Analytic Continuability of Bergman Inner Functions

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0. Introduction and Preliminaries

For $0 < p < \infty$, the Bergman space L^p_a consists of those functions f analytic in the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ for which

$$\|f\|^p_{L^p_a} = \iint_{\mathbb{D}} |f(z)|^p \frac{dA(z)}{\pi} < \infty.$$

If $p \geq 1$ then $\|\cdot\|_{L^p_a}$ is a norm making L^p_a a Banach space, and if $0 < p < 1$ then $d(f, g) = \|f - g\|^p_{L^p_a}$ is a metric making L^p_a a nonlocally convex complete metric topological vector space.

Let $\{\alpha_1, \alpha_2, \dots\}$ be an L^p_a zero sequence—that is, the sequence of zeros, repeated according to multiplicity, of some nonidentically vanishing L^p_a function—and let M be the set of L^p_a functions that vanish on the sequence $\{\alpha_n\}$ to at least the prescribed multiplicity. We let N denote the number of times that 0 appears in the sequence $\{\alpha_n\}$ and consider the following extremal problem:

$$\sup\{ \operatorname{Re} f^{(N)}(0) : f \in M,\ \|f\|_{L^p_a} \leq 1 \}. \tag{0.1}$$

It is shown in [DKSS1; DKSS2] that there is a unique extremal function φ for this problem, and that φ satisfies the following properties:

$$\iint_{\mathbb{D}} |\varphi(z)|^p u(z) \frac{dA(z)}{\pi} = u(0) \quad \text{if } u \text{ is a bounded harmonic function in } \mathbb{D}; \tag{0.2}$$

$$\text{if } f \in M \text{ then } f/\varphi \in L^p_a \text{ and } \|f/\varphi\|_{L^p_a} \leq \|f\|_{L^p_a}. \tag{0.3}$$

In particular, (0.3) says that the function φ vanishes at each point of the sequence to *exactly* the prescribed multiplicity; that is, it has no "extra zeros".

In accordance with what has become common practice, we take (0.2) to be the defining property of L^p_a *inner functions*. The reason for the terminology lies in the analogy to the case of the Hardy space H^p. There has been much interest in these functions in recent years, starting with Hedenmalm's groundbreaking paper [Hed] in which he established, among other facts, that (0.3) holds in the case $p = 2$. See [DKSS1], [DKSS2], and [ARS] for further information.

Received October 2, 1996. Revision received April 28, 1997.
The author is partially supported by NSF Grant R01105358.
Michigan Math. J. 44 (1997).

If $p = 2$ and the zero sequence $\{\alpha_n\}$ is finite, then an easy argument shows that the extremal function φ is a rational function with poles at the points $1/\bar{\alpha}_n$, and hence it continues analytically across $\partial\mathbb{D}$. A version of this fact holds also for infinite zero sequences. Suppose $I \subset \partial\mathbb{D}$ is an open arc that does not meet $\operatorname{clos}\{\alpha_n\}$. Then it is a consequence of a theorem of Akutowicz and Carleson (in [AC]; see also [S1]) that the associated extremal function φ extends analytically across I.

In [DKSS1] and [DKSS2] it is shown that, for general p, the extremal function associated to a finite zero sequence extends analytically across $\partial\mathbb{D}$. The authors asked whether it were also true for general p that the extremal function associated to a zero sequence extends analytically across any arc $I \subset \partial\mathbb{D}$ not meeting the closure of the sequence. That this is indeed true is the main result of the present paper. It had previously been shown to be true if the zero sequence were a Blaschke sequence by Duren, Khavinson, and Shapiro in [DKS].

The paper is organized as follows. In Section 1 a formula yielding the analytic continuation of extremal functions associated to finite zero sequences is derived. As a consequence we obtain estimates on these analytic continuations. In Section 2 these estimates are used to prove our main result. In Section 3 the formula derived in Section 2 is used to give a different proof of a result of MacGregor and Stessin [MS], and some related questions are discussed.

ACKNOWLEDGMENT. The author would like to thank Peter Duren for informing him of the papers [MS] and [V].

1. The Finite Case

Let $\{\alpha_1, \dots, \alpha_n\}$ be a finite sequence of points in $\mathbb{D}$ and denote by φ its associated L^p_a inner function.

