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Invariant Vector Bundles of Rank 2 on Hyperelliptic Curves

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1. Introduction

In classical projective geometry, the Segre cubic 3-fold $\mathbf{\Sigma}$ has been extensively studied in Baker [1] and Coble [4]. It is the GIT quotient $(\mathbb{P}^1)^6 /\!/ \mathrm{PGL}(2,\mathbb{C})$ of $(\mathbb{P}^1)^6$ by the diagonal action of $\mathrm{PGL}(2,\mathbb{C})$ for the natural linearization on the line bundle $\boxtimes_{i=1}^6 \mathcal{O}_{\mathbb{P}^1}(1)$. It has been shown in Baker [1] and Coble [4] that the Segre cubic 3-fold arises on considering the linear system of quadrics in $\mathbb{P}^3$ that pass through five points in general position. The variety $\mathbf{\Sigma}$ thus embedded in $\mathbb{P}^4$ as a cubic hypersurface is actually the blow-up of $\mathbb{P}^3$ at these points, but with the proper transform of all lines joining any two points blown down to the ten nodes of $\mathbf{\Sigma}$. A general point $\omega \in \mathbf{\Sigma}$ of the Segre cubic 3-fold can obviously be interpreted as a curve $C = C_\omega$ of genus $g = 2$ with level 2-structure. Indeed, Van der Geer [12] showed that the variety dual to $\mathbf{\Sigma}$, which is a quartic 3-fold, can be identified with the Satake compactification of the moduli space $\mathcal{M}_{2,2}$ of smooth projective curves of genus $g = 2$ with level 2-structure.

A beautiful classical theorem (see [1; 4]) states that if $\omega \in \mathbf{\Sigma}$ is a general point then the *apparent contour*—namely, the locus of points of contact of tangent to $\mathbf{\Sigma}$ from this point ω—is the Kummer surface $\mathrm{Kum}(C)$ of the curve $C = C_\omega$ associated to $\omega \in \mathbf{\Sigma}$. In other words, the projection from the point ω maps $\mathbf{\Sigma}$ as a $2:1$ covering of $\mathbb{P}^3$ with Kummer surface $\mathrm{Kum}(C)$ as its branch locus and the apparent contour as its ramification locus. The composition of the birational map $\mathbb{P}^3 \dashrightarrow \mathbf{\Sigma}$ and the $2:1$ rational map $\mathbf{\Sigma} \dashrightarrow \mathbb{P}^3$ yields a $2:1$ rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, which is induced by the quadrics passing through six points in $\mathbb{P}^3$ in general position. The ramification locus of this rational map is called the *Weddle surface.* The Weddle surface with six nodes is a birational model of the Kummer surface. A nice modern account of these results may be found in the book by Dolgachev and Ortland [8].

The aim of this paper is to generalize all this beautiful geometry to higher dimensions. For $g \geq 2$, we consider the GIT quotient $(\mathbb{P}^1)^{2g+2} /\!/ G$ of $(\mathbb{P}^1)^{2g+2}$ by the diagonal action of $G = \mathrm{PGL}(2,\mathbb{C})$ for the natural G-linearization on the line bundle $\mathcal{L} = \boxtimes_{i=1}^{2g+2} \mathcal{O}_{\mathbb{P}^1}(1)$; we call it a *generalized Segre variety* or the *Segre g-variety* $\mathbf{\Sigma}_g$. We show that the Segre g-variety $\mathbf{\Sigma}_g$ is obtained by the linear system Ω of g-forms on $\mathbb{P}^{2g-1}$ that vanish with multiplicity $g-1$ through $2g+1$

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points $e_1, \dots, e_{2g+1}$ in general position (cf. Theorem 4.1). In other words, the rational map ι_Ω induced by Ω maps $\mathbb{P}^{2g-1}$ birationally onto $\mathbf{\Sigma}_g$.

A general point $\omega \in \mathbf{\Sigma}_g$ represents a hyperelliptic curve of genus g together with a special level-2 structure—namely, those given rise to by an ordering of the Weierstrass points (when $g = 2$, all level 2-structures arise in this way). If $e_0 \in \mathbb{P}^{2g-1}$ such that $\iota_\Omega(e_0) = \omega$, then we consider the partial linear system Λ of g-forms in Ω that vanish with multiplicity $g-1$ at all the $2g+2$ points $e_1, \dots, e_{2g+1}, e_0 = e_{2g+2}$. The projection of $\mathbf{\Sigma}_g$ into $|\Lambda|^*$ yields a rational map of degree 2 onto its image $\mathbf{S}^i$, a connected component of the moduli space of semistable vector bundles of rank 2 with trivial determinant over $C = C_\omega$, which are invariant under the hyperelliptic involution. Also, this rational map is branched precisely along the Kummer variety $\mathrm{Kum}(C)$ in $\mathbf{S}^i$ (see Theorem 4.2). This is the precise generalization of the classical relationship between the Segre cubic 3-fold and curves of genus $g = 2$ to higher dimension. Moreover, it establishes a connection between $\mathbf{\Sigma}_g$ and certain moduli spaces of invariant vector bundles of rank 2 on hyperelliptic curves.

A part of this generalization was carried out by Coble in his two papers [5; 7] and a survey article [6]. His aim was to find a higher-dimensional analog of the Weddle surface and study its geometry relative to the geometry of Kummer variety. Coble showed that the linear system Λ is the 2θ-linear system on the Jacobian of the hyperelliptic curve $C = C_\omega$ and that it induces a rational map of degree 2 onto its image, which is branched precisely along the Kummer variety; the ramification locus of this rational map is what Coble calls the *Weddle manifold*. We have given a modern account of the work of Coble and hope that this will lead to a better understanding of his work.

