Apostol Algebras and Decomposition in Douglas Algebras

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1. Introduction

Let A be a function algebra. That is, A is a uniformly closed subalgebra of the space of continuous functions on a compact Hausdorff space Ω that separates the points in Ω and contains constant functions. We denote by $M(A)$ and ∂A the maximal ideal space and the Shilov boundary of A, respectively. We identify a function in A with its Gelfand transform on $M(A)$. For a point x in $M(A)$, there exists a probability measure μ_x on ∂A such that

$$\int_{\partial A} f\, d\mu_x = f(x) \quad \text{for every} \quad f \in A.$$

The measure μ_x is called a *representing measure* for the point x. We denote by $\operatorname{supp}\mu_x$ the closed support set for μ_x. There is a function algebra A and a point x in $M(A)$ that admits at least two representing measures. A representing measure μ_x is called a *Jensen measure* if

$$\log|f(x)| \le \int_{\partial A} \log|f|\, d\mu_x, \qquad f \in A.$$

It is known that for each $x \in M(A)$ there exists a Jensen measure in the set of representing measures for x. A closed subset E of ∂A is called a *peak set* for A if there exists $h \in A$ such that $h = 1$ on E and $|h| < 1$ on $\partial A \setminus E$. A nonempty intersection of peak sets is called a *weak* peak set. For $f \in A$ and a subset E of $M(A)$, let $f(E) = \{f(x);\ x \in E\}$. [4] is a nice reference for function algebras.

For f in A, there corresponds the multiplication operator T_f on A defined by $T_f g = fg$ for $g \in A$. For a function f in A, the operator T_f is called *decomposable* if, for every pair of open sets U and V covering the complex plane, there exist T_f-invariant closed linear subspaces A_U and A_V of A such that

$$\sigma(T_f|A_U) \subset U, \quad \sigma(T_f|A_V) \subset V, \quad \text{and} \quad A_U + A_V = A,$$

where $\sigma(T)$ denotes the spectrum of the operator T. We denote by $\operatorname{Dec}(A)$ the set of all functions f in A such that T_f is decomposable. $\operatorname{Dec}(A)$ is called an *Apostol algebra*. The subalgebra $\operatorname{Dec}(A)$ dates back to classical work of Apostol [1].

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On the other hand, Neumann [14] and Inoue and Takahasi [8] proved that any commutative Banach algebra $\mathfrak{A}$ contains a greatest regular closed subalgebra, denoted by $\mathrm{Reg}(\mathfrak{A})$. Moreover, Neumann [13] showed for a semisimple $\mathfrak{A}$ that $f \in \mathrm{Dec}(\mathfrak{A})$ if and only if f is hull-kernel continuous on $M(A)$ and $\mathrm{Reg}(\mathfrak{A}) \subset \mathrm{Dec}(\mathfrak{A})$. By this fact, $\mathrm{Dec}(\mathfrak{A})$ is a closed subalgebra of $\mathfrak{A}$. Since function algebras are semisimple commutative Banach algebras, these results hold for them.

It is an interesting problem to identify $\mathrm{Dec}(A)$ and $\mathrm{Reg}(A)$ respectively for every function algebra A. For a function f in A, we say that f has *natural spectrum* if

$$f(M(A)) = f(\partial A).$$

Note that the definition of natural spectrum is analogous to the standard definition of that in abstract harmonic analysis. Let

$$\mathrm{Ns}(A) = \{f \in A;\ f \text{ has natural spectrum}\}.$$

Since ∂A is hull-kernel dense in $M(A)$, we have $\mathrm{Dec}(A) \subset \mathrm{Ns}(A)$. Hence

$$A \cap \bar{A} \subset \mathrm{Reg}(A) \subset \mathrm{Dec}(A) \subset \mathrm{Ns}(A),$$

where $\bar{A}$ is the set of complex conjugates of functions in A.

In Section 2, we prove that a function $f \in \mathrm{Dec}(A)$ is constant on the support set for every Jensen measure. In the rest of this paper, A_0 denotes a function algebra that satisfies

(#) each point in $M(A_0)$ has a unique representing measure on ∂A_0.

Under this assumption, every representing measure is a Jensen measure. Let

$$\mathrm{Cos}(A_0) = \{f \in A_0;\ f|\operatorname{supp}\mu_x \text{ is constant for every } x \in M(A_0)\}.$$

Then $\mathrm{Cos}(A_0)$ is a closed subalgebra of A_0, and by the aforementioned result we have $\mathrm{Dec}(A_0) \subset \mathrm{Cos}(A_0)$. For each $f \in \mathrm{Cos}(A_0)$, we have $f(x) \in f(\operatorname{supp}\mu_x)$ for every $x \in M(A_0)$. For a function f in A_0, we say that f has *full range on support sets* if

$$f(x) \in f(\operatorname{supp}\mu_x) \quad \text{for every } x \in M(A_0).$$

Let

$$\mathrm{Frs}(A_0) = \{f \in A_0;\ f \text{ has full range on support sets}\}.$$

Then we have (see Corollary 2.2)

$$A_0 \cap \bar{A}_0 \subset \mathrm{Reg}(A_0) \subset \mathrm{Dec}(A_0) \subset \mathrm{Cos}(A_0) \subset \mathrm{Frs}(A_0) \subset \mathrm{Ns}(A_0).$$

When $\operatorname{supp}\mu_x = \partial A_0$ for every $x \in M(A_0) \setminus \partial A_0$, we have $\mathrm{Frs}(A_0) = \mathrm{Ns}(A_0)$. The class $\mathrm{Frs}(A_0)$ is fairly large and contains unfamiliar functions. But for the study of function algebras A_0, the classes $\mathrm{Ns}(A_0)$ and $\mathrm{Frs}(A_0)$ are interesting enough in their own rights. For instance, if $\mathcal{A}$ is the disk algebra then $\mathrm{Cos}(\mathcal{A})$ coincides with the set of all constant functions. We also see that $\mathrm{Ns}(\mathcal{A})$ coincides with $\mathrm{Frs}(\mathcal{A})$ and contains no nonconstant polynomials. On the other hand, there exists a function f in $\mathcal{A}$ such that $f(e^{i\theta})$, $0 \le \theta \le 2\pi$, gives a Peano curve that is contained in $\mathrm{Ns}(\mathcal{A})$; see [15; 16] for the existence of such an f. Generally, $\mathrm{Ns}(A_0) \ne \mathrm{Frs}(A_0)$ and $\mathrm{Ns}(A_0)$ and $\mathrm{Frs}(A_0)$ are not closed under addition. So we are interested in what the sums $\mathrm{Ns}(A_0)+\mathrm{Ns}(A_0)$ and $\mathrm{Frs}(A_0)+\mathrm{Frs}(A_0)$ are. We study these questions for certain function algebras A_0 on the unit circle.