Let g denote the *Cauchy transform* of $|\varphi|^p\mathcal{X}_{\mathbb{D}}$; that is,

$$g(z) = \iint_{w\in\mathbb{D}} \frac{|\varphi(w)|^p}{w - z}\frac{dA(w)}{\pi}. \tag{1.1}$$

We will use the following facts about g:

$$\bar{\partial}g = -|\varphi|^p\mathcal{X}_{\mathbb{D}} \text{ in the sense of distributions;} \tag{1.2}$$

$$g \text{ is continuous in all of } \mathbb{C}; \tag{1.3}$$

$$g(z) = -1/z \quad \text{for } |z| \geq 1. \tag{1.4}$$

(Here $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$ is the standard Cauchy–Riemann partial differential operator.) Of these, (1.2) is standard (see e.g. [C2, Chap. V, Thm. 3.3]); (1.3) follows from the boundedness of φ, which in turn is a consequence of the analytic continuability of φ across $\partial\mathbb{D}$ [DKSS1; DKSS2]; and (1.4) follows for $|z| > 1$ from (0.2) and for $|z| = 1$ from continuity.

We next cut out of $\mathbb{D}$ a set of nonintersecting curves $\gamma_1, \dots, \gamma_n$, each γ_j connecting α_j to a point $\beta_j \in \partial\mathbb{D}$ (if $\alpha_j = \alpha_k$ we assume that $\gamma_j = \gamma_k$). We denote by Ω the resulting simply connected region. Because, as mentioned previously,

φ has no "extra zeros", φ does not vanish in Ω and we can define $\varphi^{p/2}$ in Ω, for definiteness choosing it so that $\varphi^{p/2}(0) > 0$. We can then define its integral

$$\Phi(z) = \int_{\sigma_z} \varphi^{p/2}(w)\,dw \quad \text{for } z \in \Omega, \tag{1.5}$$

where σ_z is some rectifiable path connecting 0 to z in Ω.

We use all the functions just constructed to define

$$h(z) = g(z) + \overline{\Phi(z)}\varphi^{p/2}(z). \tag{1.6}$$

The following properties of h are important for us:

$$h \text{ is analytic in } \Omega; \tag{1.7}$$

$$h \text{ is continuous in } \Omega \cup (\partial\mathbb{D}\backslash\{\beta_1, \dots, \beta_n\}); \tag{1.8}$$

$$h(z) = -1/z + \overline{\Phi(z)}\varphi^{p/2}(z) \quad \text{for } z \in \partial\mathbb{D}\backslash\{\beta_1, \dots, \beta_n\}. \tag{1.9}$$

The first of these properties is a consequence of Weyl's lemma (see e.g. [C2, Chap. V, 2.12]), (1.2), and the definition of Φ, which implies that $\bar\partial(\bar\Phi\varphi^{p/2}) = \overline{\Phi'}\varphi^{p/2} = |\varphi|^p$. The second property is a consequence of (1.3) and the analytic continuability of φ across $\partial\mathbb{D}$. Finally, (1.9) follows from (1.4) and the continuity of $\varphi^{p/2}$ and Φ up to $\partial\mathbb{D}$.

We can now state a formula giving an expression for the analytic continuation of φ. Let $\Omega^* = \{\,1/\bar z : z \in \Omega\,\}$ and define

$$\Phi(z) = \frac{z + \overline{h(1/\bar z)}}{\overline{\varphi^{p/2}(1/\bar z)}}, \quad z \in \Omega^* \tag{1.10}$$

The continuity of h and φ up to $\partial\mathbb{D}\backslash\{\beta_1, \dots, \beta_n\}$, together with the fact that $|\varphi(z)| \geq 1$ for $z \in \partial\mathbb{D}$ [DKSS1; DKSS2], shows that we can extend (1.10) continuously to $z \in \partial\mathbb{D}\backslash\{\beta_1, \dots, \beta_n\}$. By (1.9) this extension agrees with the original definition of Φ there, and so by (1.7) we see that (1.10) gives an analytic continuation of Φ to $\tilde\Omega = \Omega \cup \Omega^* \cup (\partial\mathbb{D}\backslash\{\beta_1, \dots, \beta_n\})$. The nonvanishing of φ near $\partial\mathbb{D}$ now shows that the formula $\varphi = (\Phi')^{2/p}$ yields an analytic continuation of φ across each arc of $\partial\mathbb{D}\backslash\{\beta_1, \dots, \beta_n\}$.