We now give a brief overview of this paper. First we discuss certain moduli spaces of semistable vector bundles of rank 2 on a hyperelliptic curve C of genus $g \geq 2$. Let $K = K_C$ and h be the *canonical* and *hyperelliptic line bundles* on C, respectively. Let $W = \{w_1, \dots, w_{2g+2}\}$ be an ordered set of all Weierstrass points of C. Set $w_0 = w_{2g+2}$. Then all extensions of the form $0 \to \mathcal{O}(-w_0) \to E \to K(w_0) \to 0$ are parameterized by $H^1(C, K^{-1} \otimes h^{-1})$ and hence there is a rational extension map $\varepsilon \colon \mathbf{P} = PH^1(C, K^{-1} \otimes h^{-1}) \dashrightarrow \mathrm{SU}_C(2, K)$, where $\mathrm{SU}_C(2, K)$ is the moduli space of semistable vector bundles of rank 2 and determinant K on the curve C. Bertram [3] showed that the rational map ε, even for C nonhyperelliptic, is induced by the linear system $H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}_{\mathbf{P}}(g))$, which is canonically isomorphic to the 2θ-linear system on the Jacobian $\mathrm{Pic}^{g-1}(C)$, where $\mathcal{I}_C$ is the ideal sheaf of C in $\mathbf{P}$ and $\mathrm{Pic}^{g-1}(C)$ is the space of all line bundles of degree $g-1$ on C. Since the line bundle $K^{-1} \otimes h^{-1}$ is invariant under the hyperelliptic involution $i \colon C \to C$, there is an involution on the cohomology group $H^1(C, K^{-1} \otimes h^{-1}) \simeq H^0(C, h^{2g-1})^*$. Let $\mathbf{P}^+$ be the linear subspace of $\mathbf{P}$ corresponding to the positive eigenspace for this involution. Then $\mathbf{P}^+$ is of dimension $2g-1$, that is, $\mathbf{P}^+ \simeq \mathbb{P}^{2g-1}$. Restricting the rational map ε to $\mathbf{P}^+$ yields a rational map $\varepsilon^+ \colon \mathbf{P}^+ \dashrightarrow \mathbf{S}^{\mathrm{inv}}$, where $\mathbf{S}^{\mathrm{inv}}$ is the i-invariant locus in $\mathrm{SU}_C(2, K)$. We showed that ε^+ is generically $2{:}1$ onto its image $\mathbf{S}^i$, a connected component in $\mathbf{S}^{\mathrm{inv}}$, and it is branched along the Kummer variety $\mathrm{Kum}(C) = \mathrm{Pic}^{g-1}(C)/\pm$ in $\mathbf{S}^i$ (see Corollary 2.1). Then in the next section we give another proof of a result of Coble that the linear system Λ is isomorphic to the 2θ-linear system $H^0(\mathrm{Pic}^{g-1}(C), \mathcal{O}(2\theta))$. In the last section, we established a relationship between Segre g-variety and hyperelliptic curves of genus g that generalizes the relationship between the Segre cubic 3-fold and curves of genus $g = 2$.

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2. Invariant Vector Bundles of Rank 2

Let E be an invariant vector bundle of rank 2 on a hyperelliptic curve C of genus $g \geq 2$. Let $j: E \to E$ be a lift of i-action to E. Then (E, j) is called a *vector bundle pair*. Two vector bundle pairs (E, j) and (E', j') are said to be *equivalent* if there is a vector bundle isomorphism $f: E \to E'$ such that $j' \circ f = f \circ j$. We say that the vector bundle pair (E, j) is *semistable* (resp., *stable*) if, for every j-invariant line subbundle F of E,

$$\deg(F) = \mu(F) \leq \mu(E) = \frac{\deg(E)}{2} \quad \text{(resp., } \mu(F) < \mu(E)).$$

Let $W = \{w_1, \ldots, w_{2g+2}\}$ be the ordered set of all Weierstrass points of C. Consider a vector bundle pair (E, j). Then, for every $w \in W$, $j_w: E_w \to E_w$ is an involution on the fiber E_w. Let $\mathbf{S}_0^i$ be the moduli space of semistable vector bundle pairs (E, j) of rank 2 on the hyperelliptic curve C with $\det(E) = K$ and trace $\mathrm{Tr}(j_w) = 0$ for all $w \in W$. The existence of the moduli space $\mathbf{S}_0^i$ follows from the work of Seshadri [11] on π-vector bundles.

Let $p: \mathbf{S}_0^i \to \mathbf{S}^{\mathrm{inv}}$ be the map given by $p((E, j)) = E$ and let $\mathbf{S}^i$ be the image of p. Then we show that $\mathbf{S}_0^i$ is a ramified double cover of $\mathbf{S}^i$.

THEOREM 2.1. *The map $p: \mathbf{S}_0^i \to \mathbf{S}^i$ given by $p((E, j)) = E$ is generically* 2 : 1 *with the Kummer variety* $\mathrm{Kum}(C)$ *in* $\mathbf{S}^i$ *as its branch locus.*

Proof. If (E, j) and (E, j') are two vector bundle pairs over $E \in \mathbf{S}^i$, then $j' = Aj$ for some $A \in \mathrm{Aut}(E)$. If E is stable, then $\mathrm{Aut}(E) \simeq \mathbb{C}^*$. Thus $j' = \pm j$ and so, for every stable bundle $E \in \mathbf{S}^i$, there are two nonequivalent vector bundle pairs (E, j), $(E, -j)$ over E. This shows that p is generically 2 : 1. Now the Kummer variety $\mathrm{Kum}(C)$ of the curve C is embedded in $\mathbf{S}^i$ by the map $\alpha \mapsto \alpha \oplus i^*\alpha$, and it corresponds to strictly semistable (i.e., semistable but not stable) bundles in $\mathbf{S}^i$. If $E = \alpha \oplus i^*\alpha$ for some $\alpha \in \mathrm{Pic}^{g-1}(C)$, then any two lifts of E in $\mathbf{S}_0^i$ are equivalent. $\square$