Let L^∞ be the Banach algebra of essentially bounded measurable functions on the unit circle. For a function f in L^∞, we denote by $\|f\|_\infty$ the essential supremum norm. We denote by H^∞ the Banach algebra of boundary functions on ∂D of bounded analytic functions on the open unit disk D. Then $H^\infty \subset L^\infty$. By the corona theorem [2], $D \subset M(H^\infty)$ and D is dense in $M(H^\infty)$. A closed subalgebra B with $H^\infty \subset B \subset L^\infty$ is called a *Douglas algebra*. We can consider that $M(L^\infty) \subset M(B) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary of every Douglas algebra B. It is known that every Douglas algebra satisfies condition (#). Sarason (see [18]) showed that $H^\infty + C$ is a Douglas algebra and $M(H^\infty + C) = M(H^\infty) \setminus D$, where C is the space of continuous functions on ∂D. Let $QC_B = B \cap \bar{B}$. When $B = H^\infty + C$, we write $QC = QC_{H^\infty + C}$ in short. Let $QA = QC \cap H^\infty$. Then QA satisfies (#). References [5; 6; 18] are nice for H^∞ and Douglas algebras.

Let B be a Douglas algebra. In Section 3, we prove that $B = \mathrm{Ns}(QA)+\mathrm{Ns}(B)$ and $QA = \mathrm{Ns}(QA) + \mathrm{Ns}(QA)$. In Section 4, we show $B = \mathrm{Frs}(H^\infty) + \mathrm{Frs}(B)$. Also in Section 2, we prove that $\mathrm{Reg}(B) = \mathrm{Dec}(B) = QC_B$. We do not know whether $\mathcal{A} = \mathrm{Ns}(\mathcal{A}) + \mathrm{Ns}(\mathcal{A})$ for the disk algebra $\mathcal{A}$.

2. Apostol Algebras of Function Algebras

In this section, we prove the following theorem.

THEOREM 2.1. *Let A be a function algebra and $f \in \mathrm{Dec}(A)$. Let μ_x be a Jensen measure for a point x in $M(A)$. Then f is constant on* $\operatorname{supp}\mu_x$.

Proof. Let $f \in \mathrm{Dec}(A)$. To prove our assertion, suppose not. Then there exists a Jensen measure μ_x such that $f|\operatorname{supp}\mu_x$ is not constant. Then we may assume without loss of generality that there are two points a and b in $\operatorname{supp}\mu_x$ such that $\operatorname{Re} f(a) = 1$ and $\operatorname{Re} f(b) = -1$. Let $U = \{\operatorname{Re} z < 1/2\}$ and $V = \{\operatorname{Re} z > -1/2\}$. Since the multiplication operator T_f is decomposable on A, there are T_f-invariant closed subspaces A_U and A_V such that $\sigma(T_f|A_U) \subset U$, $\sigma(T_f|A_V) \subset V$, and $A_U + A_V = A$. Let $K = f^{-1}(U^c)$ and $L = f^{-1}(V^c)$. Then K (resp. L) is a compact neighborhood of a (resp. b), hence

$$\mu_x(K) > 0 \quad \text{and} \quad \mu_x(L) > 0. \tag{2.1}$$

Let $x_0 \in K$. Since $f(x_0) \in U^c$ and $\sigma(T_f|A_U) \subset U$, the operator $T_f|A_U - f(x_0)I$ is invertible on A_U, where I denotes the identity operator on A_U. For every $h \in A_U$, we have

$$(T_f|A_U - f(x_0)I)(h)(x_0) = 0.$$

Since $T_f|A_U - f(x_0)I$ is a surjection, we have $h(x_0) = 0$ for every $h \in A_U$. It follows that

$$A_U \subset \{g \in A;\ g|K = 0\}. \tag{2.2}$$

In the same way, we have

$$A_V \subset \{g \in A;\ g|L = 0\}. \tag{2.3}$$

Since μ_x is a Jensen measure, we have

$$\log|g(x)| \le \int_{\partial A} \log|g|\, d\mu_x \quad \forall g \in A.$$

Since $A_U + A_V = A$, there exist functions $1_U \in A_U$ and $1_V \in A_V$ such that $1 = 1_U + 1_V$. By (2.1), (2.2), and (2.3), we have

$$\log|1_U(x)| \le \int_{\partial A} \log|1_U|\, d\mu_x = -\infty,$$

$$\log|1_V(x)| \le \int_{\partial A} \log|1_V|\, d\mu_x = -\infty.$$

So $1_U(x) = 0$ and $1_V(x) = 0$, which is a contradiction. □

COROLLARY 2.2. *Let A_0 be a function algebra such that every point in $M(A_0)$ has a unique representing measure on ∂A_0. Then*

$$A_0 \cap \bar{A}_0 \subset \mathrm{Reg}(A_0) \subset \mathrm{Dec}(A_0) \subset \mathrm{Cos}(A_0) \subset \mathrm{Frs}(A_0) \subset \mathrm{Ns}(A_0).$$

Proof. Since $A_0 \cap \bar{A}_0$ is a C^*-subalgebra of A_0, $A_0 \cap \bar{A}_0$ is a regular algebra. Hence $A_0 \cap \bar{A}_0 \subset \mathrm{Reg}(A_0)$; $\mathrm{Dec}(A_0) \subset \mathrm{Cos}(A_0)$ follows from Theorem 2.1. □

Let $\mathcal{A}$ be the disk algebra. Then

$$\mathcal{A} \cap \bar{\mathcal{A}} = \mathrm{Reg}(\mathcal{A}) = \mathrm{Dec}(\mathcal{A}) = \mathrm{Cos}(\mathcal{A}) \subset \mathrm{Ns}(\mathcal{A}),$$

and $\mathrm{Dec}(\mathcal{A})$ consists of the constant functions. But $\mathrm{Ns}(\mathcal{A})$ contains a nonconstant function that is not a polynomial.