We will now derive some estimates on the functions we have constructed; these estimates will be needed in the proof of our main result in the next section. The estimate

$$|f(z)| \leq \frac{1}{(1-|z|)^{2/p}}\|f\|_{L^p_a}, \quad f \in L^p_a \tag{1.11}$$

is well known and elementary; it follows from the subharmonicity of $|f|^p$ after integration over $\{\,w : |w - z| < 1 - |z|\,\}$.

Suppose now that the curves γ_j have been chosen such that, for any $z \in \Omega$, a rectifiable path σ_z from 0 to z within Ω can be chosen along which $|dw| \leq 2d|w|$. Then, by (1.11),

$$\Phi(z) = \left|\int_{\sigma_z} \varphi^{p/2}(w)\,dw\right| \leq \int_{\sigma_2} \frac{|dw|}{1-|w|} \leq 2\int_0^{|z|} \frac{dr}{1-r},$$

yielding

$$|\Phi(z)| \leq 2\log\frac{1}{1-|z|}, \quad z \in \Omega. \tag{1.12}$$

The estimate

$$|g(z)| \leq \frac{6}{1-|z|}, \quad z \in \mathbb{D} \tag{1.13}$$

follows from (1.11) and simple estimates on the defining integral (1.1) of g obtained by dividing this integral into the integral over $|w-z| < \frac{1}{2}(1-|z|)$ and that over $|w-z| > \frac{1}{2}(1-|z|)$. Finally, we combine (1.11), (1.12), and (1.13) to obtain

$$|h(z)| \leq \frac{8}{(1-|z|)^2}, \quad z \in \Omega. \tag{1.14}$$

REMARK. These estimates can be improved somewhat. Vukotić [V] shows that (1.11) can be improved to

$$|f(z)| \leq \frac{1}{(1-|z|^2)^{2/p}} \|f\|_{L^p_a}$$

and that this is best possible. Combining his ideas with some of Richter in [Ric] yields the following estimate for L^p_a inner functions:

$$|\varphi(z)| \leq \frac{1}{(1-|z|^2)^{1/p}}.$$

In particular, this shows that (1.12) can be replaced by

$$|\Phi(z)| \leq 2\sin^{-1}|z|$$

and hence Φ has the universal bound of π.

2. The Main Result

We are now ready to prove the analytic continuability of L^p_a inner functions.

THEOREM 2.1. *Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is an L^p_a zero sequence and that $I \subset \partial\mathbb{D}$ is an arc not meeting* clos$\{\alpha_n\}$. *Then the associated L^p_a inner function φ has an analytic continuation across I.*

Proof. Since the property we wish to prove is local, it clearly suffices to prove the following: If $z_0 \in \partial\mathbb{D}$ is not a limit point of $\{\alpha_n\}$ then φ has an analytic continuation to a neighborhood of z_0. Given such a z_0, we construct nonintersecting curves γ_n in $\mathbb{D}$, each γ_n connecting α_n to a point $\beta_n \in \partial\mathbb{D}$ (again, if $\alpha_j = \alpha_k$ we set $\gamma_j = \gamma_k$) in such a way that if Ω_n is the simply connected open set $\mathbb{D}\backslash\{\gamma_1, \ldots, \gamma_n\}$, then the following properties hold:

$$\text{the closure of } \bigcup \gamma_n \text{ does not contain } z_0, \tag{2.1}$$

$$\Omega = \bigcap \Omega_n \text{ is open,} \tag{2.2}$$

and

$$\text{each } z \in \Omega_n \text{ can be connected to } 0 \text{ by a rectifiable path } \sigma_z \text{ in } \Omega_n \text{ along which } |dw| \leq 2d|w|. \tag{2.3}$$

