Invariant Subspaces of the Bergman Space and Some Subnormal Operators in $\mathbb{A}_1 \setminus \mathbb{A}_2$

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1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$. The Bergman space $L_a^2(D)$ consists of analytic functions in D with

$$\|f\|_{L_a^2} = \left(\int_D |f(z)|^2 \, dA(z)\right)^{1/2} < \infty,$$

where dA denotes the area measure in $\mathbb{C}$, normalized by a constant factor

$$dA(z) = dx\, dy/\pi.$$

Let μ be a finite positive Borel measure with compact support, and let $\operatorname{spt}\mu$ denote the support of μ. Let $P^2(\mu)$ denote the closure in $L^2(\mu)$ of analytic polynomials in z and let S_μ denote the operator of multiplication by z on $P^2(\mu)$. The operator S_μ is *pure* if $P^2(\mu)$ has no L^2 summand. A measure with support in the closed unit disk is a reverse Carleson measure for $L_a^2(D)$ if

$$\int_D |p|^2 \, dA \le C \int |p|^2 \, d\mu$$

for every polynomial p. The set E denotes a compact subset of the unit circle $\mathbb{T}$ with positive Lebesgue measure. Set $\mathbb{T} \setminus E = \bigcup J_n$, where J_n is a connected component. We say that E satisfies the *Carleson condition* if

$$\sum_{n=1}^{\infty} m(J_n) \log \frac{1}{m(J_n)} < \infty,$$

where m stands for the normalized Lebesgue measure on $\mathbb{T}$, that is, $dm = (1/2\pi)\, d\theta$. Every closed arc of $\mathbb{T}$ satisfies the Carleson condition. It is easy to construct a nowhere dense subset of $\mathbb{T}$ satisfying the Carleson condition. Put

$$\mu_E = dA|_D + m|_E. \tag{1.1}$$

The *Cauchy transform* of a finite measure μ is defined by

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$$\hat{\mu}(\lambda) = \int \frac{1}{z-\lambda}\, d\mu(z).$$

This function is locally integrable with respect to the area measure. The function P_λ is defined by

$$P_\lambda(z) = \frac{z}{z-\lambda} - \frac{z}{z-1/\bar{\lambda}}.$$

It is easy to show that $P_\lambda(e^{i\theta})$ is the Poisson kernel. For a compact subset E of $\mathbb{T}$, define the space H_E^2 as follows:

$$H_E^2 = \left\{ f : f \text{ is analytic in the open unit disk and } \liminf_{r\to 1-0} \int_E |f(re^{i\theta})|^2\, d\theta < \infty \right\}.$$

Let W be a connected component of $D \setminus \operatorname{spt} \mu$. Olin and Thomson [10] have defined the *strong boundary* B_W of W to be the set of points a in $\mathbb{T} \cap \partial W$ such that, for all $\alpha \in (0, \pi)$, there is an isosceles triangle $T_{a,\alpha}^s$ where:

(1) a is a vertex of $T_{a,\alpha}^s$ and s is the height of $T_{a,\alpha}^s$;
(2) $\operatorname{int} T_{a,\alpha}^s \subset W$;
(3) the interior angle of $T_{a,\alpha}^s$ at a is α; and
(4) the radial line segment from 0 to a bisects the interior angle of $T_{a,\alpha}^s$ at a.

The strong boundary of W is a Borel set (see [10]). If $m(B_W) > 0$, we call W an *outer hole* in the support of μ.

Let z be the coordinate function on $\mathbb{C}$, with $z(\lambda) = \lambda$ and $\lambda \in \mathbb{C}$. Let X be a Banach space of analytic functions over an open set G. A closed subspace M of X is said to be *z-invariant,* or simply invariant, if zf belongs to M whenever f is in M. One says that an invariant subspace M of X has the *codimension-n property* if M/zM is an n-dimensional space.