Next, we study the case of Douglas algebras. For a Douglas algebra B, by the Chang–Marshall theorem [3; 12] we have that

$$B = \{f \in L^\infty;\ f|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x \text{ for every } x \in M(B)\}.$$

It is not difficult to see that there are no nonconstant real functions in $H^\infty|\mathrm{supp}\,\mu_x$. By this fact, we have the next lemma.

LEMMA 2.3. *Let B be a Douglas algebra. Then $QC_B = \mathrm{Cos}(B)$.*

Hence, by Corollary 2.2, we have the following.

COROLLARY 2.4. *Let B be a Douglas algebra. Then we have that*

$$QC_B = \mathrm{Reg}(B) = \mathrm{Dec}(B) = \mathrm{Cos}(B).$$

3. $B = \mathrm{Ns}(QA) + \mathrm{Ns}(B)$ for Douglas Algebras B

Let $\{z_n\}_n$ be a sequence in D with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$. Then the function b defined by

$$b(z) = \prod_{n=1}^\infty \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D$$

is called the *Blaschke product* with zeros $\{z_n\}_n$. Then $b \in H^\infty$ and $|b| = 1$ a.e. on ∂D. A function f in H^∞ with $|f| = 1$ a.e. on ∂D is called *inner.* A sequence $\{z_n\}_n$ in D is called *interpolating* if, for every bounded sequence $\{a_n\}_n$, there exists f in H^∞ such that $f(z_n) = a_n$ for every n. The associated Blaschke product is also called interpolating. For a subset E of $M(H^\infty)$, we denote by cl E the closure of E in $M(H^\infty)$.

Let $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$ and $QA = QC \cap H^\infty$. Then $QC = QA + C$, $M(QA) = M(QC) \cup D$, and $\partial QA = M(QC)$ (see [7]). Hence QA satisfies (#). By Lemma 2.3 we have

$$QC = \{f \in L^\infty;\ f|\operatorname{supp}\mu_x \text{ is constant for every } x \in M(H^\infty + C)\}$$

For $f \in QA$, we note that $f \in \mathrm{Ns}(QA)$ if and only if $f \in \mathrm{Ns}(H^\infty)$. For a point z in D, we have that $\operatorname{supp}\mu_z = M(QC)$. Hence $\mathrm{Frs}(QA) = \mathrm{Ns}(QA)$. For $z, w \in D$, let $\rho(z, w) = |(z - w)/(1 - \bar{z}w)|$. A sequence $\{z_n\}_n$ in D is called *sparse* if

$$\lim_{k\to\infty} \prod_{n\neq k} \rho(z_n, z_k) = 1.$$

A Blaschke product b is called sparse if the zeros of b form a sparse sequence.

In this section, we prove the following decomposition of Douglas algebras.

THEOREM 3.1. *Let B be a Douglas algebra. Then $B = \mathrm{Ns}(QA) + \mathrm{Ns}(B)$.*

To prove Theorem 3.1, we need some lemmas. For a closed subset E of the complex plane, we denote by ∂E the topological boundary of E.

LEMMA 3.2. *Let A be a function algebra. Then $\partial f(M(A)) \subset f(\partial A)$ for $f \in A$. Moreover, if $f(\partial A)$ is simply connected then $f \in \mathrm{Ns}(A)$.*

Proof. Let $a \in \partial f(M(A))$. Take a sequence of complex numbers $\{a_n\}_n$ such that $a_n \notin f(M(A))$ and $a_n \to a$. Since $a_n \notin f(M(A))$, it follows that $(f - a_n)^{-1} \in A$. Then there exists x_n in ∂A such that $|f(x_n) - a_n|^{-1} \geq |a - a_n|^{-1}$. Hence $f(x_n) \to a$. Thus $a \in f(\partial A)$.

Suppose that $f(\partial A)$ is simply connected. Since $f(\partial A) \subset f(M(A))$ and $\partial f(M(A)) \subset f(\partial A)$, we have $f(\partial A) = f(M(A))$. Hence $f \in \mathrm{Ns}(A)$. □

For $f \in L^\infty$, let

$$N(f) = \operatorname{cl} \bigcup \{\operatorname{supp}\mu_x;\ x \in M(H^\infty + C),\ f|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x\}$$

and

$$\hat{f}(x) = \int_{M(L^\infty)} f\, d\mu_x \quad \text{for } x \in M(H^\infty).$$

Then $\hat{f}$ is a continuous function on $M(H^\infty)$ [6, p. 93]. If $f \in B$ for some Douglas algebra B, then $\hat{f} = f$ on $M(B)$. Let $Z(f) = \{x \in M(H^\infty + C);\ \hat{f}(x) = 0\}$.

The following lemma was proved in [9, p. 296].

