Cohomology of the Symplectic Group Sp(4, Z), Part II: Computations at the Prime 2

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0. Introduction

In this paper, we give a complete description of the integral cohomology group $H^*(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z})$ of $\mathrm{Sp}(4,\mathbf{Z})$. In our previous paper [BL], the odd torsion pieces were determined, so here we concentrate on the 2-primary components. These components are closely related to the cohomology of the mapping class group Γ_2^0 of a genus-2 Riemann surface. Using this relationship and the knowledge of $H^*(\Gamma_2^0;\mathbf{Z})$ in the recent work of Benson and Cohen, we complete the project started in [BL].

Let $\mathrm{Sp}(4;\mathbf{Z})$ denote the group of 4-by-4 integral matrices preserving the skew symmetric pairing $\begin{pmatrix} 0 & I \\ -1 & 0 \end{pmatrix}$. This is a well-known arithmetic subgroup of the real symplectic group $\mathrm{Sp}(4;\mathbf{R})$ and has been studied by various authors from the viewpoint of automorphic forms (e.g. see [G1; G2; Ba]). For $\mathrm{Sp}(4;\mathbf{R})$ is a generalization of the special linear group $\mathrm{Sp}(2;\mathbf{R}) = \mathrm{SL}(2;\mathbf{R})$, and the theory of automorphic forms on $\mathrm{Sp}(4;\mathbf{R})$ can be regarded as a natural extension of the study of elliptic functions. From a topological viewpoint, this group is of some interest because of its relation with the mapping class groups Γ_2^0 and Γ_0^6. In general, let S_g^n be a surface of genus g with n punctures, and let $\mathrm{diff}^+(S_g^n)$ be the group of orientation-preserving diffeomorphisms of the surface S_g^n which fix the punctures setwise. The mapping class group Γ_g^n is the set of isotopy classes of $\mathrm{diff}^+(S_g^n)$. Since the cohomology $H^*(\Gamma_2^0;\mathbf{Z})$ can be identified with the cohomology $H^*(B\Gamma_2^0;\mathbf{Z})$ of the classifying space $B\Gamma_2^0$, classes in $H^*(\Gamma_2^0;\mathbf{Z})$ can be regarded as characteristic classes of genus-2 surface bundles [Mo]. Motivated by this interpretation, Cohen completely determined the cohomological structure of Γ_2^0 in [Co; BC]. Since $\mathrm{Sp}(4;\mathbf{Z})$ is the fundamental group of the moduli space of genus-2 abelian varieties, the cohomology $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$ gives rise to characteristic classes of fibrations of these abelian varieties [CL]. Hence, it is an interesting question to determine the structure of $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$.

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In our previous paper [BL], we set up a Mayer–Vietoris sequence to study the cohomology of $\mathrm{Sp}(4;\mathbf{Z})$, $H^*(\mathrm{Sp}(4;\mathbf{Z});R)$, with coefficients in an arbitrary ring R. As an application of this machinery, we determined the rank of $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Q})$ and odd torsion components of $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$. More explicitly, there exists an infinite cyclic component in $H^2(\mathrm{Sp}(4;\mathbf{Z}))$ and nowhere else. The odd torsion summands are of order 3 and 5 and can be determined from the formulas

$$H^q(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z}_{(5)})=\begin{cases}\mathbf{Z}_{(5)} & q=2\\ \mathbf{Z}/5 & q>2,\ q\equiv 0 \bmod 2\\ 0 & q\equiv 1 \bmod 2\end{cases}$$

and

$$H^q(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z}_{(3)})=\begin{cases}\mathbf{Z}_{(3)} & q=0\\ \bigoplus_{\gamma_q}\mathbf{Z}/3 & q>2\end{cases}$$

where γ_q is the coefficient of t^q in $t(1-2t+3t^2+2t^3+2t^4-t^5)/(1-t)(1-t^4)$.

The object of this paper is to determine the 2-torsion piece, and hence a complete picture of $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$. Explicitly, we obtain the following theorem.

THEOREM (0.1). *The 2-primary components of* $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z}_{(2)})$ *of* $\mathrm{Sp}(4;\mathbf{Z})$ *have orders* 2, 4, 8, *and* 16. *As a graded abelian group, there exists a decomposition into the following summands:*

$$H^q(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z}_{(2)})\cong\begin{cases}\mathbf{Z}/16\oplus(\bigoplus_{e_q}\mathbf{Z}/4)\oplus(\bigoplus_{g_q-1}\mathbf{Z}/2) & q\equiv 0 \bmod 4\\ \mathbf{Z}/8\oplus(\bigoplus_{e_q-1}\mathbf{Z}/4)\oplus(\bigoplus_{g_q}\mathbf{Z}/2) & q\equiv 2 \bmod 4\\ (\bigoplus_{f_q}\mathbf{Z}/2)\oplus(\bigoplus_{g_q}\mathbf{Z}/2) & q\equiv 1 \bmod 2\end{cases}$$

and

$$H^2(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z}_{(2)})\cong\mathbf{Z}_{(2)}\oplus\mathbf{Z}/2,$$

where the e_q, f_q, g_q *are specified in* (5.12).

Compared with the odd primary components, the answer is far more complicated and so is the technique involved. A good deal of help is provided by Cohen's paper [Co], where the 2-primary components of $H^*(\Gamma_2^0;\mathbf{Z})$ are determined. Besides [BL] and [Co], our general method throughout is to perform cohomology computations with $\mathbf{F}_2$ coefficients and use these results to deduce the integral cohomology. The technique involves a careful analysis of Bockstein operations.