We claim that the rational extension map $\varepsilon^+\colon \mathbf{P}^+ \dashrightarrow \mathbf{S}^i$ lifts to the rational map $\bar{\varepsilon}\colon \mathbf{P}^+ \dashrightarrow \mathbf{S}_0^i$. For $v \in \mathbf{P}^+$, the two extensions $0 \to \mathcal{O}(-w_0) \to E_v \to K(w_0) \to 0$ and $0 \to \mathcal{O}(-w_0) \to i^*(E_v) \to K(w_0) \to 0$ are isomorphic, so E_v comes with a lift j_v of i-action. Thus (E_v, j_v) is a vector bundle pair. Also the trace $\mathrm{Tr}((j_v)_w) = 0$ for each $w \in W$. Since a generic extension is semistable, $(E_v, j_v) \in \mathbf{S}_0^i$ for a generic $v \in \mathbf{P}^+$. Thus we define a rational map $\bar{\varepsilon}\colon \mathbf{P}^+ \dashrightarrow \mathbf{S}_0^i$ by $\bar{\varepsilon}(v) = (E_v, j_v)$.

THEOREM 2.2. *The rational map $\bar{\varepsilon}\colon \mathbf{P}^+ \dashrightarrow \mathbf{S}_0^i$ is birational.*

Proof. It suffices to prove that, for a generic $(E, j) \in \mathbf{S}_0^i$, there exists a unique $v \in \mathbf{P}^+$ such that $\bar{\varepsilon}(v) = (E, j)$. Let Θ_0^i be the generalized theta divisor on $\mathbf{S}_0^i$; that is, $\mathrm{Supp}(\Theta_0^i) = \{(E, j) \in \mathbf{S}_0^i : H^0(C, E) \neq 0\}$. If $(E, j) \notin \Theta_0^i$ then, from the short exact sequence $0 \to E \to E(w_0) \to E(w_0)\big|_{w_0} \to 0$, we have $\dim(H^0(C, E(w_0))) \leq 2$. Since the Euler characteristic $\chi(E(w_0)) = 2$, we have $\dim(H^0(C, E(w_0))) = 2$. Then involution j on E induces an involution $\bar{j}$ on $H^0(C, E(w_0))$. Now, by the Atiyah–Bott fixed point theorem (see [2]), the trace $\mathrm{Tr}(\bar{j}) = 0$. Thus $\dim(H^0(C, E(w_0))^+) = \dim(H^0(C, E(w_0))^-) = 1$ and so, for each $(E, j) \notin \Theta_0^i$, there exists a unique extension $0 \to \mathcal{O}(-w_0) \to E \to K(w_0) \to 0$, where the inclusion $\mathcal{O}(w_0) \to E$ is induced by the unique invariant nonzero section of $E(w_0)$. Clearly, E and i^*E are the same as extensions. Hence there is a unique $v \in \mathbf{P}^+$ such that $\bar{\varepsilon}(v) = (E, j)$. □

COROLLARY 2.1. *The rational map $\varepsilon^+\colon \mathbf{P}^+ \dashrightarrow \mathbf{S}^i$ is generically $2:1$ with the Kummer variety $\mathrm{Kum}(C)$ in $\mathbf{S}^i$ as its branch locus.*

Proof. Since $\varepsilon^+ = p \circ \bar{\varepsilon}$, the proof follows from Theorems 2.1 and 2.2. □

3. 2θ-Linear System

In this section, we identify the 2θ-linear system on the Jacobian $\mathrm{Pic}^{g-1}(C)$ of a hyperelliptic curve C with the linear system $\Lambda_C = \Lambda$ on $\mathbf{P}^+ \simeq \mathbb{P}^{2g-1}$. From the canonical isomorphism $H^0(\mathrm{Pic}^{g-1}(C), \mathcal{O}(2\theta)) \simeq H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g))$, we obtain a linear map

$$\mathrm{res}\colon H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g)) \to H^0(\mathbf{P}^+, \mathcal{O}(g))$$

by restricting the sections of $H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g))$ to $\mathbf{P}^+$. We recall that the linear system $\Lambda_C = \Lambda$ consists of all the g-forms on $\mathbf{P}^+ \simeq \mathbb{P}^{2g-1}$ that vanish with multiplicity $g-1$ at the Weierstrass points $w_1, \ldots, w_{2g+2}$ in $\mathbf{P}^+$. We will prove that the mapping res induces an isomorphism between the 2θ-linear system $H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g))$ and the linear system Λ. But first we prove the following results.

LEMMA 3.1. *Let $Q \in H^0(\mathbb{P}^N, \mathcal{O}(n))$, and let A and B be any two distinct points on $\mathbb{P}^N$. Suppose the n-form Q vanishes with multiplicity l and m at A and B, respectively. Then Q vanishes along the line $\overline{AB}$ with multiplicity at least $l+m-n$. If $l+m-n \leq 0$, then the conclusion is vacuous.*

Proof. Let $r = l + m - n$. We need only consider the case $0 < r \leq l, m$. Let $\partial^{|r-1|}Q$ be a partial derivative of Q of order $r-1$. Then $\deg(\partial^{|r-1|}Q) = n - r + 1$ and $\partial^{|r-1|}Q$ vanishes with multiplicity $l - r + 1$ and $m - r + 1$ at A and B, respectively. Since $(l - r + 1) + (m - r + 1) = n - r + 2 > n - r + 1 = \deg(\partial^{|r-1|}Q)$, the line $\overline{AB}$ intersects $\partial^{|r-1|}Q = 0$ in a divisor greater than its degree $\deg(\partial^{|r-1|}Q)$. Hence $\partial^{|r-1|}Q$ vanishes identically on $\overline{AB}$. $\square$