LEMMA 3.3. *Let b be a sparse Blaschke product. Then*
(i) *$N(\bar{b})$ is a weak peak set for QA, and*
(ii) *$N(\bar{b}) = \mathrm{cl} \bigcup \{\mathrm{supp}\, \mu_x;\ x \in Z(b)\}$.*

When $\hat{f}$ has a radial limit at $e^{i\theta} \in \partial D$, we denote it by $f^*(e^{i\theta})$. It is well known that $\hat{f}$ has a radial limit at almost every $e^{i\theta} \in \partial D$. For $e^{i\theta} \in \partial D$, let

$$R_{e^{i\theta}} = \mathrm{cl}\{re^{i\theta};\ 0 < r < 1\} \setminus \{re^{i\theta};\ 0 < r < 1\}.$$

Let $g_\theta(z) = e^{(z+e^{i\theta})/(z-e^{i\theta})}$. Then $g_\theta(re^{i\theta}) \to 0$ as $r \to 1$ and $|g_\theta| = 1$ on $M(L^\infty)$. Hence $R_{e^{i\theta}} \cap M(L^\infty) = \emptyset$ for every $e^{i\theta} \in \partial D$ and

$$\hat{f} = f^*(e^{i\theta}) \ \text{ on } R_{e^{i\theta}} \ \text{a.e. } \theta \in \partial D.$$

LEMMA 3.4. *Let $f \in L^\infty$. Then there is a countable set $\{e^{i\theta_n}\}_n$ in ∂D such that*
(i) *the radial limit $f^*(e^{i\theta_n})$ exists for every n,*
(ii) *$\{f^*(e^{i\theta_n})\}_n$ is dense in $f(M(L^\infty))$, and*
(iii) *$f = f^*(e^{i\theta_n})$ on $\bigcup\{\mathrm{supp}\, \mu_x;\ x \in R_{e^{i\theta_n}}\}$ for every n.*

Proof. Let $h \in L^\infty$. If the radial limit $h^*(e^{i\theta})$ exists at $e^{i\theta} \in \partial D$, then

$$\hat{h} = h^*(e^{i\theta}) \ \text{ on } R_{e^{i\theta}}. \tag{3.1}$$

Moreover, if $h \in QC$ then $h = h^*(e^{i\theta})$ on $\mathrm{supp}\, \mu_x$ for $x \in R_{e^{i\theta}}$, so that

$$h = h^*(e^{i\theta}) \ \text{ on } \bigcup\{\mathrm{supp}\, \mu_x;\ x \in R_{e^{i\theta}}\}. \tag{3.2}$$

Next, suppose that $h \in QC$ and $fh \in QC$. Let $e^{i\theta} \in \partial D$ such that the radial limits $h^*(e^{i\theta})$, $f^*(e^{i\theta})$, and $(fg)^*(e^{i\theta})$ exist and $h^*(e^{i\theta}) \neq 0$. Then (3.2) holds and

$$fh = (fh)^*(e^{i\theta}) \ \text{ on } \bigcup\{\mathrm{supp}\, \mu_x;\ x \in R_{e^{i\theta}}\}. \tag{3.3}$$

Since $h \in QC$, we have $(fh)\hat{} = \hat{f}\hat{h}$ on $M(H^\infty + C) = M(H^\infty) \setminus D$. Hence, by (3.1), $(fh)^*(e^{i\theta}) = f^*(e^{i\theta})h^*(e^{i\theta})$. Since $h^*(e^{i\theta}) \neq 0$, by (3.2) and (3.3) we have

$$f = f^*(e^{i\theta}) \ \text{ on } \bigcup\{\mathrm{supp}\, \mu_x;\ x \in R_{e^{i\theta}}\}. \tag{3.4}$$

Now we shall prove our assertion. We may assume that $f \neq 0$. By Wolff's theorem [21], there exists a function $h \in QA$ such that $fh \in QC$ and $h \neq 0$. Let Γ be the set of $e^{i\theta} \in \partial D$ at which the radial limits $h^*(e^{i\theta})$, $f^*(e^{i\theta})$, and $(fh)^*(e^{i\theta})$ exist. Then $d\theta/2\pi(\Gamma) = 1$ and $\{f^*(e^{i\theta});\ e^{i\theta} \in \Gamma\}$ is dense in $f(M(L^\infty))$. Since $h \in H^\infty$ and $h \neq 0$, we may assume that $h^*(e^{i\theta}) \neq 0$ for every $e^{i\theta} \in \Gamma$. We can now apply the second paragraph of the proof. Then, by (3.4), for every $e^{i\theta} \in \Gamma$ we have

$$f = f^*(e^{i\theta}) \ \text{ on } \bigcup\{\mathrm{supp}\, \mu_x;\ x \in R_{e^{i\theta}}\}.$$

Now take a countable set $\{e^{i\theta_n}\}_n$ in Γ such that $\{f^*(e^{i\theta_n})\}_n$ is dense in $f(M(L^\infty))$. We have completed the proof. □

Proof of Theorem 3.1. Let $f \in B$. We shall prove the existence of a function F in QA such that $F \in \mathrm{Ns}(QA)$ and $f - F \in \mathrm{Ns}(B)$. We may assume that $\|f\|_\infty = 1$. Then, by Lemma 3.4, there exists a countable set $\{e^{i\theta_n}\}_n$ in ∂D such that $\hat{f}$ has the radial limit $f^*(e^{i\theta_n})$ at $e^{i\theta_n}$,

$$\{f^*(e^{i\theta_n})\}_n \text{ is dense in } f(M(L^\infty)), \tag{3.5}$$

and

$$f = f^*(e^{i\theta_n}) \text{ on } \bigcup\{\operatorname{supp}\mu_x;\ x \in R_{e^{i\theta_n}}\}. \tag{3.6}$$

Let $\{r_j\}_j$ be a sparse sequence in D consisting of positive numbers. Let $\{E_n\}_n$ be a disjoint partition of $\{r_j\}_j$ such that E_n is an infinite set for every n. For each n, let

$$E_n = \{r_{n,j}\}_j.$$

For a sequence $\{z_j\}_j$ in D with $|z_j| = r_j$, we have $\rho(z_j, z_k) \geq \rho(r_j, r_k)$. Since $\{r_j\}_j$ is sparse, so is $\{z_j\}_j$. Hence $\{r_{n,j}e^{i\theta_n};\ n, j = 1, 2, \dots\}$ is a sparse sequence.

