimes_R k$. Then, if R is equicharacteristic or ramified of mixed characteristic, there is a homomorphism of bigraded rings

$$\phi\colon G(I) \otimes_R k \to C_0[S_1, \ldots, S_d, T_0, \ldots, T_n],$$

where the S_i are variables of degree $(0, 1)$ and the T_i are variables of degree $(1, 1)$. We let C denote $C_0[S_1, \ldots, S_d, T_0, \ldots, T_n]$. (The map ϕ is induced by the differential on I/I^2, but the details are fairly complicated and we refer to the sources previously cited for the complete definition; we will not use the details of the construction in this paper.) The map ϕ defines an extension of polynomial rings locally; that is, locally on $\mathrm{Proj}(A)$, $G(I) \otimes_R k$ is a polynomial ring over C_0, ϕ is injective, and C can be obtained from the image of ϕ by adjoining indeterminates. Finally, if we let J denote the ideal of C generated by $S_1, \ldots, S_d, T_0, \ldots, T_n$ and let K be the ideal generated by the image of K_0 in C, then $\chi(R/\mathfrak{p}, R/\mathfrak{q})$ is positive if and only if $\chi(C/K, C/J)$ is positive, where the Euler characteristic $\chi(C/K, C/J)$ is defined in terms of an alternating sum of Tor modules in a way that we will now make precise.

Since K and J are bigraded ideals, the modules $\mathrm{Tor}_i^C(C/K, C/J)$ are bigraded modules for all i. In addition, they are annihilated by J, which is generated by the variables $S_1, \ldots, S_d, T_0, \ldots, T_n$, so they can be considered as graded modules over C_0. Thus the Tor modules define coherent sheaves over $\mathrm{Proj}(C_0)$; we denote the coherent sheaf defined by $\mathrm{Tor}_i^C(C/K, C/J)$ by $\mathcal{F}_i$. Since $C_0 = (A/I) \otimes_R k$ is a graded ring over the field k, the cohomology modules $H^j(\mathrm{Proj}(C_0), \mathcal{F}_i)$ are finite-dimensional k-modules for all i and j. For each i and j we let

$$\chi(\mathcal{F}_i) = \sum_j (-1)^j \dim_k(H^j(\mathrm{Proj}(C_0), \mathcal{F}_i)).$$

We then define the Euler characteristic under consideration by letting

$$\chi(C/K, C/J) = \sum_i (-1)^i \chi(\mathcal{F}_i).$$

We refer again to Berthelot [1], Hochster [5], and Roberts [10] for more details and different versions of this construction.

In the remainder of the paper we show that the Euler characteristic $\chi(C/K, C/J)$ can be expressed in terms of the Hilbert polynomial defined by the bigraded module C/K.

3. Hilbert Polynomials in Two Variables

In this section we prove the existence of Hilbert polynomials in the specific situation we are considering.

We first change the notation slightly from that of the previous section. Let C be a bigraded polynomial ring over a field k in variables $X_0, \ldots, X_s, T_0, \ldots, T_u$ and $S_1, \ldots, S_v$, where each X_i has degree $(1, 0)$, each T_i has degree $(1, 1)$, and each S_i has degree $(0, 1)$. (The ring C considered in the previous paragraph is a homomorphic image of a bigraded polynomial ring of this type and we also had that $s = u = n$ and $v = d$; it is more convenient here to restrict to polynomial rings but to allow more general conditions on the number of variables.)

We assume the basic facts about Hilbert functions of a $\mathbb{Z}$-graded ring (as presented e.g. in [8, Sec. 13]). We will often consider C as a $\mathbb{Z}$-graded ring by using the grading in the first variable, so that $C_m = \bigoplus_n C_{m,n}$. In general, C_m is not a finite-dimensional vector space over k, although if $v = 0$ (so that there are no variables of degree $(0, 1)$) then we will have $C_0 = k$ and C_m will be finite-dimensional for all k. We use the notation $\mathrm{Proj}(C)$ to denote the projective scheme associated to C using this grading, and if M is a bigraded C-module then we will sometimes consider it as a $\mathbb{Z}$-graded module in the same way.

For any finitely generated bigraded C-module M and for any integers m and n, let $M_{m,n}$ be the component of M of degree (m, n). Let H_M be the Hilbert function of M defined by the formula

$$H_M(m, n) = \sum_{i \leq n} \dim_k(M_{m,i}).$$

Just as in the classical case, the Hilbert function is not a polynomial in m and n for all (m, n), but there is a polynomial that agrees with the Hilbert function for (m, n) in a certain subset of $\mathbb{Z} \times \mathbb{Z}$. We prove that there exist integers m_0 and n_0 such that the Hilbert function is given by a polynomial for (m, n) with $m \geq m_0$ and $n \geq m + n_0$. We note that the intersection of two subsets of $\mathbb{Z} \times \mathbb{Z}$ defined by inequalities of this type is nonempty and is defined by inequalities of the same type. We will sometimes abbreviate the statement that a condition holds for (m, n) satisfying such inequalities by saying that it holds for sufficiently large m and n.