COROLLARY 3.1. *Let $Q \in H^0(\mathbb{P}^N, \mathcal{O}(n))$. Let $\{u_i : i \in \Delta\}$ be a collection of finitely many points in $\mathbb{P}^N$ in general position such that Q vanishes with multiplicity $n-1$ at the u_i. Then $Q\big|_{P(I)} = 0$, where $P(I) = \langle u_i : i \in I\rangle \subset \mathbb{P}^N$ is the linear subspace spanned by u_i with $i \in I \subset \Delta$ and $\#(I) \leq n-1$.*

Proof. Let $\#(I) = r \leq n-1$. Then we claim that the n-form Q vanishes with multiplicity $n - r$ on $P(I)$. Using Lemma 3.1, this claim can be proved by induction on r. $\square$

REMARK. With notation as in Corollary 3.1, if $Q\big|_{P(J)} = 0$ for every $J \subset \Delta$ with $\#(J) = n$, then $Q\big|_{P(\Delta)} = 0$. By induction, one proves that $Q\big|_{P(H)} = 0$ for $H \subset \Delta$ with $\#(H) \geq n$. For instance, if $\#(H) = n+1$, then by assumption $Q\big|_{P(J)} = 0$ for every $J \subset H$ with $\#(J) = n$. Thus $Q\big|_{P(H)}$ is a product of $n+1$ hyperplanes in $P(H)$. Since Q is a n-form, it is absurd unless $Q\big|_{P(H)} = 0$.

LEMMA 3.2. *The linear map* $\mathrm{res}\colon H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g)) \to H^0(\mathbf{P}^+, \mathcal{O}(g))$ *is injective, and its image is contained in Λ.*

Proof. Let $Q \in H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g))$ be such that $\mathrm{res}(Q) = Q\big|_{\mathbf{P}^+} = 0$. Let $z_1, \ldots, z_g$ be any general g points on the hyperelliptic curve C in $\mathbf{P}$. Consider the g-secant $\mathbb{P}^{g-1} = \langle z_1, \ldots, z_g\rangle$ spanned by the z_i. Since the g-form Q vanishes on the curve with multiplicity $g-1$, by Corollary 3.1 it follows that the g-form $Q\big|_{\mathbb{P}^{g-1}}$ is (up to a constant factor) a product of g hyperplanes of the form $\mathbb{P}^{g-2} = \langle z_1, \ldots, \hat{z}_i, \ldots, z_g\rangle$ in $\mathbb{P}^{g-1}$. But $\mathbb{P}^{g-1} \cap \mathbf{P}^+ \neq \emptyset$ and, for a general g-secant $\mathbb{P}^{g-1}$, we may assume that $\mathbf{P}^+$ does not meet any of these hyperplanes $\mathbb{P}^{g-2}$ in $\mathbb{P}^{g-1}$. Since $Q\big|_{\mathbf{P}^+} = 0$ and $\mathbf{P}^+$ meets $\mathbb{P}^{g-1}$ in the complement of the hyperplanes just described, we must have $Q\big|_{\mathbb{P}^g} = 0$. Thus, the g-form Q vanishes on a general g-secant to the hyperelliptic curve C in $\mathbf{P}$. Since C is nondegenerate in $\mathbf{P}$, by the remark to Corollary 3.1 we have that Q is identically zero. This proves that the mapping res is injective. Also $C \cap \mathbf{P}^+ = W$, the set of all Weierstrass points of C in $\mathbf{P}$. Thus $\mathrm{res}(Q) \in \Lambda$. $\square$

REMARKS. (i) Since $\dim(H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g))) = 2^g$ and res is injective, we have $\dim(\Lambda) \geq 2^g$. Thus, to show that res is an isomorphism onto Λ, it is enough to prove that $\dim(\Lambda) \leq 2^g$.

(ii) Every $Q \in \Lambda$ vanishes with multiplicity $g-2$ on the rational normal curve S in $\mathbf{P}^+$ besides vanishing with multiplicity $g-1$ at the Weierstrass points (see [7, Thm. 1.4]).

LEMMA 3.3. *Let $\{u_i : i \in \Delta\}$ be a finite collection of points in $\mathbb{P}^{2g-1}$ in general position. Let Q be a n-form on $\mathbb{P}^{2g-1}$ for $n \leq g$. Suppose Q vanishes at u_i with multiplicity $n-1$ for $i \in \Delta$. Let $P(I) = \langle u_i : i \in I \rangle$ for $I \subset \Delta$. Let $\mathbb{P}^{2g-n}$ be a linear subspace of $\mathbb{P}^{2g-1}$ such that $\mathbb{P}^{2g-n} \cap P(I) = \emptyset$ for $I \subset \Delta$ with $\#(I) = n-1$. If $Q\big|_{\mathbb{P}^{2g-n}} = 0$, then $Q\big|_{P(\Delta)} = 0$.*

Proof. From Corollary 3.1, we may assume that $\#(\Delta) \geq n$. Also, in view of the remark to Corollary 3.1, it is enough to prove that $Q\big|_{P(J)} = 0$ for $J \subset \Delta$ with $\#(J) = n$. But again by Corollary 3.1, Q vanishes on hyperplanes $P(I)$ in $P(J)$, $I \subset J$, with $\#(I) = n-1$. Thus $Q\big|_{P(J)}$ is a product of n hyperplanes. Since $\mathbb{P}^{2g-n}$ intersects $P(J)$ in the complement of the hyperplanes $P(I)$ and since $Q\big|_{\mathbb{P}^{2g-n}} = 0$, we must have $Q\big|_{P(J)} = 0$. □