In the first section we review material from [BL] used in this paper, including the 2-local version of the Mayer–Vietoris sequence for $\mathrm{Sp}(4;\mathbf{Z})$. In this last sequence, the main ingredients are the cohomology of three different groups Γ_2^0, G, and H. In Sections 2 and 3 we compute the cohomology of G and H with coefficients in $\mathbf{Z}$ and $\mathbf{F}_2$ and the induced homomorphism between them. In Section 4 we calculate the map in cohomology φ_2^*: $H^*(H;\mathbf{Z})\to H^*(\Gamma_2^0;\mathbf{Z})$, and in Section 5 we assemble the information to complete the computation of $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$.

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1. The Mayer–Vietoris Sequence

In [BL] it was shown that there is a Mayer–Vietoris sequence in cohomology

$$\begin{aligned}\cdots \longrightarrow H^*(\mathrm{Sp}(4,\mathbf{Z});R) &\xrightarrow{(\iota_1^*,\iota_2^*)} H^*(\Gamma_2^0;R)\oplus H^*(\mathrm{SL}(2,\mathbf{Z})\wr\mathbf{Z}/2;R)\\ &\xrightarrow{\varphi_2^*-\varphi_1^*} H^*(\mathfrak{B}\rtimes\mathbf{Z}/2;R)\longrightarrow H^{*+1}(\mathrm{Sp}(4,\mathbf{Z});R)\longrightarrow\cdots.\end{aligned}\tag{1.1}$$

The group $\mathfrak{B}$ is defined as follows: Let St(2, **Z**) denote the Steinberg group associated to SL(2, **Z**) (cf. [Mi, p. 82]). The group St(2, **Z**) has presentation

$$\mathrm{St}(2,\mathbf{Z})=\langle\sigma_1,\sigma_2\,|\,\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2\rangle.$$

There is a homomorphism $\psi\colon \mathrm{St}(2,\mathbf{Z})\to\mathrm{SL}(2,\mathbf{Z})$ given by

$$\psi(\sigma_1)=\begin{pmatrix}1&1\\0&1\end{pmatrix},\qquad \psi(\sigma_2)=\begin{pmatrix}1&0\\-1&1\end{pmatrix}.$$

The kernel of ψ is a central infinite cyclic subgroup which is generated by $\sigma=(\sigma_1\sigma_2\sigma_1)^4$, so there is a central extension

$$1\longrightarrow\mathbf{Z}[\sigma]\longrightarrow\mathrm{St}(2,\mathbf{Z})\xrightarrow{\psi}\mathrm{SL}(2,\mathbf{Z})\longrightarrow 1.\tag{1.2}$$

The group $\mathfrak{B}$ is then defined by

$$\mathfrak{B}=\frac{\mathrm{St}(2,\mathbf{Z})\times\mathrm{St}(2,\mathbf{Z})}{Z\langle\sigma\times\sigma^{-1}\rangle}.$$

We give a brief description of the group homomorphisms φ_1, φ_2, ι_1, and ι_2 which induce the maps in the Mayer–Vietoris sequence (1.1). Detailed descriptions and proofs can be found in [BL]. The relationships between the relevant groups are summarized in the following diagram:

$$\begin{tikzcd} B\rtimes\mathbf{Z}/2 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"] & \mathrm{SL}(2,\mathbf{Z})\wr\mathbf{Z}/2 \arrow[d, "\iota_1"] \\ \Gamma_2^0 \arrow[r, "\iota_2"] & \mathrm{Sp}(4,\mathbf{Z}). \end{tikzcd}\tag{1.3}$$

There is a subgroup $\Gamma(S,\gamma)$ of the mapping class group Γ_2^0, which is defined as follows. Let $S_{1,1}$ be a genus-1 surface with one boundary component. The mapping class group $\Gamma_{1,1}$ is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{1,1}$ which fix the boundary pointwise. A genus-2 surface S can be viewed as the union along γ of two halves of type $S_{1,1}$. The group $\mathfrak{B}\rtimes\mathbf{Z}/2$ is isomorphic to the subgroup $\Gamma(S,\gamma)$ of Γ_2^0 fixing the separating curve γ, and φ_2 is the inclusion of this group into Γ_2^0.

If we allow the separating curve γ to degenerate to a point P then we have a stable Riemann surface $\tilde{S}$, consisting of the union of two genus-1 surfaces, which intersect at the point P. The mapping class group of $\tilde{S}$, $\Gamma(S/\gamma)$, is isomorphic to $\mathrm{SL}(2,\mathbf{Z})\wr\mathbf{Z}/2$. The map φ_1 is then the natural map $\Gamma(S,\gamma)$ to $\Gamma(S/\gamma)$.

There is an inclusion of $\mathrm{SL}(2,\mathbf{Z})\wr\mathbf{Z}/2$ into $\mathrm{Sp}(4,\mathbf{Z})$ as matrices of the form

$$\begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & b_1 & 0 & b_2 \\ a_3 & 0 & a_4 & 0 \\ 0 & b_3 & 0 & b_4 \end{bmatrix}, \quad \begin{bmatrix} 0 & a_1 & 0 & a_2 \\ b_1 & 0 & b_2 & 0 \\ 0 & a_3 & 0 & a_4 \\ b_3 & 0 & b_4 & 0 \end{bmatrix},$$

where

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

are in $\mathrm{SL}(2,\mathbf{Z})$. This is the subgroup of $\mathrm{Sp}(4,\mathbf{Z})$ which preserves the subspace of diagonal matrices in the Siegel space $\mathcal{H}_2$. This inclusion is the map ι_1. Finally, ι_2 is the well-known epimorphism from Γ_2^0 to $\mathrm{Sp}(4,\mathbf{Z})$ given by associating to the isotopy class of a diffeomorphism its induced action on a homology basis for the surface.

Since we are considering the 2-torsion piece of the cohomology of $\mathrm{Sp}(4,\mathbf{Z})$, we can make some simplifications to the groups $\mathrm{SL}(2,\mathbf{Z})\wr\mathbf{Z}/2$ and $\mathcal{B}\rtimes\mathbf{Z}/2$.