We now let φ_n denote the L^p_a extremal function corresponding to the finite set $\{\alpha_1, \ldots, \alpha_n\}$, and use φ_n as in Section 1 to define functions g_n in $\mathbb{C}$ and functions Φ_n, h_n in Ω_n. The formula

$$\Phi_n(z) = \frac{z + \overline{h_n(1/\bar{z})}}{\varphi_n^{p/2}(1/\bar{z})}, \quad z \in \Omega_n^*, \tag{2.4}$$

gives an analytic continuation of Φ_n to $\tilde{\Omega}_n = \Omega_n \cup \Omega_n^* \cup (\partial\mathbb{D}\backslash\{\beta_1, \ldots, \beta_n\})$. Since $\varphi_n(z) \to \varphi(z)$ for every $z \in \mathbb{D}$ (see [DKSS1; DKSS2]), our theorem will be proved if we can show that the functions $\{\Phi_n\}$ form a normal family of functions analytic in $\tilde{\Omega} = \Omega \cup \Omega^* \cup (\partial\mathbb{D}\backslash\text{clos}\{\beta_n\})$.

Of course, the estimates we have do not give bounds on Φ_n at points of $\partial\mathbb{D}$, and so we cannot immediately apply Montel's theorem. The usual way around this difficulty, and the way that is used at this point in the papers [AC] and [S1], is to apply a famous theorem of Beurling (see [D]). Beurling's theorem could also be applied here, as will soon be evident, but the following much easier result will suffice.

LEMMA 2.2. *Let $U \subset \mathbb{C}$ be open and let $\mathcal{F}$ be a family of functions analytic in U. Suppose there exists a $\rho \in L^1_{\text{loc}}(U)$ such that* $\log^+|f(z)| \leq \rho(z)$ *for any $f \in \mathcal{F}$ and $z \in U$. Then $\mathcal{F}$ is a normal family.*

Proof. Let $K \subset U$ be compact, and pick a $\delta > 0$ such that $K_\delta = \{z \in \mathbb{C} : \text{dist}(z, K) \leq \delta\} \subset U$. Then if $f \in \mathcal{F}$ and $z \in K$, the subharmonicity of $\log^+|f|$ implies that

$$\log^+|f(z)| \leq \frac{1}{\pi\delta^2}\iint_{|w-z|\leq\delta} \log^+|f(w)|\, dA(w) \leq \frac{1}{\pi\delta^2}\iint_{K_\delta} \rho(w)\, dA(w).$$

Thus

$$|f(z)| \leq \exp\left[\frac{1}{\pi\delta^2}\iint_{K_\delta} \rho(w)\, dA(w)\right] \quad \text{for any } f \in \mathcal{F},\ z \in K,$$

so an application of Montel's theorem [C1, Chap. VII, 2.9]; [Rud, Chap. 14, Thm. 14.6] proves the Lemma. □

We will now show that the family $\{\Phi_n\}$ satisfies the hypothesis of Lemma 2.2 in the open set $\tilde{\Omega}$. Write (2.4) in the form

$$\Phi_n(z) = [z + \overline{h_n(1/\bar{z})}]\frac{\overline{\varphi^{p/2}(1/\bar{z})}}{\varphi_n^{p/2}(1/\bar{z})}\frac{1}{\overline{\varphi^{p/2}(1/\bar{z})}}, \quad z \in \Omega^*. \tag{2.5}$$

By (0.3) and (1.11), $|\varphi^{p/2}(1/\bar{z})/\varphi_n^{p/2}(1/\bar{z})| \leq |z|/(|z| - 1)$. We combine this estimate with (1.14) and use (2.5) and a little manipulation to derive the estimate

$$|\Phi_n(z)| \leq \frac{8|z|^2}{(|z| - 1)^2}\frac{1}{|\varphi(1/\bar{z})|^{p/2}}, \quad z \in \Omega^*. \tag{2.6}$$