To prove the result we will need to know that certain simply graded subsets of M have Hilbert polynomials. Let k be an integer, and let $D_k(M)$ be the subset of M defined by

$$D_k(M) = \bigoplus_{n \leq m+k} M_{m,n}.$$

Let B be the subring of C generated by the X_i and the T_i and by all $X_i S_j$ for $i = 0, \ldots, s$ and $j = 1, \ldots, v$. Then each generator of B has degree $(1, 0)$ or $(1, 1)$, and B is a bigraded subring of C. Like C, B can be considered as a $\mathbb{Z}$-graded ring, and $D_k(M)$ can be considered as a $\mathbb{Z}$-graded B-module.

LEMMA 1. *For each k, $D_k(M)$ is a finitely generated B-module.*

Proof. We are assuming that M is finitely generated over C; let x_i be homogeneous generators of M and let the degrees of x_i be (m_i, n_i). We assume first that $n_i \leq m_i + k$ for all i, which means that the x_i are in $D_k(M)$. For each v-tuple $(k_1, \ldots, k_v)$ of nonnegative integers, let $|K| = k_1 + \cdots + k_v$ and let S^K denote the monomial $S_1^{k_1} S_2^{k_2} \cdots S_v^{k_v}$. We claim that $D_k(M)$ is generated as a B-module by the set of $S^K x_i$ for all i and all K such that $n_i + |K| \leq m_i + k$. Note that these elements are in $D_k(M)$ and that this set is finite.

To show that these elements $S^K x_i$ generate M, we will show that each component $M_{m,n}$ of $D_k(M)$ is generated as a k-vector space by multiples of these elements by monomials in B. Fix m and n with $n \leq m + k$. The component $M_{m,n}$ is generated as a vector space over k by elements of the form $X^I T^J S^K x_i$, where I, J, and K denote s-, u-, and v-tuples of nonnegative integers and where $X^I T^J S^K$ is a monomial of the correct degree. If one factor of the form T_j or one factor of the form $X_i S_j$ occurs in this monomial, then T_j or $X_i S_j$ can be factored out and this generator is a multiple of an element in $M_{m-1,n-1}$ by an element of B, and we can conclude the result by induction on m. If no factor of T_j or $X_i S_j$ occurs in

$X^I T^J S^K x_i$, then $X^I T^J S^K x_i$ is of the form $X^I x_i$ or $S^K x_i$. In the first case, since we have assumed that $x_i \in D_k(M)$, we can divide by X_j for some j, so we can again divide by an element of B and conclude the result by induction. In the second case, the element is one of those that we have chosen to generate $D_k(M)$. Hence, in either case we can conclude that $X^I T^J S^K x_i$ is a multiple of one of the given generators by an element of B, so $D_k(M)$ is finitely generated.

If some of the x_i are not in $D_k(M)$, we choose a k' large enough so that the x_i are in $D_{k'}(M)$. Then the foregoing argument shows that $D_{k'}(M)$ is finitely generated. Since $D_k(M)$ is a sub-B-module of $D_{k'}(M)$ and B is Noetherian, $D_k(M)$ is also finitely generated. □

Our proof of the existence of Hilbert polynomials uses an inductive argument, and it is convenient to represent polynomials using binomial coefficients. We recall that a polynomial in one variable m can be uniquely written as a linear combination of the binomial coefficients $\binom{m}{i}$ for various nonnegative integers i. Since a polynomial in two variables is a sum of products of polynomials in each variable (for example, the monomials are such products), it follows that a polynomial in two variables m and n can be written as a linear combination of products $\binom{m}{i}\binom{n}{j}$ for various i and j. The reason for writing polynomials in this form comes from the fact that binomial coefficients satisfy the equation

$$\binom{n}{i} - \binom{n-1}{i} = \binom{n-1}{i-1},$$

which is useful in proving results by induction.

We can now prove the main result of this section. If M is a bigraded module, we let $M[i, j]$ denote the module M with degrees shifted by (i, j), so that $M[i, j]_{m,n} = M_{m+i,n+j}$ for all m and n.

THEOREM 1. *Let C be as above, and let M be a finitely generated bigraded C-module. Then there exist integers m_0 and n_0 and a polynomial $P_M(m, n)$ in two variables such that we have*

$$P_M(m, n) = H_M(m, n)$$

for all (m, n) with $m \geq m_0$ and $n \geq n_0 + m$.