PROPOSITION (1.4). *Let R be a ring in which primes $p\neq 2$ are invertible. Then*

$$H^*(\mathrm{SL}(2,\mathbf{Z})\wr\mathbf{Z}/2;R)\cong H^*(\mathbf{Z}/4\wr\mathbf{Z}/2;R)$$

and

$$H^*(\mathcal{B}\times\mathbf{Z}/2;R)\cong H^*\left(\frac{\mathbf{Z}\times\mathbf{Z}}{\mathbf{Z}\langle 4,-4\rangle}\rtimes\mathbf{Z}/2;R\right).$$

For notational convenience, we define

$$G=\mathbf{Z}/4\wr\mathbf{Z}/2 \quad\text{and}\quad H=\frac{\mathbf{Z}\times\mathbf{Z}}{\mathbf{Z}\langle 4,-4\rangle}\rtimes\mathbf{Z}/2.$$

The proof is very similar to computations in [BL] for the projective symplectic group $\mathrm{PSp}(4,\mathbf{Z})$ (cf. Lemma 6.1); we omit the details. Let $\mathbf{Z}_{(2)}$ denote the local ring $\mathbf{Z}[\frac{1}{3},\frac{1}{5},\frac{1}{7},\dots]$ and $\mathbf{F}_2$ the field with two elements. Assembling these results, we have the following.

PROPOSITION (1.5). *Let $R=\mathbf{Z}_{(2)}$ or $\mathbf{F}_2$. Then there is a Mayer–Vietoris sequence in cohomology with trivial R coefficients,*

$$\cdots\longrightarrow H^*(\mathrm{Sp}(4,\mathbf{Z});R)\xrightarrow{(\iota_1^*,\,\iota_2^*)} H^*(\Gamma_2^0;R)\oplus H^*(G;R)$$
$$\xrightarrow{\varphi_2^*-\varphi_1^*} H^*(H;R)\longrightarrow H^{*+1}(\mathrm{Sp}(4,\mathbf{Z});R)\longrightarrow\cdots.$$

For simplicity, we retain the original names from (1.3) for the maps between the summands.

Finally, we review another result of [BL] which will be needed in Section 4. The pushout diagram (1.3) resulted from a $\mathrm{Sp}(4;\mathbf{Z})$-Borel construction.

An analogous Borel construction was also performed for the projective symplectic group, $\mathrm{PSp}(4,\mathbf{Z}) \cong \mathrm{Sp}(4,\mathbf{Z})/\pm I$. The resulting pushout diagram is

$$\begin{array}{ccc} \dfrac{\mathrm{St}(2,\mathbf{Z})\times \mathrm{St}(2,\mathbf{Z})}{\mathbf{Z}\langle \mu\times\mu^{-1}\rangle} \rtimes \mathbf{Z}/2 & \xrightarrow{\phi_1} & \dfrac{\mathrm{SL}(2,\mathbf{Z})\times \mathrm{SL}(2,\mathbf{Z})}{\mathbf{Z}/2\langle I\times -I\rangle} \rtimes \mathbf{Z}/2 \\ \downarrow \phi_2 & & \downarrow \iota_1' \\ \Gamma_0^6 & \xrightarrow{\iota_2'} & \mathrm{PSp}(4,\mathbf{Z}), \end{array} \tag{1.6}$$

where $\mu = (\sigma_1\sigma_2)^3 \in \mathrm{St}(2,\mathbf{Z})$. We define the group $\mathcal{B}'$ by

$$\mathcal{B}' = \frac{\mathrm{St}(2,\mathbf{Z})\times \mathrm{St}(2,\mathbf{Z})}{\mathbf{Z}\langle \mu\times\mu^{-1}\rangle}.$$

As in (1.4), $\mathcal{B}'$ can be simplified for calculating cohomology over the coefficient rings $\mathbf{Z}_{(2)}$ and $\mathbf{F}_2$. Setting

$$H' = \frac{\mathbf{Z}\times\mathbf{Z}}{\mathbf{Z}\langle 2,-2\rangle},$$

we have the following proposition.

Proposition (1.7). *Let $R = \mathbf{Z}_{(2)}$ or $\mathbf{F}_2$. Then there is an isomorphism in cohomology with trivial R coefficients:*

$$H^*(H';R) \cong H^*(\mathcal{B}';R).$$

2. The Integral Cohomology of G

In this section, we determine the cohomology $H^*(G,R)$ of G for the coefficient rings $R = \mathbf{Z}, \mathbf{F}_2$.

The mod 2 cohomology of the wreath product $G = \mathbf{Z}/4 \wr \mathbf{Z}/2$ is well known in the literature. As pointed out by the referee, for a general wreath product group G, the Steenrod algebra structure of its cohomology can be found in the book of Steenrod and Epstein [SE], *Cohomology Operations*; earlier calculation of its co-algebra structure can be traced back to the work of Smith and Richardson in fixed point theory. More recently, its homology operations, together with the higher Bocksteins, have been obtained by May in [Ma]. As we need to understand these classes explicitly, we redo part of the computation.