Take a dense countable subset $\{a_k\}_k$ in $\{|z| \leq 1\}$. Let $\{\alpha_j\}_j$ be a sequence such that each a_k appears in $\{\alpha_j\}_j$ infinitely many times. One such example is

$$a_1\, a_1\, a_2\, a_1\, a_2\, a_3\, a_1\, a_2\, a_3\, a_4\, a_1 \dots .$$

Since $\{r_{n,j}e^{i\theta_n};\ n, j = 1, 2, \dots\}$ is sparse, by the theorem of Sundberg and Wolff [20] there exists a function g in QA such that

$$g(r_{n,j}e^{i\theta_n}) = \alpha_j \quad \text{for every } n \text{ and } j.$$

Let b be the sparse Blaschke product with zeros $\{r_{n,j}e^{i\theta_n};\ n, j = 1, 2, \dots\}$. Then $Z(b) = \mathrm{cl}\{r_{n,i}e^{i\theta_n};\ i, n = 1, 2, \dots\} \setminus \{r_{n,i}e^{i\theta_n};\ i, n = 1, 2, \dots\}$ [6, p. 205]. Hence

$$g(Z(b)) = \bigcap_{n=1}^{\infty} \overline{\{\alpha_j;\ j \geq n\}} = \overline{\{a_k\}_k} = \{|z| \leq 1\}, \tag{3.7}$$

where the bar indicates closure in the complex plane. By Lemma 2.3, $QA = \{f \in H^\infty;\ f|\operatorname{supp}\mu_x \text{ is constant for every } x \in M(H^\infty + C)\}$. Since $g \in QA$, $g = g(x)$ on $\operatorname{supp}\mu_x$ for $x \in Z(b)$. Hence, by (3.7) and Lemma 3.3(ii),

$$g(N(\bar{b})) = \{|z| \leq 1\}. \tag{3.8}$$

For each n, let b_n be the sparse Blaschke product with zeros $\{r_{n,j}e^{i\theta_n};\ j = 1, 2, \dots\}$. In the same way, we have

$$g(N(\bar{b}_n)) = \{|z| \leq 1\}. \tag{3.9}$$

Since $Z(b_n) \subset R_{e^{i\theta_n}}$, $N(\bar{b}_n) \subset \mathrm{cl}\bigcup\{\operatorname{supp}\mu_x;\ x \in R_{e^{i\theta_n}}\}$. Then, by (3.6), we have

$$f = f^*(e^{i\theta_n}) \text{ on } N(\bar{b}_n). \tag{3.10}$$

By (3.8) and Lemma 3.3(i), there exists $F \in QA$ such that

$$F = g \text{ on } N(\bar{b}) \quad \text{and} \quad \|F\|_\infty = 1; \tag{3.11}$$

see [4, p. 58].

Since $N(\bar{b}_n) \subset N(b_n)$, by (3.9) and (3.11) we have

$$F(M(L^\infty)) = F(N(\bar{b}_n)) = \{|z| \le 1\}. \tag{3.12}$$

By Lemma 3.2, $F \in \mathrm{Ns}(H^\infty)$. Since $F \in QA$, it follows that $F \in \mathrm{Ns}(QA)$.

On the other hand, we have

$$\begin{aligned}(f-F)(M(L^\infty)) &\supset \overline{\bigcup_{n=1}^{\infty}(f-F)(N(\bar{b}_n))}\\ &= \overline{\bigcup_{n=1}^{\infty}\left((f^*(e^{i\theta_n}) + \{|z| \le 1\}\right)} \quad \text{by (3.10) and (3.12)}\\ &= f(M(L^\infty)) + \{|z| \le 1\} \quad \text{by (3.5)}\\ &\supset (f-F)(M(L^\infty)) \quad \text{by (3.12).}\end{aligned}$$

Hence

$$(f-F)(M(L^\infty)) = f(M(L^\infty)) + \{|z| \le 1\}. \tag{3.13}$$

Since $\|f\|_\infty = 1$, by (3.13) it is easy to see that $(f-F)(M(L^\infty))$ is simply connected. Hence, by Lemma 3.2, $f - F \in \mathrm{Ns}(B)$. □

When $B = H^\infty$, we have the following corollary.

COROLLARY 3.5. $H^\infty = \mathrm{Ns}(QA) + \mathrm{Ns}(H^\infty)$.

COROLLARY 3.6. $QA = \mathrm{Ns}(QA) + \mathrm{Ns}(QA)$.

Proof. Let $f \in QA$. By Corollary 3.5, there exists F in $\mathrm{Ns}(QA)$ such that $f - F \in \mathrm{Ns}(H^\infty)$. Since $f - F \in QA$, $f - F \in \mathrm{Ns}(QA)$. □

4. $B = \mathrm{Frs}(H^\infty) + \mathrm{Frs}(B)$ for Douglas Algebras B

In this section, we study another decomposition of Douglas algebras. For a point x in $M(H^\infty + C)$, $\mathrm{supp}\,\mu_x$ is a weak peak set for H^∞ [6, p. 207]. Let

$$H^\infty_{\mathrm{supp}\,\mu_x} = \{f \in L^\infty;\ f|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x\}.$$

Then $H^\infty_{\mathrm{supp}\,\mu_x}$ is a Douglas algebra and

$$M(H^\infty_{\mathrm{supp}\,\mu_x}) = M(L^\infty) \cup \{\zeta \in M(H^\infty + C);\ \mathrm{supp}\,\mu_\zeta \subset \mathrm{supp}\,\mu_x\}.$$

For a Douglas algebra B and a subset Λ of L^∞, we denote by $B[\Lambda]$ the Douglas algebra generated by B and Λ. For $f \in H^\infty$, put $\{|f| < 1\} = \{\zeta \in M(H^\infty + C);\ |f(\zeta)| < 1\}$. We note that $H^\infty \ne \mathrm{Ns}(QA) + \mathrm{Frs}(H^\infty)$, because

$$\mathrm{Ns}(QA) + \mathrm{Frs}(H^\infty) = \mathrm{Frs}(H^\infty) \ne H^\infty.$$

The following is the main theorem in this section.

THEOREM 4.1. *Let B be a Douglas algebra. Then $B = \mathrm{Frs}(H^\infty) + \mathrm{Frs}(B)$.*

To prove our theorem, we need some lemmas.