Proof. We prove this result by induction on the number of variables S_i of degree $(0, 1)$. We first suppose that there are no variables of this type. Let M be generated by elements x_i of degree (m_i, n_i), and let n_0 be an integer greater than the maximum value of $n_i - m_i$. For every (m, n), the component of M of degree (m, n) is generated by products of the x_i with monomials in X_i and T_i. Since each X_i has degree $(1, 0)$ and each T_i has degree $(1, 1)$, if $X^I T^J x_i$ has degree (m, n) then $n - m \leq n_i - m_i$. Since n_0 was chosen greater than all of the $n_i - m_i$, we thus have $M_{m,n} = 0$ when $n - m \geq n_0$. Therefore,

$$H_M(m, n) = \sum_{i \leq n} \dim_k M_{m,i} = \sum_{i \leq m+n_0} \dim_k M_{m,i} = H(m, m + n_0)$$

when $n \geq m + n_0$. Thus $H_M(m, n)$ is constant in n for large n, and its value is the sum $\sum \dim_k(M_{m,i})$, where the sum runs over all $i \in \mathbb{Z}$. Thus, if we let $\tilde{H}_M(m)$ be the Hilbert function of the $\mathbb{Z}$-graded module M, we have

$$H_M(m, n) = \tilde{H}_M(m)$$

for $n \geq m + n_0$. Combining these results with the usual theory of Hilbert polynomials, we conclude that there exists an m_0 such that $H_M(m, n)$ is a polynomial in m (it does not involve n in this case) when $m \geq m_0$ and $n \geq m + n_0$.

Now assume that $v > 0$, so that there are variables of type S_i. Then, letting M_{S_v} denote the submodule of M consisting of elements annihilated by S_v, we have the exact sequence

$$0 \to M_{S_v}[0, -1] \to M[0, -1] \xrightarrow{S_v} M \to M/S_vM \to 0.$$

By induction on v, the theorem holds for $M_{S_v}[0, -1]$ and M/S_vM. Taking the difference of the Hilbert functions of these modules and using the above short exact sequence, we obtain an equation

$$H_M(m, n) - H_M(m, n-1) = G_1(m, n) \tag{$*$}$$

for some polynomial G_1 and for large enough m and n. Let m_0 and n_0 be integers such that equation $(*)$ holds when $m \geq m_0$ and $n \geq m + n_0$. We consider the B-module $D_{n_0}(M)$ as defined before. By Lemma 1, $D_{n_0}(M)$ is a finitely generated graded B-module, so there is a polynomial $G_2(m)$ such that $G_2(m) = \dim_k(D_{n_0}(M))_m$ for large m. Choose m_0 large enough so that both this equality and equation $(*)$ hold. Represent $G_1(m, n)$ as $\sum c_{ij}\binom{m}{i}\binom{n}{j}$. Then we claim that

$$H_M(m, n) = \sum c_{ij}\binom{m}{i}\binom{n+1}{j+1} - \sum c_{ij}\binom{m}{i}\binom{m+n_0+1}{j+1} + G_2(m)$$

for (m, n) in this range. The first two terms on the right-hand side cancel when $n = m + n_0$, leaving $G_2(m)$, which is the value of $H_M(m, m + n_0)$. Hence this equality holds when $n = m + n_0$. If $n > m + n_0$ then, letting $P(m, n)$ denote the polynomial on the right-hand side, we have

$$\begin{aligned} &P(m, n) - P(m, n-1) \\ &\quad = \sum c_{ij}\binom{m}{i}\binom{n+1}{j+1} - \sum c_{ij}\binom{m}{i}\binom{n}{j+1} \\ &\quad = \sum c_{ij}\binom{m}{i}\binom{n}{j} = G_1(m, n) = H_M(m, n) - H_M(m, n-1). \end{aligned}$$

Hence these polynomials agree for all (m, n) with $m \geq m_0$ and $n \geq m + n_0$, as was to be shown. □

For convenience, we define the Hilbert polynomial of a bounded complex of finitely generated modules to be the alternating sum of the Hilbert polynomials of the modules. By the additivity of Hilbert functions (and thus of Hilbert polynomials), we have the following.

PROPOSITION 1. *If $F_\bullet$ is a bounded complex of finitely generated bigraded modules, then*

$$P_{F_\bullet}(m,n) = \sum (-1)^i P_{H_i(F_\bullet)}(m,n).$$

4. Difference Formulas

In applications of Hilbert polynomials, the most important information is usually contained in the terms of the polynomial of highest degree. If $P_M(m,n)$ has degree less than or equal to r, we use $P_M^r(m,n)$ to denote the homogeneous component of P_M of degree r, and we use the same notation for the Hilbert polynomial of a bounded complex. If x is a homogeneous element of C of degree (i, j) then we denote by $K_\bullet(x)$ the Koszul complex on x, so that $K_0(x) = C$, $K_1(x) = C[-i,-j]$, $K_i(x) = 0$ for $i \neq 0$ or 1, and the map from $K_1(x)$ to $K_0(x)$ is multiplication by x.