Proposition (2.1). *The mod 2 cohomology of G, $H^*(G;\mathbf{F}_2)$, is generated by the classes $\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5$, with $|\gamma_0|=|\gamma_1|=1$, $|\gamma_2|=|\gamma_3|=2$, $|\gamma_4|=3$, and $|\gamma_5|=4$. These generators have the relations $\gamma_0\gamma_1=\gamma_1^2=\gamma_1\gamma_2=\gamma_2^2=\gamma_0\gamma_3=\gamma_0\gamma_4=\gamma_2\gamma_3+\gamma_1\gamma_4=\gamma_4^2=\gamma_2\gamma_4=0$; that is,*

$$H^*(G;\mathbf{F}_2) = \mathbf{F}_2[\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5] \text{ modulo the ideal}$$
$$\langle \gamma_0\gamma_1, (\gamma_1)^2, \gamma_1\gamma_2, (\gamma_2)^2, \gamma_0\gamma_3, \gamma_0\gamma_4, \gamma_2\gamma_3+\gamma_1\gamma_4, (\gamma_4)^2, \gamma_2\gamma_4\rangle.$$

The Bockstein operation, Sq^1, *is given by* $\mathrm{Sq}^1(\gamma_0)=(\gamma_0)^2$, $\mathrm{Sq}^1(\gamma_2)=\gamma_0\gamma_2$, $\mathrm{Sq}^1(\gamma_1)=\mathrm{Sq}^1(\gamma_3)=\mathrm{Sq}^1(\gamma_4)=\mathrm{Sq}^1(\gamma_5)=0$.

Consider the Lyndon–Hochschild–Serre spectral sequence associated to the group extension

$$1 \longrightarrow \mathbf{Z}/4 \times \mathbf{Z}/4 \longrightarrow G \longrightarrow \mathbf{Z}/2 \longrightarrow 1 \tag{2.2}$$

of the wreath product G. The $E_2^{p,q}$ terms of this spectral sequence with F_2 coefficients are given by

$$H^p(\mathbf{Z}/2; H^q(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{F}_2)).$$

The mod 2 cohomology of $\mathbf{Z}/4$ is well known and is given by

$$H^*(\mathbf{Z}/4; \mathbf{F}_2) \cong \mathbf{F}_2[\alpha, \beta]/\langle \alpha^2 \rangle, \quad |\alpha| = 1, \ |\beta| = 2.$$

Hence,

$$H^*(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{F}_2) \cong \Lambda[\alpha_1, \alpha_2] \otimes \mathbf{F}_2[\beta_1, \beta_2], \tag{2.3}$$

where $|\alpha_i| = 1$ and $|\beta_i| = 2$. The action of $\mathbf{Z}/2$ switches the generators—that is, $\alpha_1 \leftrightarrow \alpha_2$ and $\beta_1 \leftrightarrow \beta_2$. From this description, it is not difficult to calculate the $(\mathbf{Z}/2)$-invariant subalgebra $H^*(\mathbf{Z}/4 \otimes \mathbf{Z}/4; \mathbf{F}_2)^{\mathbf{Z}/2}$. Setting $\gamma_1 = \alpha_1 + \alpha_2$, $\gamma_2 = \alpha_1\alpha_2$, $\gamma_3 = \beta_1 + \beta_2$, $\gamma_4 = \alpha_1\beta_2 + \alpha_2\beta_1$, and $\gamma_5 = \beta_1\beta_2$, we have

$$H^0(\mathbf{Z}/2; H^*(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{F}_2)) \cong \frac{\mathbf{F}_2[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5]}{\langle (\gamma_1)^2, \gamma_1\gamma_2, (\gamma_2)^2, \gamma_2\gamma_3 + \gamma_1\gamma_4, (\gamma_4)^2, \gamma_2\gamma_4 \rangle}. \tag{2.4}$$

As a $\mathbf{Z}/2$ module, $H^{\mathrm{odd}}(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{F}_2)$ is free and $H^{\mathrm{even}}(\mathbf{Z}/4 \times \mathbf{Z}/4)$ is the direct sum of a free module and a trivial $\mathbf{Z}/2$ module. This gives

$$H^*(\mathbf{Z}/2; H^{\mathrm{odd}}(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{F}_2)) = 0,$$

and $H^*(\mathbf{Z}/2; H^{\mathrm{even}}(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{F}_2))$ is a free $H^*(\mathbf{Z}/2; \mathbf{F}_2)$ module on one generator. Putting this information together, we get the E_2 terms of the spectral sequence shown in Diagram (2.5).

$$\begin{array}{r|cccc}
\gamma_3\gamma_4, \gamma_1\gamma_5, \gamma_1\gamma_3^2 & \cdot 0 & \cdot 0 & \cdot 0 & \cdot 0 \\
\gamma_5, \gamma_2\gamma_3, \gamma_3^2 & \cdot \gamma_0\gamma_5 & \cdot \gamma_0^2\gamma_5 & \cdot \gamma_0^3\gamma_5 & \cdot \gamma_0^4\gamma_5 \\
\gamma_4, \gamma_1\gamma_3 & \cdot 0 & \cdot 0 & \cdot 0 & \cdot 0 \\
\gamma_2, \gamma_3 & \cdot \gamma_0\gamma_2 & \cdot \gamma_0^2\gamma_2 & \cdot \gamma_0^3\gamma_2 & \cdot \gamma_0^4\gamma_2 \\
\gamma_1 & \cdot 0 & \cdot 0 & \cdot 0 & \cdot 0 \\
\hline
 & \gamma_0 & \gamma_0^2 & \gamma_0^3 & \gamma_0^4
\end{array}$$

Diagram (2.5)

From (2.1), we see that all of the generators of the E_2 terms must be permanent cocycles, and hence the spectral sequence collapses at E_2. Routine computations give the following two results.

COROLLARY (2.6). *The Poincaré series*

$$\chi(G;\mathbf{F}_2)=\Sigma\dim[H^q(G;\mathbf{F}_2)]t^q$$

of the mod 2 cohomology $H^*(G;\mathbf{F}_2)$ *is given by*

$$1+2t+3t^2+\cdots+(n-1)t^n+\cdots.$$

COROLLARY (2.7). *Let* $\mathcal{E}_2$ *denote the second stage of the Bockstein spectral sequence for G. Then* $\mathcal{E}_2$ *is given by*

$$\Lambda[\gamma_1,\gamma_4]\otimes\mathbf{F}_2[\gamma_3,\gamma_5],$$

where $|\gamma_1|=1$, $|\gamma_3|=2$, $|\gamma_4|=3$, *and* $|\gamma_5|=4$.