This estimate, together with (1.12), shows that the function

$$\rho(z) = \begin{cases} \log^+\bigl(2\log\frac{1}{1-|z|}\bigr), & |z| < 1, \\ \log\frac{8|z|^2}{(|z|-1)^2} + \frac{p}{2}\log^+\frac{1}{|\varphi(1/\bar{z})|}, & |z| > 1, \end{cases} \tag{2.7}$$

dominates $\log^+|\Phi_n(z)|$ for all n. It remains only to show that $\rho \in L^1_{\mathrm{loc}}(\mathbb{C})$, and this is trivial except for the term $\log^+(1/|\varphi(1/\bar{z})|)$. To handle this we write

$$\begin{aligned} \iint_{1<|z|<R} \log^+\frac{1}{|\varphi(1/\bar{z})|}\,dA(z) &= \iint_{1/R<|w|<1} \frac{1}{|w|^4}\log^+\frac{1}{|\varphi(w)|}\,dA(w) \\ &\le R^4 \iint_{|w|<1} \log^+\frac{1}{|\varphi(w)|}\,dA(w) \\ &= R^4 \iint_{\mathbb{D}} \log^+|\varphi(w)|\,dA(w) \\ &\quad - R^4 \iint_{\mathbb{D}} \log|\varphi(w)|\,dA(w) \\ &\le R^4\frac{3\pi}{2p} - R^4\pi\log|\varphi(0)|, \end{aligned}$$

where the first term comes from (1.11) and an integration; the second term comes from the inequality $\log|\varphi(0)| \le \iint_{\mathbb{D}} \log|\varphi(w)|\,dA(w)/\pi$, which follows from the subharmonicity of $\log|\varphi|$.

This completes the proof of Theorem 2.1. □

3. Some Related Results

It was shown by Horowitz in [Hor] that the union of two L^p_a zero sequences is not necessarily an L^p_a zero sequence. However, the following is an immediate consequence of Theorem 2.1.

COROLLARY 3.1. *Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two L^p_a zero sequences such that $\mathrm{clos}\{\alpha_n\} \cap \mathrm{clos}\{\beta_n\} \cap \partial\mathbb{D} = \emptyset$. Then $\{\alpha_n\} \cup \{\beta_n\}$ is an L^p_a zero sequence.*

Proof. Let φ_1 and φ_2 be the L^p_a inner functions associated with $\{\alpha_n\}$ and $\{\beta_n\}$, respectively. Then, by Theorem 2.1 and an easy compactness argument, $\varphi_1\varphi_2 \in L^p_a$. □

In [MS], MacGregor and Stessin give the following formula for the L^p_a inner function φ associated with a finite zero sequence $\{\alpha_1, \dots, \alpha_n\}$:

$$\varphi(z) = B(z)\Biggl[b - \sum_k \frac{a_k}{1-\bar{\alpha}_{j_k}z}\Biggr]^{2/p}. \tag{3.1}$$

Here

$$B(z) = \prod_{j=1}^n \frac{|\alpha_j|}{\alpha_j}\frac{\alpha_j - z}{1-\bar{\alpha}_j z}$$

is the Blaschke product associated with $\{\alpha_j\}$ (where we adopt the convention $|0|/0 = -1$), and $\{\alpha_{j_1}, \ldots, \alpha_{j_s}\}$ is a listing of the distinct nonzero elements of $\{\alpha_1, \ldots, \alpha_n\}$. We here give an alternate proof of this formula based on the ideas in Section 1. Using the notation in that section, it is easy to see from (1.6) and (1.5) that

$$h(z) = h_j(z) + \overline{\Phi(\alpha_j)}\varphi^{p/2}(z), \tag{3.2}$$

where h_j is analytic in a neighborhood of α_j. It follows from (3.2) and (1.10) that, if $\alpha_j \neq 0$,

$$\Phi(z) = \Phi(\alpha_j) + \frac{z + \overline{h_j(1/\bar{z})}}{\overline{\varphi^{p/2}(1/\bar{z})}(1 - \bar{\alpha}_j z)} \tag{3.3}$$

and hence

$$\varphi^{p/2}(z) = \Phi'(z) = \frac{H_j(z)}{\overline{\varphi^{p/2}(1/\bar{z})}(1 - \bar{\alpha}_j z)}, \tag{3.4}$$

where H_j is analytic in a neighborhood of $1/\bar{\alpha}_j$. Thus $(\varphi/B)^{p/2}$ is meromorphic near $1/\bar{\alpha}_j$ with at worst a simple pole there. Similar reasoning shows that φ/B is analytic and nonvanishing in a neighborhood of ∞. Since φ/B does not vanish in clos $\mathbb{D}$, we can finally conclude that $(\varphi/B)^{p/2}$ is rational with simple poles or removable singularities at those α_j that are nonzero (in fact, it is not difficult to show it must have poles at these points). The formula (3.1) follows. □