LEMMA 3.4. *Let $Q \in \Lambda$. Suppose $\{w_i : i \in \Delta\}$ is a subset of W and $\mathbb{P}^{g+r}$ is a linear subspace of $\mathbf{P}^+ \simeq \mathbb{P}^{2g-1}$ such that $\mathbb{P}^{g+r} \cap P(I) = \emptyset$ for $I \subset \{w_i : i \in \Delta\}$ with $\#(I) = g-r-1$. If $Q\big|_{\mathbb{P}^{g+r}} = 0$ then $Q\big|_{S^r(\Delta)} = 0$, where $S^r(\Delta) = \mathrm{Sec}^r(S) * P(\Delta)$ is the join of rth-order secant variety to the rational normal curve S in $\mathbf{P}^+$ and the linear space $P(\Delta)$. For $r = 0$, $S^0(\Delta) = P(\Delta)$.*

Proof. We proceed by an induction on r. For $r = 0$, it follows from Lemma 3.3 that $Q\big|_{P(\Delta)} = 0$. Thus, by induction we assume that $Q\big|_{S^{r-1}(\Delta)} = 0$. Now consider r general points $z_1, \ldots, z_r$ on S. Let $P(z_1, \ldots, z_r, \Delta) = z_1 * \cdots * z_r * P(\Delta)$. Then, by induction assumption, $Q\big|_{P(z_1,\ldots,z_r,\Delta)}$ is a product of r hyperplanes of the form $P(z_1, \ldots, \hat{z}_i, \ldots, z_r, \Delta)$ and a $(g-r)$-form Q' in $P(z_1, \ldots, z_r, \Delta)$. Since every $Q \in \Lambda$ vanishes with multiplicity $g-2$ along the rational normal curve S (see [7, Thm. 1.4]), the $(g-r)$-form Q' vanishes with multiplicity $g-r-1$ at $z_1, \ldots, z_r$ and w_i $(i \in \Delta)$. Because $z_1, \ldots, z_r$ are general points of S, it follows from Lemma 3.3 that $Q'\big|_{P(z_1,\ldots,z_r,\Delta)} = 0$. This implies that $Q\big|_{S^r(\Delta)} = 0$. □

We now proceed to show that the dimension of the linear system $\Lambda_C = \Lambda$ is 2^g. Let $I_n = \{1, \ldots, n\}$ for $n \leq 2g$ and let $P(I_n) = \langle w_i \in W : i \in I_n \rangle \subset \mathbf{P}^+$. Then $P(I_n) \simeq \mathbb{P}^{n-1}$ and we have a complete flag

$$P(I_1) \subset P(I_2) \subset \cdots \subset P(I_{2g}) \simeq \mathbf{P}^+$$

for the projective space $\mathbf{P}^+ \simeq \mathbb{P}^{2g-1}$. We define a decreasing filtration on Λ as follows. Let $F_k\Lambda = \left\{Q \in \Lambda : Q\big|_{P(I_{g+k-1})} = 0\right\}$ for $0 \leq k \leq g+1$. Since $Q \in \Lambda$ vanishes with multiplicity $g-1$ at $w \in W$, we have $Q\big|_{P(I_{g-1})} = 0$. Thus, $F_0\Lambda = \Lambda$; also, $F_k\Lambda \supset F_{k+1}\Lambda$ and $F_{g+1}\Lambda = 0$. Hence we have a finite decreasing filtration

$$\Lambda = F_0\Lambda \supset F_1\Lambda \supset \cdots \supset F_g\Lambda \supset F_{g+1}\Lambda = 0$$

of the linear system Λ. The associated graded linear space for this filtration is given by $\bigoplus_{k=0}^{g} \mathrm{Gr}_k\,\Lambda = \bigoplus_{k=0}^{g}(F_k\Lambda/F_{k+1}\Lambda)$. Therefore, $\dim(\Lambda) = \sum_{k=0}^{g} \dim(\mathrm{Gr}_k\,\Lambda)$. Let $\Lambda_k = \left\{Q\big|_{P(I_{g+k})} : Q \in F_k\Lambda\right\}$. Then we have a short exact sequence $0 \to F_{k+1}\Lambda \to F_k\Lambda \to \Lambda_k \to 0$, where $F_k\Lambda \to \Lambda_k$ is the natural restriction map. Thus $\dim(\mathrm{Gr}_k\,\Lambda) = \dim(\Lambda_k)$.

LEMMA 3.5. $\dim(\mathrm{Gr}_k\,\Lambda) \leq \binom{g}{g-k}$.

Proof. Since $\dim(\mathrm{Gr}_k\,\Lambda) = \dim(\Lambda_k)$, we show that $\dim(\Lambda_k) \le \binom{g}{g-k}$. For $I \subset I_g = \{1, \ldots, g\}$ with $\#(I) = g-k$, we define linear subspaces $P(I; g-k)$ of $P(I_{g+k})$ by $P(I; g-k) =$ span of $\{w_i : i \in I\}$ and $\{w_{g+1}, \ldots, w_{g+k}\}$. Then each $P(I; g-k)$ is isomorphic to a $\mathbb{P}^{g-1}$, and the number of such $P(I; g-k)$-subspaces is precisely $\binom{g}{g-k}$. Let $\Lambda_{P(I;g-k)} = \{Q\big|_{P(I;g-k)} : Q \in \Lambda\}$ and consider the natural restriction map $r\colon \Lambda_k \to \bigoplus_{\#(I)=g-k}(\Lambda_{P(I;g-k)})$, where the direct sum is taken over all $I \subset I_g$ with $\#(I) = g-k$. We claim that r is injective. Let $Q \in \Lambda_k$ be such that $r(Q) = 0$. Then $Q \in \Lambda$, $Q\big|_{P(I_{g+k-1})} = 0$, and $Q\big|_{P(I;g-k)} = 0$ for every $I \subset I_g$ with $\#(I) = g-k$.