Next we examine the integral cohomology, $H^*(G;\mathbf{Z})$, of G. Since this group is finite, the universal coefficient theorem implies that

$$\operatorname{rank}_{\mathbf{F}_2}H^n(G;\mathbf{F}_2)=\operatorname{rank}H^n(G;\mathbf{Z})+\operatorname{rank}H^{n+1}(G;\mathbf{Z}),$$

where rank $H^*(G;\mathbf{Z})$ means $\operatorname{rank}_{\mathbf{F}_2}H^*(G;\mathbf{Z})\otimes\mathbf{F}_2$.

Using the results of the $\mathbf{F}_2$ computation, this formula allows us to inductively compute the rank of $H^n(G;\mathbf{Z})$. In low dimensions, we have:

n	rank $H^n(G;\mathbf{F}_2)$	rank $H^n(G;\mathbf{Z})$	generators
1	2	0	0
2	3	2	g_0, g_1
3	4	1	g_2
4	5	3	g_0^2, g_1^2, g_3
5	6	2	g_4, g_2g_0
6	7	4	$g_0^3, g_1^3, g_3g_0, g_3g_1$

Table (2.8)

More generally, rank $H^{2k}(G;\mathbf{Z})=k+1$ and rank $H^{2k+1}(G;\mathbf{Z})=k$.

Using this data, we now calculate the LHS spectral sequence of the group extension (2.2) with integral coefficients. The $E_2^{p,q}$ term is given by

$$E_2^{p,q}=H^p(\mathbf{Z}/2;H^q(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z})),$$

and so the coefficient term is given by the degree-q subspace of the algebra

$$H^*(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z})\cong\mathbf{Z}/4[\bar{\beta}_1.\bar{\beta}_2.\chi]/\langle\chi^2\rangle, \tag{2.9}$$

where $|\bar{\beta}_i|=2$ and $|\chi|=3$. The notation indicates that the reduction modulo 2 of the class $\bar{\beta}_i$ is β_i of (2.3). The reduction of the class χ is $\alpha_1\beta_2+\alpha_2\beta_1$. As for the $\mathbf{Z}/2$ action, it exchanges $\bar{\beta}_1$ and $\bar{\beta}_2$ and sends χ to $-\chi$. The latter result can be deduced from a similar argument given for $H^*(\mathbf{Z}/2\otimes\mathbf{Z}/2;\mathbf{Z})$ in [L].

The subalgebra of invariants

$$E_2^{0,*}=H^*(\mathbf{Z}/4\otimes\mathbf{Z}/4;\mathbf{Z})^{\mathbf{Z}/2}$$

is generated by $g_1=\bar{\beta}_1+\bar{\beta}_2$, $g_2=2\chi$, $g_3=\overline{\beta_1\beta_2}$, and $g_4=\chi(\bar{\beta}_1-\bar{\beta}_2)$, satisfying the relations $(g_2)^2=0$, $(g_4)^2=0$, and $g_1g_2=2g_4$. The cohomology of $\mathbf{Z}/2$ with twisted coefficients is well known (see e.g. [Br]). In our case, for $p>0$, we have

$$H^p(\mathbf{Z}/2;\mathrm{H}^q(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z}))=\begin{cases}0 & \text{if } q\equiv 1,2 \bmod 4,\\ \mathbf{Z}/2 & \text{if } q\equiv 3,4 \bmod 4,\ q>0.\end{cases}$$

Finally, the bottom row is a polynomial algebra on a degree-2 generator of order 2 which we denote by g_0. The terms $p>0$ in the spectral sequence are periodic with respect to cup product by g_0. Let g_5 and g_6 be respectively the generator of $H^1(\mathbf{Z}/2;H^3(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z}))$ and $H^1(\mathbf{Z}/2;H^4(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z}))$. Then the product of these classes gives rise to g_5g_0, $g_5g_6^2,\dots$ and $g_6g_0^2,\dots$ in the following diagram for E_2.

$\mathbf{Z}/4g_1g_3, \mathbf{Z}/4g_1^3$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$
$\mathbf{Z}/4g_4$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$
$\mathbf{Z}/4g_3, \mathbf{Z}/4g_1^2$	$\cdot\mathbf{Z}/2g_6$	$\cdot\mathbf{Z}/2g_3g_0$	$\cdot\mathbf{Z}/2g_6g_0$	$\cdot\mathbf{Z}/2g_3g_0^2$	$\cdot\mathbf{Z}/2g_6g_0^2$	$\cdot\mathbf{Z}/2g_3g_0^3$
$\mathbf{Z}/2g_2$	$\cdot\mathbf{Z}/2g_5$	$\cdot\mathbf{Z}/2g_2g_0$	$\cdot\mathbf{Z}/2g_5g_0$	$\cdot\mathbf{Z}/2g_2g_0^2$	$\cdot\mathbf{Z}/2g_5g_0^2$	$\cdot\mathbf{Z}/2g_2g_0^3$
$\mathbf{Z}/4g_1$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$
0	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$	$\cdot 0$
0	0	g_0	0	g_0^2	0	g_0^3

Diagram (2.10)

The bottom row consists of permanent cocycles, because the extension (2.2) is split. The classes g_1 and g_2 must also be permanent cocycles since no nontrivial differential can affect them. To determine the differentials of the spectral sequence, we compare it with the corresponding spectral sequence with $\mathbf{F}_2$ coefficients. By their definitions, it is clear that the reduction modulo 2 of g_0 and g_1 are $(\gamma_0)^2$ and γ_3, respectively. The class g_2 must reduce to $\gamma_0\gamma_2$, since $\mathrm{Sq}^1\colon H^2(G;\mathbf{Z}/2)\to H^3(G;\mathbf{Z}/2)$ factors through $H^3(G;\mathbf{Z})\cong \mathbf{Z}/2$. By examining the map

$$H^*(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z})\to H^*(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z}/2),$$

we see that the reductions of g_3 and g_4 are γ_5 and $\gamma_3\gamma_4$, respectively.