LEMMA 4.2 [19]. *Let $\{q_n\}_n$ be a sequence of inner functions. Then there exists a Blaschke product b such that $b = 0$ on $\bigcup_{n=1}^{\infty}\{|q_n| < 1\}$.*

LEMMA 4.3. *Let $f \in L^\infty$. For each complex number a in $f(M(L^\infty))$, there exists a point x in $M(H^\infty + C) \setminus M(L^\infty)$ such that* $\operatorname{supp}\mu_x \subset \{\zeta \in M(L^\infty);\ f(\zeta) = a\}$.

Proof. Let $a \in f(M(L^\infty))$. Put

$$E = \{\zeta \in M(L^\infty);\ f(\zeta) = a\}.$$

Then E is a closed G_δ-subset of $M(L^\infty)$. Hence there exists a function g in L^∞ such that

$$g = 1 \ \text{ on } E \quad \text{and} \quad 0 \le g \le 1 \ \text{ on } M(L^\infty) \setminus E. \tag{4.1}$$

Recall that

$$\hat{g}(\zeta) = \int_{M(L^\infty)} g\, d\mu_\zeta \quad \text{for } \zeta \in M(H^\infty) \tag{4.2}$$

and that $\hat{g}$ is a continuous function on $M(H^\infty)$. By the corona theorem, there exists a sequence $\{z_n\}_n$ in D such that $\hat{g}(z_n) \to 1$. By considering a subsequence, we may assume that $\{z_n\}_n$ is an interpolating sequence. Let x be one of the cluster points of $\{z_n\}_n$ in $M(H^\infty)$. Then $x \in M(H^\infty + C) \setminus M(L^\infty)$ and $\hat{g}(x) = 1$. By (4.1) and (4.2), we obtain $\operatorname{supp}\mu_x \subset E$. □

LEMMA 4.4. *Let B be a Douglas algebra with $B \ne L^\infty$ and $f \in L^\infty \setminus B$. Then*
(i) *$M(B[f, \bar{f}]) = \{x \in M(B);\ f|\operatorname{supp}\mu_x$ is constant$\}$,*
(ii) *$M(B[f, \bar{f}])$ is a closed but not open subset of $M(B)$.*

Proof. (i) follows from the Chang–Marshall theorem. Since $f \notin B$, $M(B[f, \bar{f}])$ is a proper closed subset of $M(B)$. Suppose that $M(B[f, \bar{f}])$ is open. Then, by the Shilov idempotent theorem (see [4, p. 88]), there exists $g \in B$ such that $g = 0$ on $M(B[f, \bar{f}])$ and $g = 1$ on $M(B) \setminus M(B[f, \bar{f}])$. Since $M(L^\infty) \subset M(B[f, \bar{f}])$, we have $g = 0$. This is a contradiction. □

LEMMA 4.5. *Let B be a Douglas algebra with $B \ne L^\infty$, and let b be an inner function with $\bar{b} \notin B$. Then $b(M(B)) = \{|z| \le 1\}$.*

Proof. Suppose not. Then there exists $z_0 \in D$ such that $z_0 \notin b(M(B))$. Then $(b - z_0)/(1 - \bar{z}_0 b)$ is an inner function and invertible in B. Hence $|(b - z_0)/(1 - \bar{z}_0 b)| = 1$ on $M(B)$, so that $|b| = 1$ on $M(B)$. Therefore $\bar{b} \in B$. This is a contradiction. □

Proof of Theorem 4.1. When $B = L^\infty$, we have $\mathrm{Frs}(L^\infty) = L^\infty$, so that $L^\infty = \mathrm{Frs}(H^\infty) + \mathrm{Frs}(L^\infty)$. Hence we may assume that $B \ne L^\infty$. Let $f \in B$. It is sufficient to prove the existence of F in $\mathrm{Frs}(H^\infty)$ such that $f - F \in \mathrm{Frs}(B)$. We may assume that $\|f\|_\infty = 1$. When $f \in QC$ we may take $F = 0$, so that we may assume $f \notin QC$.

Take a dense countable subset of $\{a_n\}_n$ in $f(M(L^\infty))$. By Lemma 4.3, for each n there exists x_n in $M(H^\infty + C) \setminus M(L^\infty)$ such that

$$\operatorname{supp} \mu_{x_n} \subset \{\zeta \in M(L^\infty);\ f(\zeta) = a_n\}. \tag{4.3}$$

By [7, p. 177] and Lemma 4.2, there exists an inner function I_n such that

$$I_n(x_n) = 0. \tag{4.4}$$

Put

$$\Gamma = \{x \in M(H^\infty + C);\ f|\operatorname{supp} \mu_x \text{ is not a constant}\}. \tag{4.5}$$

Since $f \notin QC$, by Lemma 2.3 we have $\Gamma \neq \emptyset$, $\Gamma \cap M(L^\infty) = \emptyset$, and $H^\infty[f, \bar{f}] \supset H^\infty + C$. Then, by Lemma 4.4(i), $M(H^\infty[f, \bar{f}]) = M(H^\infty + C) \setminus \Gamma$. By [10, Lemma 2.2], there exists a sequence of inner functions $\{J_n\}_n$ such that $H^\infty[f, \bar{f}] = H^\infty[\bar{J}_n;\ n = 1, 2, \dots]$. Then

$$\Gamma = \bigcup_{n=1}^{\infty} \{|J_n| < 1\}. \tag{4.6}$$

By Lemma 4.2, there exists a Blaschke product b_1 such that

$$b_1 = 0 \ \text{ on } \bigcup_{n=1}^{\infty} (\{|I_n| < 1\} \cup \{|J_n| < 1\}).$$

Then by (4.4) and (4.6),

$$b_1 = 0 \ \text{ on } \Gamma \cup \{x_n\}_n. \tag{4.7}$$

Applying Lemma 4.2 inductively, we can find a sequence of Blaschke products $\{b_n\}_n$ such that

$$b_{n+1} = 0 \ \text{ on } \{|b_n| < 1\}. \tag{4.8}$$

Let

$$F = \sum_{n=1}^{\infty} \frac{b_n}{2^{n-1}}. \tag{4.9}$$

Then

$$F \in H^\infty \quad \text{and} \quad \|F\|_\infty \le 2. \tag{4.10}$$

The following two claims will be proved later.