We need to show that $Q\big|_{P(I_{g+k})} = 0$. For $k \le 1$ this is trivial, so assume that $k \ge 2$. Since $P(I_{g+k-1}) \simeq \mathbb{P}^{g+k-2}$ and $Q\big|_{\mathbb{P}^{g+k-2}} = 0$, we deduce from Lemma 3.3 that $Q\big|_{S^{k-1}(W')} = 0$, where $W' = \{w_i \in W : i \notin I_{g+k-1}\}$. Now consider $\mathbb{P}^g =$ span of $\{w_j;\, j \in J\}$ and $\{w_{g+1}, \ldots, w_{g+k}\}$, where $J \subset I_g$ with $\#(J) = g-k+1$. By assumption, $Q\big|_{P(I;g-k)} = 0$ for $I \subset J$ with $\#(I) = g-k$, and $Q\big|_{P(J;g-k+1)} = 0$ because $Q\big|_{P(I_{g+k-1})} = 0$. This shows that $Q\big|_{\mathbb{P}^g}$ is a product of $g-k+2$ hyperplanes and a $(k-2)$-form Q' on $\mathbb{P}^g$. Also, Q' vanishes with multiplicity $k-2$ at the points w_{g+k}, w_j $(j \in J)$ whereas it vanishes with multiplicity $k-3$ at the remaining points $w_{g+1}, \ldots, w_{g+k-1}$. This implies that Q' must be a cone over a $(k-2)$-form Q'' on $\mathbb{P}^{k-2} = \langle w_{g+1}, \ldots, w_{g+k-1}\rangle$. Now, for a general $k-2$ points $z_1, \ldots, z_{k-2} \in S$ we have $\mathbb{P}^g \cap P(z_1, \ldots, z_{k+2}, W'') \neq \emptyset$, where $W'' = W' - \{w_{g+k}\}$ and $P(z_1, \ldots, z_{k-2}, W'') \simeq \mathbb{P}^{g-1}$. Since $Q\big|_{S^{k-1}(W'')} = 0$, it follows that $Q\big|_{\mathbb{P}^g} = 0$ contains a $(k-2)$-dimensional subvariety of $\mathbb{P}^g$. The same is true for $Q'\big|_{\mathbb{P}^g} = 0$ and hence also for $Q'' = 0$ in $\mathbb{P}^{k-2}$, since Q' is a cone over Q''. Thus we must have $Q'' \equiv 0$, and so $Q\big|_{\mathbb{P}^g} \equiv 0$.

On similar lines, we can deduce that $Q\big|_{\mathbb{P}^{g+i}} \equiv 0$, where $\mathbb{P}^{g+i} =$ span of $\{w_j : j \in J\}$ and $\{w_{g+1}, \ldots, w_{g+k}\}$, and that $J \subset I_g$ with $\#(J) = g-k+1+i$. Thus, for $i = k-1$, we have $Q\big|_{P(I_{g+k})} = 0$ and hence r is injective. Now, in view of Corollary 3.1, $\dim(\Lambda_{P(I;g-k)}) \le 1$ and so $\dim(\Lambda_k) \le \dim\big(\bigoplus_{\#I=g-k}(\Lambda_{P(I;g-k)})\big) \le \binom{g}{g-k}$. $\square$

THEOREM 3.1 (Coble). *The linear system $\Lambda_C = \Lambda$ on $\mathbf{P}^+$ is isomorphic to the 2θ-linear system on the Jacobian* $\mathrm{Pic}^{g-1}(C)$ *of the hyperelliptic curve C.*

Proof. Since $\dim(H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g))) = 2^g$ and the linear map $\mathrm{res}\colon H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g)) \to \Lambda$ is injective, it follows that $\dim(\Lambda) \ge 2^g$. But from Lemma 3.5 we have $\dim(\Lambda) = \sum_{k=0}^{g} \dim(\mathrm{Gr}_k\,\Lambda) \le \sum_{k=0}^{g}\binom{g}{g-k} = 2^g$. Thus $\dim(\Lambda) = 2^g$ and res induces an isomorphism of the 2θ-linear system $H^0(\mathbf{P}, \mathcal{I}_C^{g-1} \otimes \mathcal{O}(g)) \simeq H^0(\mathrm{Pic}^{g-1}(C), \mathcal{O}(2\theta))$ with Λ. $\square$

REMARK. Since $\dim(\Lambda) = 2^g$, we have $\dim(\mathrm{Gr}_k\,\Lambda) = \binom{g}{g-k}$.

THEOREM 3.2. *The rational map $\iota_\Lambda : \mathbf{P}^+ \dashrightarrow |\Lambda|^*$ induced by the linear system $\Lambda_C = \Lambda$ is generically $2:1$ onto $\mathbf{S}^i$, and its branch locus is the Kummer variety* $\mathrm{Kum}(C)$ *in $\mathbf{S}^i$.*

Proof. From Theorem 3.1, the pull-back of the linear system $H^0(\mathbf{S}^i, \Theta^i)$, which is isomorphic to the 2θ-linear system $H^0(\mathrm{Pic}^{g-1}(C), \mathcal{O}(2\theta))$ under the rational map $\varepsilon^+ : \mathbf{P}^+ \dashrightarrow \mathbf{S}^i$, is isomorphic to the linear system Λ, where Θ^i is the generalized theta divisor on $\mathbf{S}^i$. Since $\mathbf{S}^i$ is embedded in the linear system $|\Theta^i|^*$, the rational map $\varepsilon^+ : \mathbf{P}^+ \dashrightarrow \mathbf{S}^i$ is induced by the linear system Λ. Now the theorem follows from Corollary 2.1. $\square$