PROPOSITION (2.11). *The integral cohomology* $H^4(G;\mathbf{Z})$, $\ell>0$, *contains a* $\mathbf{Z}/8$ *summand generated by* $[g_3]^\ell$.

From the discussion above, $H^4(G;\mathbf{Z})$ must have rank 3. There are classes

$$\mathbf{Z}/4[g_3]\oplus\mathbf{Z}/4[(g_1)^2]\oplus\mathbf{Z}/2[g_5]\oplus\mathbf{Z}/2[(g_0)^2]$$

on the appropriate diagonal of the spectral sequence (2.10). The class $\mathbf{Z}/4[g_1]$ generates a $\mathbf{Z}/4$ direct summand which accounts for $(g_1)^2$. Further, from the discussion of the reduction modulo 2 of these classes, we see that no nontrivial differential can affect them. Thus the only possibility is that there is a $\mathbf{Z}/8$ extension between $\mathbf{Z}/2[g_5]$ and $\mathbf{Z}/4[g_3]$.

PROPOSITION (2.12). *In the spectral sequence* (2.10) *we have* $d_2(g_6) = g_5 g_0$.

From the rank calculation, we know that the rank of $H^5(G; \mathbf{Z})$ is 2. By considering reduction modulo 2, we find that g_4 and $g_2 g_0$ are permanent cocycles. To show that $H^5(G; \mathbf{Z})$ contains a class of order 4 given by g_4, we consider the Bockstein spectral sequence whose $\mathcal{E}_2$ term was computed in Corollary (2.7). For $i > 1$, we denote the ith Bockstein operation by β_i. The first Bockstein operation is, of course, Sq^1. To account for the class $\mathbf{Z}/4[g_1]$ we must have $\beta_2(\gamma_1) = \gamma_3$, hence $\beta_2(\gamma_3) = 0$. Since γ_5 is the mod 2 reduction of $\mathbf{Z}/8[g_3]$, it must survive to $\mathcal{E}_3$; that is, $\beta_2(\gamma_5) = 0$. This also shows that $\mathcal{E}_2^2 \neq 0$, so $\beta_2(\gamma_4) = 0$ up to lower filtration terms. This determines all the differentials β_2 for $\mathcal{E}_2$. The image of β_2 in $\mathcal{E}_2^5$ consists of the class $\gamma_3 \gamma_4$. As this is the reduction of g_4, we must have that g_4 is precisely a $\mathbf{Z}/4$ class. We note that the remaining differentials in the Bockstein spectral sequence are now determined by $\beta_3(\gamma_4) = \gamma_5$. Thus

$$H^5(G; \mathbf{Z}) \cong \mathbf{Z}/4[g_4] \oplus \mathbf{Z}/2[g_2 g_0],$$

so the class g_6 must perish at E_2; that is, $d_2(g_6) = g_5 g_0$.

COROLLARY (2.13). (i) *There is a unique* $\mathbf{Z}/8$ *summand in the cohomology* $H^{4\ell}(G; \mathbf{Z})$ *generated by* g_3 *and no other* $\mathbf{Z}/8$ *summands.* (ii) *The* $\mathbf{Z}/4$ *summands in* $H^*(G; \mathbf{Z})$ *are given by the* $\mathbf{Z}/4$ *summands in the subalgebra*

$$(\mathbf{Z}/4[g_1] \otimes \mathbf{Z}/4[g_4] \otimes \mathbf{Z}/8[g_3]) / \langle (g_4)^2 \rangle.$$

3. The Cohomology of H

In this section we determine the cohomology of the group H with $\mathbf{Z}$ and $\mathbf{F}_2$ coefficients. Further, we examine the induced map in cohomology

$$\phi_1^*: H^*(G; \mathbf{Z}) \longrightarrow H^*(H; \mathbf{Z}),$$

which is one of the maps in the Mayer–Vietoris sequence (1.5).

Recall that H is isomorphic to the semidirect product

$$\frac{\mathbf{Z} \times \mathbf{Z}}{\mathbf{Z}\langle 4, -4 \rangle} \rtimes \mathbf{Z}/2,$$

and has a presentation given by generators $A = (1, -1)$, $B = (0, 1)$, and $C =$ switch, satisfying the relations

$$A^4 = C^2 = CAC^{-1} = [A, B] = 1, \qquad [C, B] = A. \tag{3.1}$$

From this there is clearly a split group extension

$$1 \longrightarrow \mathbf{Z}/4[A] \times \mathbf{Z}[B] \longrightarrow H \longrightarrow \mathbf{Z}/2[C] \longrightarrow 1 \tag{3.2}$$

with associated LHS spectral sequence

$$E_2^{p,q} = H^p(\mathbf{Z}/2; H^q(\mathbf{Z}/4\times\mathbf{Z}; R)) \tag{3.3}$$

for coefficient module R. As in the previous section, we are interested in the 2-torsion part of the cohomology, so here R is $\mathbf{Z}$ or $\mathbf{F}_2$.