Let $\mathcal{H}$ be a separable Hilbert space; denote the bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ and the trace class operators by $\mathcal{B}_1(\mathcal{H})$. The weak-* topology on $\mathcal{B}(\mathcal{H})$ is that induced on $\mathcal{B}(\mathcal{H})$ as the dual of $\mathcal{B}_1(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, let $\mathcal{A}(T)$ denote the weak-* closed subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T and I, and $\mathcal{A}_\perp(T)$ the annihilators of $\mathcal{A}(T)$. Then $\mathcal{A}(T)$ is the dual of the quotient space $\mathcal{A}_*(T) = \mathcal{B}_1(\mathcal{H})/\mathcal{A}_\perp(T)$. If $B \in \mathcal{B}_1(\mathcal{H})$, let $[B]$ denote the image of B in $\mathcal{A}_*(T)$. For $x, y \in \mathcal{H}$, $x \otimes y$ denotes the rank-1 operator $(x\otimes y)z = \langle z, y\rangle x$, $z \in \mathcal{H}$; thus, if $A \in \mathcal{A}(T)$, $[x\otimes y](A) = \langle Ax, y\rangle$. Bercovici, Foiaş, and Pearcy [2] have defined a decreasing sequence $\mathbb{A}_n$ of classes of operators; $T \in \mathbb{A}_n$, $1 \le n \le \aleph_0$, if for $\{L_{ij}\}_{0\le i,j<n} \subset \mathcal{A}_*(T)$ there exist sequences $\{x_i\}_{i<n}$ and $\{y_j\}_{j<n}$ in $\mathcal{H}$ such that $L_{ij} = [x_i \otimes y_j]$ whenever $0 \le i, j < n$. Operators in $\mathbb{A}_1$ have nontrivial lattices of invariant subspaces, while if $T \in \mathcal{A}_{\aleph_0}$ then the lattice of invariant subspaces of T is huge. It is easy to show, from the definition of $\mathbb{A}_n$, that if every invariant subspace S_{μ_E} has the codimension-1 property then $S_{\mu_E} \in \mathbb{A}_1 \setminus \mathbb{A}_2$.

The structure of the lattice of invariant subspaces in a Banach space of analytic functions X, especially the Bergman space, has attracted a lot of attention from both operator theorists and function theorists. However, most results have been disappointing, in the sense that for most spaces X there are no simple characterizations such as are known for the Hardy space $H^2(D)$. The famous theorem of Beurling on invariant subspaces of $H^2(D)$ asserts that either every z-invariant subspace M of $H^2(D)$ is trivial (i.e., $M = 0$) or there exists an inner function u such that $M = uH^2(D)$. The Bergman spaces do have some z-invariant subspaces similar to the Hardy-space cases. For instance, Hedenmalm, Korenblum, and Zhu [6] studied Beurling-type invariant subspaces for the Bergman spaces. However, in contrast to the Hardy-space situation, it has been discovered by Apostol, Bercovici, Foiaş, and Pearcy [1] that there exist in the Bergman space z-invariant subspaces having the codimension-n property. Hedenmalm [5] gave a simple concrete example of an invariant subspace having the codimension-n property. In this paper, we look at a class of functions that violate the conditions of the example in [5]. We mainly show that the invariant subspace generated by this class of functions has the codimension-1 property (Theorem C).

In Section 2, we prove the following two theorems.

THEOREM A. *Let a closed subset $E \subset \mathbb{T}$ satisfy the Carleson condition. Then there exists an outer hole W in the support of $\mu_E^W = \mu_E|_{W^c}$ so that, for every polynomial p, the following inequality holds:*

$$\int |p|^2\, d\mu_E \le C \int |p|^2\, d\mu_E^W \le C \int |p|^2\, d\mu_E.$$

From Theorem A, we see that the measure μ_E^W is a reverse Carleson measure for the Bergman space $L_a^2(D)$.

THEOREM B. *Let a closed subset E of $\mathbb{T}$ with positive Lebesgue measure satisfy the Carleson condition. Suppose that S_{μ_E} is pure on $P^2(\mu_E)$; then S_{μ_E} is in $\mathbb{A}_1 \backslash \mathbb{A}_2$.*

Our main theorem is proved in Section 3.

THEOREM C. *Let a compact subset $E \subset \mathbb{T}$ of positive measure satisfy the Carleson condition, and let $\mathcal{B}$ be a subset of $L_a^2(D)$. Suppose that for each $f \in \mathcal{B}$ there exists a bounded analytic function ϕ such that $\phi f \in H_E^2$. Let M be the invariant subspace generated by $\mathcal{B}$. Then M has the codimension-1 property.*

2. A Reverse Carleson Measure and a Subnormal Operator in $\mathbb{A}_1 \backslash \mathbb{A}_2$

In this section, we construct a reverse Carleson measure for the Bergman space $L_a^2(D)$. The measure, which has an outer hole inside the open unit disk, will be used to prove our main theorem in Section 3. We fix a compact subset $E \subset \mathbb{T}$, $m(E) > 0$, that satisfies the Carleson condition. The next lemma is due to Hruscev [7].