4. Higher-Dimensional Segre Varieties

In this section we discuss a higher-dimensional analog of the Segre cubic 3-fold. As in Section 1, we consider the GIT quotient $(\mathbb{P}^1)^{2g+2}//G$ of $(\mathbb{P}^1)^{2g+2}$ by the diagonal action of $G = \mathrm{PGL}(2, \mathbb{C})$ for the natural G-linearization on $\mathcal{L} = \boxtimes_{i=1}^{2g+2} \mathcal{O}_{\mathbb{P}^1}(1)$ and call it the *Segre g-variety* $\mathbf{\Sigma}_g$.

Using the theory of associated point sets [8], we have a duality isomorphism

$$(\mathbb{P}^1)^{2g+2}//G \simeq (\mathbb{P}^{2g-1})^{2g+2}//G',$$

where $G' = \mathrm{PGL}(2g, \mathbb{C})$ acts diagonally on $(\mathbb{P}^{2g-1})^{2g+2}$ for the natural G'-linearization on the line bundle $\mathcal{M} = \boxtimes_{i=1}^{2g+2} \mathcal{O}_{\mathbb{P}^{2g-1}}(g)$. Moreover, we have $H^0(\mathcal{L})^G \simeq H^0(\mathcal{M})^{G'}$. Now let $e_1, \dots, e_{2g+1}$ be any $2g+1$ points in general position in $\mathbb{P}^{2g-1}$. Without loss of generality, we may assume that $e_j = [0 : \dots : 1 : \dots : 0]$ for $j = 1, \dots, 2g$ and $e_{2g+1} = [1 : \dots : 1]$. Then we define an inclusion $f : \mathbb{P}^{2g-1} \to (\mathbb{P}^{2g-1})^{2g+2}$ by $e \mapsto (e_1, \dots, e_{2g+1}, e)$. On composing f with the GIT quotient map and using the preceding duality isomorphism, we derive a rational map $\bar{f} : \mathbb{P}^{2g-1} \dashrightarrow \mathbf{\Sigma}_g$. Any two general points $t = (t_1, \dots, t_{2g+2}) \in (\mathbb{P}^1)^{2g+2}$ and $z = (z_1, \dots, z_{2g+2}) \in (\mathbb{P}^{2g-1})^{2g+2}$ are associated to each other under the above duality isomorphism if and only if there is a rational normal curve $\gamma : \mathbb{P}^1 \to \mathbb{P}^{2g-1}$ such that $\gamma(t_j) = z_j$ for $1 \le j \le 2g+2$ (see [8]). Any $2g+1$ points in general positions in $\mathbb{P}^{2g-1}$ can be mapped to $e_1, \dots, e_{2g+1}$ by an automorphism T of $\mathbb{P}^{2g-1}$, so if $T(\gamma(t_{2g+2})) = e$ then $\bar{f}(e)$ is the image of $t = (t_1, \dots, t_{2g+2})$ under the GIT quotient map. For a general point $e \in \mathbb{P}^{2g-1}$, there is a unique rational normal curve through $e_1, \dots, e_{2g+1}, e$. This shows that the rational map $\bar{f} : \mathbb{P}^{2g-1} \dashrightarrow \mathbf{\Sigma}_g$ is birational.

Let Ω be the linear system of g-forms on $\mathbb{P}^{2g-1}$ that vanish with multiplicity $g-1$ at $2g+1$ points $e_1, \dots, e_{2g+1}$ in $\mathbb{P}^{2g-1}$. We then show that the rational map $\bar{f}$ is induced by the linear system Ω.

THEOREM 4.1. *The linear system Ω on $\mathbb{P}^{2g-1}$ is isomorphic to $H^0(\mathcal{L})^G$, and the rational map $\iota_\Omega : \mathbb{P}^{2g-1} \dashrightarrow |\Omega^*|$ induced by the linear system Ω is birational onto $\mathbf{\Sigma}_g$.*

Proof. We consider the birational map $\bar{f} : \mathbb{P}^{2g-1} \dashrightarrow \mathbf{\Sigma}_g$ induced by the above duality isomorphism. By the Hilbert–Mumford numerical criterion for semistability (see [10]), we check that the indeterminacy locus of $\bar{f}$ consists of all the $(g-1)$-planes $\langle e_{j_1}, \dots, e_{j_g} \rangle$ spanned by e_j $(j = 1, \dots, 2g+1)$. The Segre g-variety $\mathbf{\Sigma}_g$ embeds in $P(H^0(\mathcal{L})^G)^*$ and so, for a section $s \in H^0(\mathbf{\Sigma}_g, \mathcal{O}_{\mathbf{\Sigma}_g}(1)) \simeq H^0(\mathcal{L})^G$, the pull-back section $\bar{f}^*(s) \in H^0(\mathbb{P}^{2g-1}, \mathcal{O}_{\mathbb{P}^{2g-1}}(g))$ is a g-form that vanishes on the indeterminacy locus of $\bar{f}$. In other words, the g-form $\bar{f}^*(s)$ vanishes on all the

$(g-1)$-planes spanned by the e_j. But these conditions are equivalent to the condition that $\bar{f}^*(s)$ vanish with multiplicity $g-1$ at the $2g+1$ points $e_1, \ldots, e_{2g+1}$. Thus the pull-back $\bar{f}^*$ yields a linear map $\rho\colon H^0(\mathcal{L})^G \to \Omega$. Since $\bar{f}$ is birational, ρ is nontrivial. Hence, to complete this proof we need only show that ρ is an isomorphism.