PROPOSITION 2. *Let $F_\bullet$ be a bounded complex of bigraded modules, and let i, j, and k be integers between* 0 *and s,* 0 *and u, and* 1 *and v, respectively. Assume that the degree of $P_{F_\bullet}$ is at most r. Then the degrees of $P_{F_\bullet \otimes K_\bullet(X_i)}$, $P_{F_\bullet \otimes K_\bullet(T_j)}$, and $P_{F_\bullet \otimes K_\bullet(S_k)}$ are at most $r-1$, and we have*

$$P^{r-1}_{F_\bullet \otimes K_\bullet(X_i)} = \frac{\partial P^r_{F_\bullet}}{\partial m},$$

$$P^{r-1}_{F_\bullet \otimes K_\bullet(T_j)} = \frac{\partial P^r_{F_\bullet}}{\partial m} + \frac{\partial P^r_{F_\bullet}}{\partial n},$$

$$P^{r-1}_{F_\bullet \otimes K_\bullet(S_k)} = \frac{\partial P^r_{F_\bullet}}{\partial n}.$$

Proof. We prove the second statement; the other two are proven in the same way. The degree of T_j is $(1,1)$. Thus the modules in the complex $F_\bullet \otimes K(T_j)$ consist of a copy of the modules in $F_\bullet$ together with a second copy of the modules of $F_\bullet$ but with the position in the complex shifted by 1 and the degrees of the graded modules shifted by $(-1,-1)$. Thus we have

$$P_{F_\bullet \otimes K_\bullet(T_j)}(m,n) = P_{F_\bullet}(m,n) - P_{F_\bullet}(m-1,n-1).$$

To complete the proof it suffices to show that, if $G(m,n)$ is any polynomial of degree at most r in two variables m and n, then (a) the polynomial $G'(m,n) = G(m,n) - G(m-1,n-1)$ has degree at most $r-1$ and (b) the component of degree $r-1$ is $\partial G^r/\partial m + \partial G^r/\partial n$. To see this, it suffices to check the formula for a monomial of degree r. A simple computation shows that

$$\begin{aligned} m^i n^j - (m-1)^i(n-1)^j &= i m^{i-1} n^j + j m^i n^{j-1} + \text{lower-degree terms} \\ &= \frac{\partial(m^i n^j)}{\partial m} + \frac{\partial(m^i n^j)}{\partial n} + \text{lower-degree terms.} \end{aligned}$$

This completes the proof. □

We state a similar proposition for the degree of $P_M(m,n)$ in the variable n.

PROPOSITION 3. *Let $F_\bullet$ be a bounded complex of bigraded modules, and let k be an integer between* 1 *and* v. *Assume that the degree of $P_{F_\bullet}$ in n is r. Then the degree of $P_{F_\bullet \otimes K_\bullet(S_k)}$ in n is $r-1$.*

Proof. The proof of this result is the same as Proposition 2—using the fact that, if $G(m,n)$ is any polynomial of degree r in n, then the polynomial $G'(m,n) = G(m,n) - G(m,n-1)$ has degree $r-1$ in n. □

We note that if S_k, for example, is not a zero divisor on a module M, then Proposition 3 together with Proposition 1 implies that $P^{r-1}_{M/S_kM} = \partial P^r_M/\partial n$, provided that P_M has degree at most r.

Proposition 2 also makes it easy to compute the Tor modules with C/J that we need for the positivity criterion. Let J be the ideal of C generated by the elements $T_0, \dots, T_h, S_1, \dots, S_g$. Then we have the following proposition.

PROPOSITION 4. *Let $G_i(m,n)$ be the Hilbert polynomial of the bigraded module* $\mathrm{Tor}_i(M, C/J)$ *for each i, and let*

$$G(m,n) = \sum_{i \geq 0} (-1)^i G_i(m,n).$$

Let P_M denote the Hilbert polynomial of M. Assume that the degree r of P_M is at least $g+h+1$. Then

$$G^{r-g-h-1}(m,n) = \left(\frac{\partial}{\partial m} + \frac{\partial}{\partial n}\right)^{h+1} \left(\frac{\partial}{\partial n}\right)^{g} P^r_M(m,n).$$

This proposition follows from the fact that the resolution of C/J is a Koszul complex, which is a tensor product of the $K_\bullet(T_j)$ and the $K_\bullet(S_k)$, together with a repeated application of Proposition 2.