LEMMA 2.1. *There exists a nontrivial bounded function ϕ on E such that the Cauchy transform $\int_E \phi(w)/(w-z)\,dw$ is infinitely differentiable in the closed unit disk* (see [7, Thm. 5]).

LEMMA 2.2. *There exists a function $g \in L^\infty(\mu_E)$, $g|_E$ is not the zero function a.e. $m|_E$, and $\int p\bar{g}\,d\mu_E = 0$ for every polynomial p.*

Proof. Define

$$g(z) = \begin{cases} \dfrac{d}{dz}\left(\displaystyle\int_E \frac{\phi(w)}{w-z}\,dw\right), & z \in D, \\ -2\pi i \bar{z}\phi(z), & z \in E. \end{cases}$$

From Lemma 2.1, we see that $g \in L^\infty(\mu_E)$ and $g|_E$ is nontrivial. It remains to show $g \perp P^2(\mu_E)$. In fact,

$$\begin{aligned} \int_D \bar{z}^n g(z)\,dA(z) &= \frac{1}{\pi}\int_0^1 r^{n+1}\int_0^{2\pi} e^{-in\theta}\frac{d}{dz}\left(\int_E \frac{\phi(w)\bar{w}}{1-z\bar{w}}\,dw\right)\bigg|_{z=re^{in\theta}} d\theta\,dr \\ &= \frac{1}{\pi}\int_0^1 r^{n+1}\int_0^{2\pi}(n+1)r^n\left(\int_E \bar{w}^{n+2}\phi(w)\,dw\right)d\theta\,dr \\ &= 2\pi i\int_E \bar{w}^{n+1}\phi(w)\,dm. \end{aligned}$$

The lemma is proved. □

From now on, we will fix the functions ϕ in Lemma 2.1 and g in Lemma 2.2 and assume $\|\phi\|_\infty > 1$. Set

$$E_0 = \{e^{i\theta} : |\phi(e^{i\theta})| > 1\}.$$

Then $m(E_0) > 0$. Let $T_{e^{i\theta},\alpha}$ be a Stoltz angle with vertex $e^{i\theta}$ and opening α (see [3, p. 110, Fig. 1]), where $0 < \alpha < \pi/2$. Using Fatou's theorem, we have that

$$\lim_{\lambda \in T_{e^{i\theta},\alpha} \to e^{i\theta}} \left|\int P_\lambda(e^{it})\bar{\phi}(e^{it})e^{it}\,dm(t)\right| > 1 \text{ a.e. } m|_{E_0}.$$

For $n = 1, 2, 3, \ldots$, set

$$E_n = \left\{e^{i\theta} : \left|\int P_\lambda(e^{it})\bar{\phi}(e^{it})e^{it}\,dm(t)\right| > \frac{1}{2} \text{ for all } \lambda \in T_{e^{i\theta},\alpha} \cap \left(|\lambda| > 1 - \frac{1}{n}\right)\right\}.$$

LEMMA 2.3. *There exists an n such that $m(E_n) > 0$.*

Proof. It is clear that $E_0 \subset \bigcup_{n=1}^\infty E_n$ and that $\{E_n\}$ is an increasing sequence. Hence, $\lim_{n\to\infty} m(E_n) \ge m(E_0) > 0$. □

We now fix the set E_n in Lemma 2.3 and set

$$T_k^n = \bigcup_{e^{i\theta}\in E_n}\left(T_{e^{i\theta},\alpha}\cap\left(|\lambda|>1-\frac{1}{k}\right)\right),$$

where $k\geq n$.

LEMMA 2.4. *There exists an N such that, for each $\lambda\in T_N^n$,*

$$\left|\int P_\lambda(z)\bar{g}(z)\,d\mu_E(z)\right|\geq 1.$$

Proof. The Cauchy transform $\int_D (z/(z-\lambda))\bar{g}\,dA$ is a continuous function throughout $\mathbb{C}$, since g is bounded. Hence there is an N such that, for $\lambda\in T_N^n$,

$$\left|\int_D P_\lambda(z)\bar{g}\,dA\right| = \left|\int_D \frac{z}{z-\lambda}\bar{g}\,dA - \int_D \frac{z}{z-1/\bar{\lambda}}\bar{g}\,dA\right| < 1.$$

Therefore, for each $\lambda\in T_N^n$, we have

$$\begin{aligned}\left|\int P_\lambda(z)\bar{g}(z)\,d\mu_E\right| &\geq 2\pi\left|\int P_\lambda(e^{it})\bar{\phi}(e^{it})e^{it}\,dm\right| - \left|\int_D P_\lambda(z)\bar{g}(z)\,dA\right| \\ &\geq 1. \qquad \square\end{aligned}$$

We will let C denote an absolute constant which may change from one step to the next.