First we calculate the spectral sequence (3.3) with $\mathbf{F}_2$ coefficients. Since $\mathbf{F}_2$ is a field,

$$E_2^{*,*} = H^*(\mathbf{Z}[A]\times\mathbf{Z}/4[B]; \mathbf{F}_2) \cong \Lambda[\rho_1, \rho_2]\otimes\mathbf{F}_2[\sigma], \tag{3.4}$$

where $|\rho_1| = |\rho_2| = 1$ and $|\sigma| = 2$. The classes ρ_1 and ρ_2 are the duals of the abelian group $\mathbf{Z}\times\mathbf{Z}/4$, that is, they are defined on the generators A, B by

$$\rho_1(A) = 0, \quad \rho_2(A) = 1,$$
$$\rho_1(B) = 1, \quad \rho_2(B) = 0.$$

From (3.1) and a straightforward computation, the action of C takes ρ_1 to itself, ρ_2 to $\rho_1+\rho_2$, and fixes σ. The subalgebra of invariants is given by

$$H^*(\mathbf{Z}\times\mathbf{Z}/4; \mathbf{F}_2)^{\mathbf{Z}/2} \cong \Lambda[\delta_1, \delta_2]\otimes\mathbf{F}_2[\delta_3]/\langle\delta_1\delta_2\rangle,$$

where we define $\delta_1 = \rho_1$, $\delta_2 = \rho_1\rho_2$, and $\delta_3 = \sigma$. Since there is a commutative diagram of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \dfrac{\mathbf{Z}\times\mathbf{Z}}{\mathbf{Z}\langle 4,-4\rangle} & \longrightarrow & H & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{Z}/4\times\mathbf{Z}/4 & \longrightarrow & G & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 1, \end{array} \tag{3.5}$$

we have a natural induced map

$$H^*(\mathbf{Z}/4\times\mathbf{Z}/4; \mathbf{F}_2)^{\mathbf{Z}/2} \to H^*(\mathbf{Z}\times\mathbf{Z}/4; \mathbf{F}_2)^{\mathbf{Z}/2}$$

which is given by

$$\gamma_1 = \alpha_1+\alpha_2 \mapsto \delta_1, \quad \gamma_2 = \alpha_1\alpha_2 \mapsto \delta_2, \quad \gamma_3 = \beta_1+\beta_2 \mapsto 0,$$
$$\gamma_4 = \alpha_1\beta_2+\alpha_2\beta_1 \mapsto \delta_1\delta_3, \quad \gamma_5 = \beta_1\beta_2 \mapsto (\delta_3)^2.$$

In all degrees, $H^*(\mathbf{Z}\times\mathbf{Z}/4; \mathbf{F}_2)$ has rank 2, and is given by

$$H^k(\mathbf{Z}\times\mathbf{Z}/4; \mathbf{F}_2) = \begin{cases} \mathbf{F}_2[\rho_1\rho_2\sigma^{i-1}]\oplus\mathbf{F}_2[\sigma^i] & \text{if } k = 2i, \\ \mathbf{F}_2[\rho_2\sigma^i]\oplus\mathbf{F}_2[(\rho_1+\rho_2)\sigma^i] & \text{if } k = 2i+1. \end{cases}$$

Furthermore, as a $\mathbf{Z}/2$ module, $H^*(\mathbf{Z}\times\mathbf{Z}/4; \mathbf{F}_2)$ is trivial in even degrees and free in odd degrees. Finally, we have

$$E_2^{*,0} = H^*(\mathbf{Z}/2; \mathbf{F}_2) \cong \mathbf{F}_2[\delta_0], \quad |\delta_0| = 1.$$

We put this information together in the spectral sequence diagram (3.6) (see next page). Since H is a semidirect product, the bottom row must survive to E_∞, and so all generators at the E_2 level are permanent cocycles. The algebra

$$
\begin{array}{r|llll}
 & \delta_0\delta_2\delta_3 & \delta_0^2\delta_2\delta_3 & & \\
\delta_2, \delta_3, \delta_3^2 & \cdot\delta_0\delta_3^2 & \cdot\delta_0^2\delta_3 & \cdots & \\
\delta_1\delta_3 & \cdot 0 & \cdot 0 & \cdot 0 & \cdot 0 \\
 & \delta_0\delta_2 & \delta_0^2\delta_3 & & \\
\delta_2, \delta_3 & \cdot\delta_0\delta_3 & \cdot\delta_0^2\delta_3 & \cdots & \\
\delta_1 & \cdot 0 & \cdot 0 & \cdot 0 & \cdot 0 \\
\hline
 & \delta_0 & \delta_0^2 & \delta_0^3 & \delta_0^4
\end{array}
$$

Diagram (3.6)

structure of $H^*(H; \mathbf{F}_2)$ and all but one of the Bocksteins follow from (2.1) via the map φ_1^*:

$$H^*(H; \mathbf{F}_2) \cong \mathbf{F}_2[\delta_0, \delta_1, \delta_2, \delta_3]/\langle \delta_0\delta_1, \delta_1\delta_2, (\delta_1)^2, (\delta_2)^2 \rangle,$$
$$\mathrm{Sq}^1(\delta_0) = (\delta_0)^2, \quad \mathrm{Sq}^1(\delta_1) = 0, \quad \mathrm{Sq}^1(\delta_2) = \delta_0\delta_2.$$

PROPOSITION (3.7). *The Bockstein* $\mathrm{Sq}^1(\delta_3)$ *is given by* $\mathrm{Sq}^1(\delta_3) = \delta_0\delta_3$.

There is a central extension

$$1 \longrightarrow \mathbf{Z}[AB]^2 \longrightarrow H \longrightarrow D_{16} \longrightarrow 1, \tag{3.8}$$

where D_{16} is the dihedral group of order 16. This leads to an exact sequence of group extensions

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \frac{\mathbf{Z}\times\mathbf{Z}}{\mathbf{Z}\langle 4, -4\rangle} & \longrightarrow & H & \longrightarrow & \mathbf{Z}/2[C] & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \\
1 & \longrightarrow & \mathbf{Z}/8[\bar{B}] & \longrightarrow & D_{16} & \longrightarrow & \mathbf{Z}/2[\bar{C}] & \longrightarrow & 1,
\end{array}
\tag{3.9}
$$

where $C \mapsto \bar{C}$, $A \mapsto \bar{B}^{-2}$, and $B \mapsto \bar{B}$. The cohomology of the dihedral group is well known; a detailed exposition is given in [Sn].