5. Hilbert Polynomials and Dimension

Just as in the classical case, the dimension of a bigraded module is given by the degree of the Hilbert polynomial. We now prove this fact, together with a similar result for a different type of dimension that we shall define.

We first specify the precise definition of dimension that we are using. As mentioned above, we can consider a bigraded module M as a $\mathbb{Z}$-graded module, and as such M defines a coherent sheaf on $\mathrm{Proj}(C)$, where C is given its $\mathbb{Z}$-grading as described in Section 3. By the dimension of M we mean the dimension of this coherent sheaf. Usually (if M has no components supported at the ideal generated by the X_i and the T_i), this dimension is one less than the Krull dimension of the module M. We note that if M_m is a finite-dimensional vector space over k for all m, then the theory of Hilbert polynomials of $\mathbb{Z}$-graded rings implies that the degree of its Hilbert polynomial is equal to the dimension of M in the sense we are considering.

We next introduce the second type of dimension that we will use. Let M be a bigraded module as before. Then M_m is a finitely generated $k[S_1, \dots, S_v]$-module for each m. Let $\mathfrak{a}_m$ be the annihilator of M_m. Since M is finitely generated, we have $\mathfrak{a}_m \subseteq \mathfrak{a}_{m+1}$ for large m, and since $k[S_1, \ldots, S_v]$ is Noetherian, we thus have that $\mathfrak{a}_m = \mathfrak{a}_{m+1}$ for large m. We let $\mathfrak{a} = \mathfrak{a}_m$ for large m, and we define the S-*dimension* of M to be the dimension of $k[S_1, \ldots, S_v]/\mathfrak{a}$; note that $\mathfrak{a}$ is a graded ideal of $k[S_1, \ldots, S_v]$. Equivalently, we can define the S-dimension of M to be the dimension of the $k[S_1, \ldots, S_v]$-module M_m for large m.

In the proof of Theorem 2 we wish to take a filtration of a bigraded module M with quotients of the form $C/Q[i, j]$, where Q is a bigraded prime ideal of C. This is, of course, a standard procedure; we prove that it works properly also for bigraded modules.

LEMMA 2. *Let M be a finitely generated bigraded module. Then there is a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of M consisting of bigraded modules such that, for each i, we have

$$M_i/M_{i-1} \cong C/Q_i[j_i, k_i]$$

for certain bigraded prime ideals Q_i of C and integers j_i and k_i.

Proof. It suffices to show that if $M \neq 0$ then there is a submodule M_1 of the correct form; the lemma follows by dividing by M_1 and repeating the process (it eventually stops since the module is Noetherian). Let Q be a maximal annihilator of a nonzero homogeneous element x of M. If a and b are homogeneous elements not in Q then, following a standard argument (see e.g. [8, Thm. 6.1]), we use the maximality of Q to conclude first that a does not annihilate x and then that b does not annihilate ax, so that $ab \notin Q$. We next show that Q is prime. If a and b are not in Q, we take the maximal nonzero homogeneous components $a_{i,j}$ and $b_{k,l}$ with respect to the lexicographic order on $\mathbb{Z} \times \mathbb{Z}$; by the preceding argument, their product is not in Q, so the maximality of the indices implies that the product of a and b is not in Q. $\square$

We note that it follows from this lemma that the associated prime ideals of M and, in particular, the minimal prime ideals in the support of M are bigraded. We refer to a module of the form $C/Q_i[j_i, k_i]$ in the filtration of dimension equal to the dimension of M as a *component* of M.

There are two other properties of bigraded modules that can be seen easily using Lemma 2. First, the Hilbert polynomial of a bigraded module is zero if and only if the associated coherent sheaf is zero. To see this, let M be of the form C/Q; the associated coherent sheaf is zero if and only if all the X_i and the T_i are in Q, and this holds if and only if the component $(C/Q)_m$ is zero for large m, which in turn holds if and only if the Hilbert polynomial of C/Q is zero.

The second property is that the S-dimension is always less than or equal to the dimension. Again we assume that $M = C/Q$. If the associated coherent sheaf is zero, then $(C/Q)_m = 0$ for large m and both dimensions are the dimension of the zero module. Assume that $(C/Q)_m \neq 0$ for large m, and let $\mathfrak{a} = Q \cap k[S_1, \ldots, S_v]$.

Since Q is the annihilator of every nonzero element of C/Q, it follows that the annihilator of the $k[S_1, \ldots, S_v]$-module $(C/Q)_m$ is $\mathfrak{a}$ for all m and hence the S-dimension is the dimension of $k[S_1, \ldots, S_v]/\mathfrak{a}$. Assume that $X_i \notin Q$ (the case where one of the T_i is not in Q is similar). Since the map from $k[S_1, \ldots, S_v]/\mathfrak{a}$ to $(C/Q)_{(X_i)}$ is injective, and since both rings are of finite type over the field k, the dimension of $k[S_1, \ldots, S_v]/\mathfrak{a}$ is less than or equal to the dimension of $(C/Q)_{(X_i)}$. Thus the S-dimension of C/Q is less than or equal to its dimension.