LEMMA 2.5. *There exist an integer K and a constant $C>0$ such that*

$$\int_{T_K^n} |p|^2\,dA \leq C\left(\int_E |p|^2\,dm + \int_{D\setminus T_K^n} |p|^2\,dA\right).$$

Proof. For each $\lambda\in D$ and each polynomial p, we see that

$$p(\lambda)\int \frac{z}{z-\lambda}\bar{g}\,d\mu_E = \int \frac{z}{z-\lambda}p(z)\bar{g}\,d\mu_E$$

since $\bar{z}g\perp P^2(\mu_E)$. Hence,

$$p(\lambda)\int P_\lambda(z)\bar{g}\,d\mu_E = \int P_\lambda(z)\,p(z)\bar{g}\,d\mu_E$$

since

$$\int \frac{z}{z-1/\bar{\lambda}}p(z)\bar{g}(z)\,d\mu_E = 0 \quad \text{for } \lambda\in D.$$

Let $\lambda\in T_K^n$, $K>N$; K will be determined later. Using Lemma 2.4, we see that

$$\begin{aligned}|p(\lambda)| &\leq |p(\lambda)|\left|\int P_\lambda\bar{g}\,d\mu_E\right| \\ &\leq 2\pi\left|\int P_\lambda p\bar{\phi}z\,dm\right| + \left|\int_D P_\lambda p\bar{g}\,dA\right| \leq\end{aligned}$$

$$\leq 2\pi\left|\int P_\lambda p\bar{\phi} z\,dm\right| + \left(\int_D |P_\lambda|\,dA\right)^{1/2}\left(\int_D |P_\lambda||pg|^2\,dA\right)^{1/2}$$
$$= 2\pi I_1 + I_2.$$

The area measure restricted to the unit disk is a Carleson measure for the Hardy space, so

$$\int |I_1|^2\,dA \leq C\int |p\phi|^2\,dm \leq C\int_E |p|^2\,dm$$

(see [12, Thm. 8.2.2]). On the other hand,

$$\int_D |P_\lambda(z)|\,dA(z) \leq \int_D \left|\frac{1}{z-\lambda}\right| dA(z) + \int_D \left|\frac{1}{z-1/\bar{\lambda}}\right| dA(z) \leq C.$$

Hence,

$$\int_{T_K^n} |p(\lambda)|^2\,dA(\lambda) \leq C\int_E |p|^2\,dm + C\int_D |p(z)|^2\,dA(z)\int_{T_K^n} |P_\lambda(z)|\,dA(\lambda).$$

However,

$$\int_{T_K^n} |P_\lambda(z)|\,dA(\lambda) \leq \int_{T_K^n} \left|\frac{1}{z-\lambda}\right| dA(\lambda) + \int_{T_K^n} \left|\frac{1}{1/\bar{z}-\lambda}\right| dA(\lambda) \leq C\sqrt{\text{area}(T_K^n)}$$

(see [3, p. 167, similar proof of Prop. 2.2]). Thus,

$$(1 - C\sqrt{\text{area}(T_K^n)})\int_{T_K^n} |p|^2\,dA \leq C\left(\int_E |p|^2\,dm + \int_{D\setminus T_K^n} |p|^2\,dA\right).$$

Choose an integer K such that

$$1 - C\sqrt{\text{area}(T_K^n)} \geq 1/2,$$

and we have the desired inequality. □

Let $T_K^n = \bigcup W_k$, where each W_k is a connected component of T_K^n.

LEMMA 2.6. *There exists a W_k which is an outer hole in the support of $\mu_E^{W_k} = \mu_E|_{W_k^c}$. (We will denote this W_k by W.)*

Proof. Let $\lambda \in E_n$. By definition, we see that $T_{\lambda,\alpha} \cap (|\lambda| > 1-1/K) \subset T_K^n$. Since $T_{\lambda,\alpha} \cap (|\lambda| > 1-1/K)$ is a connected set, we can find a W_k such that $T_{\lambda,\alpha} \cap (|\lambda| > 1-1/K) \subset W_k$. Thus, $\lambda \in \partial W_k \cap \mathbb{T}$ and

$$E_n \subset \bigcup(\partial W_k \cap \mathbb{T}).$$

We can find a W_k satisfying $m(\partial W_k \cap \mathbb{T}) > 0$.