Claim 1. If $|b_1(x)| < 1$ and $x \in M(H^\infty)$, then $F(\operatorname{supp} \mu_x) = \{|z| \le 2\}$.

Claim 2. $F \in \operatorname{Frs}(H^\infty)$.

For now, we shall continue with the proof of our theorem. We need to prove that $f - F \in \operatorname{Frs}(B)$. Let $x \in M(B)$. We separate the proof into the following three cases:

$$x \in M(H^\infty + C) \setminus \Gamma; \quad x \in \Gamma; \quad x \in D.$$

We note that the case $x \in D$ happens only when $B = H^\infty$.

First, suppose that $x \in M(H^\infty + C) \setminus \Gamma$. Then, by (4.5),

$$f = f(x) \text{ on } \operatorname{supp} \mu_x. \tag{4.11}$$

By Claim 2, there exists y in $\operatorname{supp} \mu_x$ such that $F(y) = F(x)$. By (4.11), $f(y) = f(x)$, so that $(f - F)(y) = (f - F)(x)$.

Next, suppose that $x \in \Gamma$. Since $x \in M(B) \setminus M(L^\infty)$, we have $B|\operatorname{supp} \mu_x = H^\infty|\operatorname{supp} \mu_x \neq L^\infty$. Then, by Lemma 4.4(i),

$$M(H^\infty_{\operatorname{supp} \mu_x}[f, \bar{f}]) = M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \Gamma.$$

In this case, $M(H^\infty_{\operatorname{supp} \mu_x}) \cap \Gamma \neq \emptyset$ because $x \in M(H^\infty_{\operatorname{supp} \mu_x})$. Hence, by Lemma 4.4(ii),

$$\operatorname{cl}(M(H^\infty_{\operatorname{supp} \mu_x}) \cap \Gamma) \neq M(H^\infty_{\operatorname{supp} \mu_x}) \cap \Gamma.$$

Let y_0 be a point in $\operatorname{cl}(M(H^\infty_{\operatorname{supp} \mu_x}) \cap \Gamma) \setminus [M(H^\infty_{\operatorname{supp} \mu_x}) \cap \Gamma]$. Since

$$M(H^\infty_{\operatorname{supp} \mu_x}) \cap \Gamma \subset \{\zeta \in M(H^\infty + C);\ \operatorname{supp} \mu_\zeta \subset \operatorname{supp} \mu_x\},$$

we have

$$\operatorname{supp} \mu_{y_0} \subset \operatorname{supp} \mu_x. \tag{4.12}$$

Since $x \in M(B)$, $y_0 \in M(B)$. By definition, $y_0 \notin \Gamma$. Hence, by (4.5),

$$f = f(y_0) \text{ on } \operatorname{supp} \mu_{y_0}. \tag{4.13}$$

By (4.7), $b_1 = 0$ on Γ. Since $y_0 \in \operatorname{cl} \Gamma$, we have

$$b_1(y_0) = 0. \tag{4.14}$$

Because $\|f\|_\infty \leq 1$, $|f(y_0) - f(x)| \leq 2$. By (4.14) and Claim 1, there exists a point ζ_0 in $\operatorname{supp} \mu_{y_0}$ such that

$$F(\zeta_0) = f(y_0) - f(x). \tag{4.15}$$

By (4.12),

$$\zeta_0 \in \operatorname{supp} \mu_x.$$

Since $x \in \Gamma$, it follows that $b_1(x) = 0$. Then, by (4.8) and (4.9), we have $F(x) = 0$. Since $\zeta_0 \in \operatorname{supp} \mu_{y_0}$, by (4.13) we have $f(\zeta_0) = f(y_0)$. Therefore, by (4.15), we obtain

$$(f - F)(x) = f(x) = (f - F)(\zeta_0).$$

Suppose, finally, that $x \in D$. In this case, $B = H^\infty$ and $\operatorname{supp} \mu_x = M(L^\infty)$. By (4.3), $f = a_n$ on $\operatorname{supp} \mu_{x_n}$. By (4.7), $b_1(x_n) = 0$. Hence, by Claim 1, $F(\operatorname{supp} \mu_{x_n}) = \{|z| \leq 2\}$. Therefore, for each n,

$$(f - F)(\operatorname{supp} \mu_{x_n}) = a_n + \{|z| \leq 2\}. \tag{4.16}$$

Since $\{a_n\}_n$ is dense in $f(M(L^\infty))$,

$$\overline{\bigcup_{n=1}^{\infty} (a_n + \{|z| \leq 2\})} = f(M(L^\infty)) + \{|z| \leq 2\}. \tag{4.17}$$

Now we have

$$\begin{aligned}(f-F)(\operatorname{supp}\mu_x) &= \overline{(f-F)(M(L^\infty))}\\ &\supset \bigcup_{n=1}^{\infty}(a_n+\{|z|\le 2\}) \quad \text{by (4.16)}\\ &= f(M(L^\infty))+\{|z|\le 2\} \quad \text{by (4.17)}\\ &= f(M(B))+\{|z|\le 2\} \quad \text{by Lemma 3.2 and } \|f\|_\infty=1\\ &\ni (f-F)(x) \quad \text{by (4.10).}\end{aligned}$$

This completes the proof. □

Now we need to prove our two claims.