With a filtration as in Lemma 2, the dimension of M is the maximum of the dimensions of the C/Q_i, and similarly for the S-dimension. We need to know that the same properties hold for the degrees of the Hilbert polynomials.

PROPOSITION 5. *Let M be a bigraded module, and let M have a filtration with quotients $C/Q_i[j_i, k_i]$ as in Lemma 2.*

(1) *The degree of P_M is the maximum of the degrees of the P_{C/Q_i}.*
(2) *The degree of $P_M(m, n)$ in the variable n is the maximum of the degrees of the $P_{C/Q_i}(m, n)$ in the variable n.*

Proof. We note first that both the total degree and the degree in n are the same for $C/Q[i, j]$ as they are for C/Q. Since Hilbert polynomials are additive on short exact sequences, what must be shown is that the terms of highest degree on the modules in the filtration cannot cancel out. We show that this holds first for the degree in n. If the maximum value of the degree in n on any module in the filtration is t, suppose that this value is attained for C/Q_i, and let $\phi_t(m)$ be the polynomial in m that is the coefficient of n^t in the expansion of $P_{C/Q_i}(m, n)$ as a polynomial in n with coefficients that are polynomials in m. Since $\phi_t(m)$ is a nonzero polynomial, there exists an m_0 such that $\phi_t(m) \neq 0$ for $m \geq m_0$. Let m be an integer with $m \geq m_0$ and such that $P_M(m, n) = H_M(m, n)$ for n sufficiently large. Then

$$\lim_{n\to\infty} \frac{P_M(m, n)}{n^t} = \phi_t(m),$$

so this limit is nonzero. On the other hand,

$$\frac{P_M(m, n)}{n^t} = \frac{H_M(m, n)}{n^t} \geq 0$$

for sufficiently large n, so the limit cannot be negative. Hence $\phi_t(m) > 0$ for sufficiently large m. Since this is true for every module in the filtration with Hilbert polynomial of degree t in n, the sum must also have this property.

For the total degree, the proof is similar. Let u be the maximum value of the total degrees of modules in the filtration, and suppose that it is attained for C/Q_i. Then, for k sufficiently large, the polynomial in m given by $\phi(m) = P_{C/Q_i}(m, km)$ has degree u and so its leading coefficient (which is nonzero) is the coefficient of m^u. Representing this leading coefficient as a limit of

$$\frac{P_{C/Q_i}(m, km)}{m^u} = \frac{H_{C/Q_i}(m, km)}{m^u}$$

shows that it must be positive. Hence, as before, the leading polynomials cannot cancel and the total degree of $P_M(m, n)$ is u. □

THEOREM 2. *Let M be a bigraded module, and let P_M be its Hilbert polynomial. Then*:

(1) *the total degree of $P_M(m, n)$ is the dimension of M*;

(2) *the degree of $P_M(m, n)$ in n is the S-dimension of M; and*

(3) *if M has a component $C/Q[i, j]$ such that the S-dimension of C/Q is t and the total dimension of C/Q (which is equal to the total dimension of M) is $s + t$, then the coefficient of $m^s n^t$ in P_M is positive.*

Proof. We prove all three statements by induction on the dimension of M. For fixed dimension, we use induction on the S-dimension of M.

We first prove all three results when the S-dimension of M is zero. Using Proposition 5, we assume that $M = C/Q$ for a bigraded prime ideal Q. Let $\mathfrak{a}$ be the intersection of Q with $k[S] = k[S_1, \ldots, S_v]$. Since $\dim(k[S]/\mathfrak{a}) = 0$ and $\mathfrak{a}$ is a graded prime ideal, each S_i must be in $\mathfrak{a}$, so the S_i annihilate M. Hence M is a finitely generated module over $k[X, T]$. In this case, as shown in the proof of Theorem 1, the ordinary theory of Hilbert polynomials applies, and $P_M(m, n)$ is a polynomial in m of degree equal to the dimension of M. If this degree is s, then statement (3) says that the coefficient of m^s is positive; this is clear because the leading coefficient of a Hilbert polynomial of a $\mathbb{Z}$-graded module is always positive. Also, the degree in n is zero, which is the S-dimension. Hence all three statements hold in this case.