It should be mentioned that our method does provide a little more information than that of MacGregor and Stessin, namely that $\varphi^{p/2}$ possesses a primitive in Ω. This is of course equivalent to the statement that

$$\int_{\Gamma_j} \varphi^{p/2}(z)\, dz = 0$$

if Γ_j is any rectifiable simple closed curve enclosing α_j and $1/\bar{\alpha}_j$, and not enclosing α_k or $1/\bar{\alpha}_k$ if $\alpha_k \neq \alpha_j$. In fact, it can be shown that this condition, together with $\|\varphi\|_{L^p_a} = 1$, determines the coefficients b and a_k.

Finally, we discuss an interpolation result of Akutowicz and Carleson. Suppose $\{\alpha_n\}$ is an L^2_a zero sequence of *distinct* points in $\mathbb{D}$ and that $\{w_n\}$ is a sequence of points in $\mathbb{C}$ such that there exists an $f \in L^2_a$ such that $f(\alpha_n) = w_n$ for all n. Let ψ be the L^2_a function of minimal norm accomplishing this interpolation. Then Akutowicz and Carleson show in [AC] that ψ continues analytically across any boundary arc not meeting clos$\{\alpha_n\}$ (this is the result alluded to in our Section 0). It is natural to ask if this result holds in L^p_a for $p \neq 2$. The following example shows that it does not, even in the case of two interpolation points.

Let $0 < r < 1$ and set

$$\psi(z) = 1 - \frac{(1-r)^2}{(1-rz)^2}.$$

Since 1 and $1/(1-rz)^2$ are the reproducing kernels for the points 0 and r, we see that ψ is the minimal L^2_a interpolating function taking 0 to $1-(1-r)^2$ and r to

$1 - 1/(1+r)^2$. The function ψ has a simple zero at $z = 1$ and no other zeros in $\operatorname{clos}\mathbb{D}$. Let $B(z) = z[(r-z)/(1-rz)]$. By minimality,

$$\frac{d}{dt}\bigg|_{t=0} \|\psi + tF\psi B\|_{L^2_a} = 0 \quad \forall F \in L^2_a,$$

which leads to

$$\iint_{\mathbb{D}} |\psi|^2 BF \frac{dA}{\pi} = 0 \quad \forall F \in L^2_a. \tag{3.5}$$

For $p > 1$ we now argue as in the proof of Theorem 4.2.1 of [S2, p. 55]: if F is a polynomial, then

$$\begin{aligned}
\|\psi^{2/p}\|^p_{L^p_a} &= \|\psi\|^2_{L^2_a} \\
&= \iint_{\mathbb{D}} (\psi^{2/p} + \psi^{2/p}FB)\frac{|\psi|^2}{\psi^{2/p}}\frac{dA}{\pi} \qquad \text{(by (3.5))} \\
&\le \left[\iint_{\mathbb{D}} |\psi^{2/p} + \psi^{2/p}FB|^p \frac{dA}{\pi}\right]^{1/p} \\
&\quad \times \left[\iint_{\mathbb{D}} |\psi|^{(2-2/p)(p/(p-1))}\frac{dA}{\pi}\right]^{(p-1)/p} \\
&= \|\psi^{2/p} + \psi^{2/p}FB\|_{L^p_a} \|\psi^{2/p}\|^{p-1}_{L^p_a}.
\end{aligned}$$

Dividing by $\|\psi^{2/p}\|^{p-1}_{L^p_a}$, we see that

$$\|\psi^{2/p}\|_{L^p_a} \le \|\psi^{2/p} + \psi^{2/p}FB\|_{L^p_a}$$

for any polynomial F. Since the functions of the form $\psi^{2/p}F$ (F a polynomial) are clearly dense in L^p_a, this shows that $\psi^{2/p}$ is the L^p_a minimal interpolating function taking 0 to $[1-(1-r)^2]^{2/p}$ and r to $[1-1/(1+r)^2]^{2/p}$. Of course, $\psi^{2/p}$ has a zero of order $2/p$ at 1 and hence does not extend analytically around 1 if $p > 1$ and $p \ne 2$.

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