By the construction of T_K^n, we know that ∂W_k is a rectifiable Jordan curve. Let f be the Riemann map from D to W_k; then the harmonic measure $\omega = m \circ f^{-1}$ of W_k and the arc-length measure are mutually absolutely continuous (see [3, p. 199]). Hence, $\omega(\partial W_k \cap \mathbb{T}) > 0$. Using the argument in [4, p. 45], we see that $\omega|_{\partial W_k \cap \mathbb{T}}$ and $m|_{B_{W_k}}$ are mutually absolutely continuous. Thus $m(B_{W_k}) > 0$. □

THEOREM 2.7. *There exists an outer hole W in the support of $\mu_E^W = \mu_E|_{W^c}$ such that, for every polynomial p, the following inequality holds:*

$$\int |p|^2 \, d\mu_E \leq C \int |p|^2 \, d\mu_E^W \leq C \int |p|^2 \, d\mu_E.$$

Proof. The inequality follows from Lemma 2.5 and Lemma 2.6. □

COROLLARY 2.8. *The measure μ_E^W in Theorem 2.7 is a reverse Carleson measure for the Bergman space $L_a^2(D)$.*

THEOREM 2.9. *Let a closed subset E of $\mathbb{T}$ with positive Lebesgue measure satisfy the Carleson condition. Suppose that S_{μ_E} is pure on $P^2(\mu_E)$; then S_{μ_E} is in $\mathbb{A}_1 \backslash \mathbb{A}_2$.*

Proof. It follows from Theorem 2.7 that S_{μ_E} is similar to $S_{\mu_E^W}$. The minimal normal extension of $S_{\mu_E^W}$ is $N_{\mu_E^W}$, the operator of multiplication by z on $L^2(\mu_E^W)$. Using Theorem 1(i) in [9], one sees that $S_{\mu_E^W}$ is in $\mathbb{A}_1 \backslash \mathbb{A}_2$. Since membership in the classes $\mathbb{A}_n$ is preserved by similarity [2, Prop. 2.09], S_{μ_E} belongs to $\mathbb{A}_1 \backslash \mathbb{A}_2$. □

Theorem 2.9 is a generalization of Theorem 2 in [9].

3. Invariant Subspaces of the Bergman Space

Before proving our main theorem, we need the following lemma.

LEMMA 3.1. *Suppose that W is the outer hole in Theorem 2.7. Then the space $L_a^2(D) \cap H_E^2$ is contained in $P^2(\mu_E^W)$ in the sense that, for every function $f \in L_a^2(D) \cap H_E^2$, there exists $g \in P^2(\mu_E^W)$ such that f equals g on D.*

Proof. Let $f_r(z) = f(rz)$; then $f_r \in P^2(\mu_E^W)$. We have the following computation:

$$\begin{aligned}\int |f_r(z)|^2 \, d\mu_E^W &\leq \int_D |f(rz)|^2 \, dA + \int_E |f(re^{i\theta})|^2 \, dm \\ &\leq \|f\|^2_{L_a^2(D)} + \int_E |f(re^{i\theta})|^2 \, dm.\end{aligned}$$

Since f is in H_E^2, we can choose r_n converging to 1 such that

$$\int |f_{r_n}(z)|^2 \, d\mu_E^W \leq M < \infty.$$

By choosing a subsequence (if necessary) we may assume that $\{f_{r_n}\}$ converges weakly to a function g in $P^2(\mu_E^W)$. It is routine to check that f and g are equal on D. □

The following theorem is our main result.

THEOREM 3.2. *Let a compact subset $E \subset \mathbb{T}$ of positive measure satisfy the Carleson condition, and let $\mathcal{B}$ be a subset of $L_a^2(D)$. Suppose that for each*

$f \in \mathcal{B}$ there exists a bounded analytic function ϕ such that $\phi f \in H_E^2$. Let M be the invariant subspace generated by $\mathcal{B}$. Then M has the codimension-1 property.