Assume now that the S-dimension of M is $t > 0$. Take a filtration as in Lemma 2. If every quotient in the filtration has either dimension or S-dimension less than that of M, then parts (1) and (2) of the theorem follow by induction and the third statement does not apply. Hence we may assume that $M = C/Q$, where the S-dimension of C/Q is t, and let the dimension of C/Q be $s + t$. It suffices to show that the degree of the Hilbert polynomial of C/Q is at most $s + t$, that the degree in n is at most t, and that the coefficient of $m^s n^t$ is positive. This will establish all three statements.

We first prove the two inequalities. Let $M = C/Q$ as before. If all the S_i were in Q, the S-dimension would be zero. Hence there is an i for which $S_i \notin Q$. Both the dimension and the S-dimension decrease by at least 1 when we replace M by M/S_iM, so by induction the degree of P_{M/S_iM} is at most the dimension of M/S_iM, and similarly for the S-dimension. By Proposition 3, the degree of P_{M/S_iM} in n is exactly one less than the degree of P_M; hence, denoting the degree in n by ndeg and the S-dimension by Sdim, we have the relations

$$\operatorname{ndeg}(P_M) - 1 = \operatorname{ndeg}(P_{M/S_iM}) \leq \operatorname{Sdim}(M/S_iM) \leq \operatorname{Sdim}(M) - 1.$$

Thus the degree of (P_M) in n is less than or equal to the S-dimension of M. We can argue similarly using M/S_iM to conclude the corresponding inequality for dimension, except in the case where the total degree of P_{M/S_iM} is strictly less than $\operatorname{degree}(P_M) - 1$. However, the only way this can happen is for the component of P_M of highest degree to be a power of m. In this case we can replace S_i by one of the X_i or T_i and, using Proposition 2, complete the argument as before.

Thus we may assume that $M = C/Q$, where the S-dimension of M is t and the total dimension is $s+t$. We need to show that the coefficient of $m^s n^t$ is positive. As before, we may assume that S_i is not a zero divisor on M. We claim that M/S_iM has a component with S-dimension $t-1$ and total dimension $s+t-1$. Since the rings involved are finitely generated over a field, every component of M/S_iM has total dimension $s+t-1$, so it suffices to show that there is a component with S-dimension equal to $t-1$. Let $\mathfrak{a}$ be $Q \cap k[S_1, \dots, S_v]$, and let $\mathfrak{p}$ be a prime ideal of $k[S_1, \dots, S_v]$ that is mimimal over $(\mathfrak{a}, S_i)$; the dimension of $k[S_1, \dots, S_v]/\mathfrak{p}$ is then $t-1$. Let S be the multiplicatively closed set $k[S_1, \dots, S_v] - \mathfrak{p}$. Then, for all m, the component of (M_S) of degree m is the localization $(M_m)_S$, which is not zero (we use here that $M = C/Q$ for some bigraded prime ideal Q). Since $(M_m)_S$ is a finitely generated module over the local ring $k[S_1, \dots, S_v]_S$ and since S_i is in the maximal ideal of $k[S_1, \dots, S_v]_S$, this implies that $\mathfrak{p}$ is in the support of $(M/S_iM)_m$ for large m. Hence the S-dimension of M/S_iM, which is clearly less than or equal to $t-1$, is equal to $t-1$.

We now complete the proof. Let $r = s+t$, and let $P_M^r(m, n)$ be the component of the Hilbert polynomial of M of degree r. We have shown that the degree of P_M is at most r. By Proposition 2, the component of degree $r-1$ of the Hilbert polynomial of M/S_iM is $\partial P_M^r/\partial n$. We have also shown that M/S_iM has a component of S-dimension $t-1$ and total dimension $s+t-1$. By induction, we have that the coefficient of $m^s n^{t-1}$ in $P_{M/S_iM}(m, n)$ is positive. Thus the coefficient of $m^s n^t$ in P_M must be positive. $\square$

We conclude this section by recalling a standard fact that relates Hilbert polynomials to Euler characteristics of coherent sheaves.

LEMMA 3. *Let $\mathcal{F}$ be a coherent sheaf on* $\mathrm{Proj}(A)$, *where A is a $\mathbb{Z}$-graded ring over a field. If $\mathcal{F}$ is defined by a graded module M, then for all n in $\mathbb{Z}$ we have*

$$\chi(\mathcal{F}(n)) = P_M(n).$$

Proof. See for example Hartshorne [4, Ex. III.5.2].

6. A Criterion for Positivity

In this section we prove the criterion (mentioned in the introduction) for intersection multiplicities to be positive.