Proof of Claim 1. Suppose that $|b_1(x)|<1$ and $x\in M(H^\infty)$. First, we prove the case $x\notin D$. Since b_1 is a Blaschke product, $|b_1|=1$ on $\operatorname{supp}\mu_x$. Since $|b_1(x)|<1$, b_1 is not constant on $\operatorname{supp}\mu_x$. Hence $\bar{b}_1\notin H^\infty_{\operatorname{supp}\mu_x}$ so that, by Lemma 4.5,

$$b_1(M(H^\infty_{\operatorname{supp}\mu_x}))=\{|z|\le 1\}. \tag{4.18}$$

Let

$$E=\{\zeta\in M(H^\infty_{\operatorname{supp}\mu_x});\ |b_1(\zeta)|<1\}.$$

Since $E=M(H^\infty_{\operatorname{supp}\mu_x})\setminus M(H^\infty_{\operatorname{supp}\mu_x}[\bar{b}_1])$, by Lemma 4.4(ii) we have $\operatorname{cl}E\ne E$. Then, by (4.18),

$$b_1(\operatorname{cl}E\setminus E)=\{|z|=1\}. \tag{4.19}$$

By (4.8),

$$b_2=0 \text{ on } \operatorname{cl}E. \tag{4.20}$$

Therefore, by (4.19) and (4.20), for each complex number c with $|c|=1$ there exists ζ_c in $M(H^\infty_{\operatorname{supp}\mu_x})$ such that

$$b_1=c \text{ on } \operatorname{supp}\mu_{\zeta_c} \quad\text{and}\quad b_2(\zeta_c)=0.$$

Repeating this argument, for each sequence $\{c_n\}_n$ of complex numbers with $|c_n|=1$ there exists a sequence $\{\zeta_n\}_n$ in $M(H^\infty_{\operatorname{supp}\mu_x})$ such that

$$\operatorname{supp}\mu_{\zeta_{n+1}}\subset\operatorname{supp}\mu_{\zeta_n}\subset\operatorname{supp}\mu_x$$

and

$$b_n=c_n \text{ on } \operatorname{supp}\mu_{\zeta_n}. \tag{4.21}$$

Take a point y_0 such that

$$y_0\in\bigcap_{n=1}^{\infty}\operatorname{supp}\mu_{\zeta_n}\subset\operatorname{supp}\mu_x.$$

Then we have

$$\begin{aligned}F(y_0) &= \sum_{n=1}^{\infty}\frac{b_n(y_0)}{2^{n-1}} \quad \text{by (4.9)}\\ &= \sum_{n=1}^{\infty}\frac{c_n}{2^{n-1}} \quad \text{by (4.21).}\end{aligned}$$

Since

$$\left\{\sum_{n=1}^{\infty}\frac{c_n}{2^{n-1}};\ |c_n|=1 \text{ for every } n\right\}=\{|z|\le 2\},$$

we have $F(\operatorname{supp}\mu_x)=\{|z|\le 2\}$.

Next, suppose that $x\in D$. Then $\operatorname{supp}\mu_x=M(L^\infty)$. By (4.7), $b_1(x_n)=0$ and $x_n\in M(H^\infty+C)$. By the fact just proved, $F(\operatorname{supp}\mu_{x_n})=\{|z|\le 2\}$. Hence, by (4.10), $F(\operatorname{supp}\mu_x)=\{|z|\le 2\}$. □

Proof of Claim 2. Let $x\in M(H^\infty)$. Suppose that $|b_1(x)|<1$. Then, by (4.10) and Claim 1, we have

$$F(x)\in\{|z|\le 2\}=F(\operatorname{supp}\mu_x).$$

Hence we may assume that $|b_1(x)|=1$.

Suppose that $|b_n(x)|=1$ for every n. Then $b_n=b_n(x)$ on $\operatorname{supp}\mu_x$, so that

$$F=\sum_{n=1}^{\infty}\frac{b_n(x)}{2^{n-1}}=F(x)\ \text{ on } \operatorname{supp}\mu_x.$$

This implies that

$$F(x)\in F(\operatorname{supp}\mu_x).$$

Next, suppose that $|b_n(x)|<1$ for some n. Then by (4.8), there exists a positive integer n_0 such that

$$|b_n(x)|=1 \quad \text{for } 1\le n\le n_0 \tag{4.22}$$

and

$$|b_n(x)|<1 \quad \text{for } n>n_0. \tag{4.23}$$

By (4.22), $b_n=b_n(x)$ on $\operatorname{supp}\mu_x$ for $1\le n\le n_0$, so that

$$\sum_{n=1}^{n_0}\frac{b_n}{2^{n-1}}=\left(\sum_{n=1}^{n_0}\frac{b_n}{2^{n-1}}\right)(x)\ \text{ on } \operatorname{supp}\mu_x. \tag{4.24}$$

By (4.19) and using the same argument as in the proof of Claim 1, we obtain

$$\left(\sum_{n=n_0+1}^{\infty}\frac{b_n}{2^{n-1}}\right)(\operatorname{supp}\mu_x)=\{|z|\le (1/2)^{n_0-1}\}. \tag{4.25}$$

Since $\|\sum_{n=n_0+1}^{\infty}(b_n/2^{n-1})\|_\infty=(1/2)^{n_0-1}$, we obtain

$$\begin{aligned}
F(x)&=\left(\sum_{n=1}^{n_0}\frac{b_n}{2^{n-1}}\right)(x)+\left(\sum_{n=n_0+1}^{\infty}\frac{b_n}{2^{n-1}}\right)(x)\\
&\in\left(\sum_{n=1}^{n_0}\frac{b_n}{2^{n-1}}\right)(x)+\{|z|\le (1/2)^{n_0-1}\}\\
&=\left(\sum_{n=1}^{n_0}\frac{b_n}{2^{n-1}}+\sum_{n=n_0+1}^{\infty}\frac{b_n}{2^{n-1}}\right)(\operatorname{supp}\mu_x)\quad \text{by (4.24) and (4.25)}\\
&=F(\operatorname{supp}\mu_x).
\end{aligned}$$

This completes the proof of Claim 2. □

Because $\mathrm{Frs}(H^\infty) \subset \mathrm{Frs}(B)$, we have the following corollaries.

COROLLARY 4.6. *Let B be a Douglas algebra. Then $B = \overline{\mathrm{Frs}(B)} + \mathrm{Frs}(B)$.*

COROLLARY 4.7. $H^\infty = \overline{\mathrm{Frs}(H^\infty)} + \mathrm{Frs}(H^\infty)$.

REMARK. $\mathrm{Frs}(H^\infty)$ is strictly smaller than $\mathrm{Ns}(H^\infty)$.

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