Proof. Without loss of generality, we assume that zero is in the outer hole W of Theorem 2.7. Using Corollary 3.15 of [11], we may assume that M is generated by two functions f_1 and f_2, and that there exist two bounded analytic functions ϕ_1 and ϕ_2, $\phi_1(0) \neq 0 \neq \phi_2(0)$, such that $\phi_i f_i \in H_E^2$ for $i = 1, 2$. Suppose that $\dim(M/zM) \geq 2$. Let x_1, x_2 be two unit vectors in $M \ominus zM$ with $\langle x_1, x_2 \rangle = 0$ and let $g_i = \bar{x}_i$ for $i = 1, 2$. Then

(1) $\int f g_i \, d\mu = 0$ for every $f \in zM$; and
(2) $\int x_i g_j \, dA = \delta_{ij}$.

Since

$$\int \phi_i x_i g_j \, dA = \phi_i(0) \delta_{ij},$$

one sees that the invariant subspace generated by $\phi_1 f_1$ and $\phi_2 f_2$ has the codimension-2 property. Therefore, we may assume that $\phi_1 = \phi_2 = 1$, that is, $f_i \in H_E^2$ for $i = 1, 2$.

Claim:

$$\det \begin{bmatrix} \int f_1 g_1 \, dA, & \int f_1 g_2 \, dA \\ \int f_2 g_1 \, dA, & \int f_2 g_2 \, dA \end{bmatrix} \neq 0.$$

Otherwise, there exists a nonzero constant c such that

$$\int f_i (g_1 + c g_2) \, dA = 0$$

for $i = 1, 2$. On the other hand,

$$\begin{aligned} &\int (p_1 f_1 + p_2 f_2)(g_1 + c g_2) \, dA \\ &\quad = p_1(0) \int f_1 (g_1 + c g_2) \, dA + p_2(0) \int f_2 (g_1 + c g_2) \, dA \\ &\quad = 0 \end{aligned}$$

and $\{p_1 f_1 + p_2 f_2\}$ are dense in M. Hence,

$$\int x_i (g_1 + c g_2) \, dA = 0.$$

This is a contradiction, so the claim is established.

Using Corollary 2.8, we see that

$$\left| \int p g_i \, dA \right| \leq \|p\|_{L_a^2} \|g_i\|_{L^2(D)} \leq C \|g_i\|_{L^2(D)} \|p\|_{P^2(\mu_E^W)}$$

for every polynomial p. Using the Hahn–Banach theorem, we can find $h_i \in L^2(\mu_E^W)$ so that

$$\int pg_i\,dA = \int ph_i\,d\mu_E^W.$$

From Lemma 3.1, we conclude that $f_i \in P^2(\mu_E^W)$. Choose a sequence of polynomials $\{p_n\}$ converging to f_i in $P^2(\mu_E^W)$. From Corollary 2.8, the sequence should converge to f_i in L_a^2. Therefore,

$$\int pf_i g_j\,dA = \int pf_i h_j\,d\mu_E^W.$$

Hence,

$$p(0)\begin{bmatrix} \int f_1 g_1\,dA, & \int f_1 g_2\,dA \\ \int f_2 g_1\,dA, & \int f_2 g_2\,dA \end{bmatrix} = \begin{bmatrix} \int p(z) f_1 h_1\,d\mu_E^W, & \int p(z) f_1 h_2\,d\mu_E^W \\ \int p(z) f_2 h_1\,d\mu_E^W, & \int p(z) f_2 h_2\,d\mu_E^W \end{bmatrix}.$$

Denote

$$\begin{bmatrix} \int f_1 g_1\,dA, & \int f_1 g_2\,dA \\ \int f_2 g_1\,dA, & \int f_2 g_2\,dA \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let $f^1 = af_1 + bf_2$ and $f^2 = cf_1 + df_2$, then

$$\int pf_i h_j\,d\mu_E^W = \begin{cases} p(0) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where p is a polynomial and $i, j = 1, 2$. Therefore, the invariant subspace generated by f^1 and f^2 in $P^2(\mu_E^W)$ has the codimension-2 property. This contradicts Theorem 2.9. □

It is interesting to ask about the following possible generalization of Theorem 3.2.

QUESTION 3.3. Does Theorem 3.2 still remain valid if E is replaced by any compact subset K ($m(K) > 0$) of the unit circle?

Our method will not work in this case, because in [8], Kegejan constructed an example as follows: There exists a compact subset E of $\mathbb{T}$ with positive Lebesgue measure such that

$$P^2(\mu_E) = L_a^2(D) \oplus L^2(m\,|_E).$$

Hence, we are not able to construct an outer hole for this E.

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