THEOREM 3. *Let $C = k[X_0, \dots, X_s, T_0, \dots, T_u, S_1, \dots, S_v]$, and let Q be a bigraded prime ideal of C. Assume that the dimension of C/Q is $u+v+1$, and let $J = (T_0, \dots, T_u, S_1, \dots, S_v)$. Then the following are equivalent.*

(1) $\chi(C/Q, C/J) > 0$.
(2) *The $m^{u+1}n^v$ coefficient of the Hilbert polynomial $P_{C/Q}$ is positive.*
(3) $Q \cap k[S_1, \dots, S_v] = 0$.

Proof. By Theorem 2, the degree of the Hilbert polynomial of C/Q is $u+v+1$. Let $G(m, n)$ be the alternating sum of the Hilbert polynomials of the $\mathrm{Tor}_i(C/Q, C/J)$. By Proposition 4, $G(m, n)$ is constant and its value is

$$\left(\frac{\partial}{\partial m}+\frac{\partial}{\partial n}\right)^{u+1}\left(\frac{\partial}{\partial n}\right)^{v} P_{C/Q}(m,n).$$

(We note that Proposition 4 gives this formula with $P_M^{u+v+1}(m,n)$ instead of $P_M(m,n)$, but in this case, since the degree of $P_{C/Q}$ is $u+v+1$, all components of lower degree vanish after applying the partial derivatives.) Since the S-dimension of C/Q is at most v, the degree of $P_{C/Q}$ in n is at most v. If we first apply $\left(\frac{\partial}{\partial n}\right)^{v}$ to $P_{C/Q}(m,n)$ we thus obtain a polynomial in m of degree at most $u+1$. If we then apply $\left(\frac{\partial}{\partial m}+\frac{\partial}{\partial n}\right)^{u+1}$, which is the same as $\left(\frac{\partial}{\partial m}\right)^{u+1}$ for polynomials in m, we end up with $v!(u+1)!$ times the coefficient of $m^{u+1}n^{v}$ in $P_{C/Q}(m,n)$. Using parts (2) and (3) of Theorem 2, we see that this coefficient is positive if and only if the S-dimension of C/Q is v; this, in turn, is equivalent to the condition that $Q\cap k[S_1,\ldots,S_v]=0$. On the other hand, by Proposition 3 the constant value of $G(m,n)$ is equal to $\chi(C/Q,C/J)$. Hence the three conditions are equivalent. □

The equivalence of (1) and (3) in this theorem appears in Kurano and Roberts [7]. It implies, as shown there, that if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of a regular local ring R with $\dim(R/\mathfrak{p})+\dim(R/\mathfrak{q})=\dim(R)$ and such that $R/\mathfrak{p}\otimes_R R/\mathfrak{q}$ has finite length, then the Serre positivity conjecture implies that $\mathfrak{p}^{(n)}\cap\mathfrak{q}\subseteq\mathfrak{m}^{n+1}$ for all n, where $\mathfrak{m}$ is the maximal ideal of R.

We conclude by giving another version of the third criterion and applying it to the situation that arises in considering the Serre positivity conjecture.

PROPOSITION 6. *Let $C=k[X_0,\ldots,X_n,T_0,\ldots,T_n,S_1,\ldots,S_d]$, and let K be a bigraded ideal of C such that the dimension of C/Q is equal to $n+d+1$ for all minimal prime ideals Q containing K. Let*

$$\bar{K}=\{c\in C \mid (X_0,\ldots,X_n)^k c\subseteq K \text{ for some } k\}.$$

Then the following statements are equivalent.

(1) $\bar{K}\cap k[S_1,\ldots,S_d]=0$.
(2) *There exists a component C/Q of C/K such that some X_i is not in Q and $Q\cap k[S_1,\ldots,S_d]=0$.*
(3) $\chi(C/K,C/J)>0$.

Proof. The equivalence of the second and third statements follows from Theorem 3 together with the fact that, if Q is a prime ideal containing K and either the dimension of C/Q is less than $n+d+1$ or the S-dimension is less than d, then $\chi(C/Q,C/J)=0$. To see that statement (1) implies (2), suppose that every minimal prime ideal Q over K contains all the X_i or a nonzero element α_Q of $k[S_1,\ldots,S_d]$. Let α be the product of the α_Q. Then the product of all minimal primes contains the product of α with a power of X_i for each i. Since the product of minimal prime ideals is nilpotent modulo K, a power of α will then be in $\bar{K}$. Conversely, suppose that some minimal prime Q meets $k[S_1,\ldots,S_v]$ trivially and that there exists an X_i not in Q. Then, for all $\alpha\neq 0$ in $k[S_1,\ldots,S_v]$, $X_i^k\alpha\notin Q$ and hence $X_i^k\alpha\notin K$. Thus $\bar{K}\cap k[S_1,\ldots,S_v]=0$. □

We note that the hypotheses of Proposition 6 hold in the situation arising from the Serre multiplicity conjectures, since they are constructed from graded rings over an integral domain.

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