We now compute the dimension of Ω. Let $\mathbb{N}_k = \{1, \ldots, k\}$, $\mathcal{R} = \{I \subset \mathbb{N}_{2g} : \#(I) = g\}$, and $\mathcal{C} = \{J \subset \mathbb{N}_{2g} : \#(J) = g-2\}$, and let x_I denote the monomial $x_{i_1} \ldots x_{i_g}$ if $I = \{i_1, \ldots, i_g\}$. The g-form Q vanishes with multiplicity $g-1$ at the points $e_1, \ldots, e_{2g}$ if and only if it is expressed as $Q = \sum_{I \in \mathcal{R}} a_I x_I$ with $a_I \in \mathbb{C}$. If Q also vanishes with multiplicity $g-1$ at e_{2g+1} then we have the condition that, for each $J \in \mathcal{C}$, $\sum_{J \subset I \in \mathcal{R}} a_I = 0$. Therefore

$$\Omega = \left\{ \sum_{I \in \mathcal{R}} a_I x_I : \sum_{J \subset I \in \mathcal{R}} a_I = 0 \ \forall J \in \mathcal{C} \right\}.$$

The incidence matrix $(\lambda_{IJ})_{I \in \mathcal{R}, J \in \mathcal{C}}$, given by $\lambda_{IJ} = 1$ if $J \subset I$ and $\lambda_{IJ} = 0$ if $J \not\subset I$, is of maximal rank, so all conditions among the generators $\{x_I : I \in \mathcal{R}\}$ of the linear system Ω are independent. Thus $\dim(\Omega) = \#(\mathcal{R}) - \#(\mathcal{C}) = \binom{2g}{g} - \binom{2g}{g-2}$.

Now let $\mathcal{W}_k$ be the symmetric group on k symbols. We recall that $H^0(\mathcal{L})^G$ is an irreducible $\mathcal{W}_{2g+2}$-module corresponding to the Young tableau consisting of 2-rows and $(g+1)$-columns; by the Hook length formula, $\dim(H^0(\mathcal{L})^G) = \frac{(2g+2)!}{(g+2)!\,(g+1)!}$ (see [8]). Forgetting the last symbol, $H^0(\mathcal{L})^G$ is also an irreducible $\mathcal{W}_{2g+1}$-module. For every $\sigma \in \mathcal{W}_{2g+1}$, there is a unique automorphism T_σ of $\mathbb{P}^{2g-1}$ such that $T_\sigma(e_j) = e_{\sigma(j)}$ for $j = 1, \ldots, 2g+1$. Now the maps $Q \mapsto T_\sigma^*(Q)$ for $\sigma \in \mathcal{W}_{2g+1}$ define an action of $\mathcal{W}_{2g+1}$ on Ω, and it can be checked that ρ is equivariant for these $\mathcal{W}_{2g+1}$-actions. Since $H^0(\mathcal{L})^G$ is an irreducible $\mathcal{W}_{2g+1}$-module, ρ must be injective. Also, since $\dim(H^0(\mathcal{L})^G) = \dim(\Omega)$, ρ must be an isomorphism. □

A general point on the Segre g-variety $\omega \in \mathbf{\Sigma}_g$ represents a hyperelliptic curve $C = C_\omega$ with a special level 2-structure as mentioned in Section 1. If $e_0 \in \mathbb{P}^{2g-1}$ such that $\iota_\Omega(e_0) = \omega$, then we consider the linear system Λ of g-forms on $\mathbb{P}^{2g-1}$ that pass with multiplicity $g-1$ through $2g+2$ points $e_1, \ldots, e_{2g+1}, e_0 = e_{2g+2}$. Then Λ is a partial linear system of Ω. We can identify $\mathbb{P}^{2g-1}$ with $\mathbf{P}^+$ by a unique projective transformation taking e_i to w_i for $i = 1, \ldots, 2g+2$. In view of Theorem 3.2, we now have our main theorem.

THEOREM 4.2. *The Segre g-variety $\mathbf{\Sigma}_g$ embeds in the projective space $|\Omega|^*$. Projecting $\mathbf{\Sigma}_g$ into the linear system $|\Lambda|^*$ yields a rational map of degree 2 onto its image $\mathbf{S}^i$, and it is branched precisely along the Kummer variety* $\mathrm{Kum}(C)$ *in* $\mathbf{S}^i$.

Proof. By Theorem 4.1, the Segre g-variety $\mathbf{\Sigma}_g$ embeds into $|\Omega|^*$. Also, the linear system Λ corresponds to the linear system Λ_C under the foregoing identification of $\mathbb{P}^{2g-1}$ with $\mathbf{P}^+$. The result then follows from Theorem 3.2. □

As an application of Theorem 4.2, we give an alternative proof of a result of Narasimhan and Ramanan [9].

THEOREM 4.3 (Narasimhan–Ramanan). *The moduli space* $\mathrm{SU}_C(2, K)$ *is isomorphic to* $\mathbb{P}^3$ *for a smooth projective curve* C *of genus* $g = 2$.

Proof. For $g = 2$, $i^*E = E$ for all $E \in \mathrm{SU}_C(2, K)$; thus $\mathbf{S}^i = \mathrm{SU}_C(2, K)$. The Segre cubic 3-fold $\mathbf{\Sigma}$ is a cubic in $|\Omega|^* \simeq \mathbb{P}^4$, and projecting away from a general point $\omega \in \mathbf{\Sigma}$ yields a rational map of degree 2 from $\mathbf{\Sigma}$ onto $|\Lambda|^* \simeq \mathbb{P}^3$. Thus, from Theorem 4.2, we derive that $\mathbf{S}^i \simeq \mathbb{P}^3$. □

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