PROPOSITION (3.10). *The cohomology of the dihedral group of order* 2^n, $H^*(D_{2^n}, \mathbf{F}_2)$, *is given by*

$$H^*(D_{2^n}, \mathbf{F}_2) \cong \mathbf{F}_2[x_0, x_1, \omega]/\langle (x_1)^2 = x_0x_1 \rangle,$$

where $|x_i| = 1$ *and* $|\omega| = 2$. *The nontrivial Bockstein is given by* $\mathrm{Sq}^1(\omega) = x_0\omega$.

For the spectral sequence associated to the bottom row of (3.10),

$$H^*(\mathbf{Z}/8[\bar{B}]; \mathbf{F}_2) \cong \Lambda[x_1] \otimes \mathbf{F}_2[\omega]$$

and

$$H^*(\mathbf{Z}/2[\bar{C}]; \mathbf{F}_2) \cong \mathbf{F}_2[x_0].$$

The morphism of group extensions (3.9) leads to a map in cohomology given by $x_0 \mapsto \delta_0$, $x_1 \mapsto \delta_1$, $\omega \mapsto \delta_3$. The claim of (3.7) thus follows by naturality.

Using the preceding result, a routine computation yields the next proposition.

PROPOSITION (3.11). *The $\mathcal{E}_2$-term of the Bockstein spectral sequence for H is given by*

$$\Lambda[\epsilon_1, \epsilon_2, \epsilon_3] \otimes \mathbf{F}_2[\epsilon_4] / \langle \epsilon_1\epsilon_2, \epsilon_1\epsilon_3, \epsilon_2\epsilon_3 \rangle,$$

where $|\epsilon_1| = 1$, $|\epsilon_2| = 3$, and $|\epsilon_3| = |\epsilon_4| = 4$. The classes $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ have representatives δ_1, $\delta_1\delta_3$, $\delta_2\delta_3$, and $(\delta_3)^2$, respectively.

We now look at the spectral sequence associated to the group extension (3.2) with integral coefficients. Here the $E_2^{p,q}$ term is given by

$$E_2^{p,q}(H) = H^p(\mathbf{Z}/2; H^q(\mathbf{Z} \times \mathbf{Z}/4; \mathbf{Z}))$$

and the coefficient cohomology by

$$\begin{aligned} H^*(\mathbf{Z} \times \mathbf{Z}/4; \mathbf{Z}) &= H^*(\mathbf{Z}; \mathbf{Z}) \otimes H^*(\mathbf{Z}/4; \mathbf{Z}) \\ &= \mathbf{Z}[\rho, \delta] / \langle 4\delta, \rho^2 \rangle, \end{aligned}$$

where $|\rho| = 1$ and $|\delta| = 2$. The $\mathbf{Z}/2$ action on $\mathbf{Z}[\rho, \delta]/\langle 4\delta, \rho^2\rangle$ preserves the generator ρ, but sends δ to its negative. Thus the invariant subalgebra $H^0(\mathbf{Z}/2; H^*(\mathbf{Z} \times \mathbf{Z}/4; \mathbf{Z}))$ of $H^*(\mathbf{Z} \times \mathbf{Z}/4; \mathbf{Z})$ is generated by the three elements $h_1 = \rho$, $h_2 = 2\delta$, and $h_3 = \delta^2$; periodically, the invariant subalgebra of a particular degree q alternates between two successive copies of $\mathbf{Z}/2(h_2 h_3^\ell)$, $\mathbf{Z}/2(h_1 h_2 h_3^\ell)$ for $q \equiv 2, 3 \bmod 4$ and copies of $\mathbf{Z}/4(h_3^\ell)$, $\mathbf{Z}/4(h_1 h_3^\ell)$ for $q \equiv 0, 1 \bmod 4$.

The bottom row $E_2^{*,0}$ of the spectral sequence is isomorphic to a polynomial algebra $H^*(\mathbf{Z}/2; \mathbf{Z}) \cong \mathbf{Z}/2[h_0]$, $* > 0$, as in the first row $E_2^{*,1}$, which is isomorphic to $\mathbf{Z}/2[h_1 h_0]$. Above these two bottom rows, we see that each of the terms in $E_2^{p,q}$, $p, q > 0$, is isomorphic to $\mathbf{Z}/2$. As a $H^*(\mathbf{Z}/2, \mathbf{Z})$ module, the horizontal row $E_2^{*,p}$, $p > 2$, is freely generated by classes in $E_2^{*,1}$ and $E_2^{*,2}$.

To sum up the situation, we have the following diagram of the E_2 terms.

$$\begin{array}{r|llll}
\mathbf{Z}/4 h_1 h_3 & \cdot h_1 h_5 & \cdot h_1 h_3 h_0 & \cdot h_1 h_5 h_0 & \cdot h_1 h_3 h_0^2 \\
\mathbf{Z}/4 h_3 & \cdot h_5 & \cdot h_3 h_0 & \cdot h_5 h_0 & \cdot h_3 h_0^2 \\
h_2 h_1 & \cdot h_4 h_1 & \cdot h_2 h_1 h_0 & \cdot h_1 h_4 h_0 & \cdot h_1 h_2 h_0^2 \\
h_2 & \cdot h_4 & \cdot h_2 h_0 & \cdot h_4 h_0 & \cdot h_2 h_0^2 \\
\mathbf{Z} h_1 & \cdot 0 & \cdot h_1 h_0 & \cdot 0 & \cdot h_1 h_0^2 \\
\hline
 & 0 & h_0 & 0 & h_0^2
\end{array}$$

Diagram (3.12)

Using (3.6) and the universal coefficient theorem, we deduce the rank of $H^*(H; \mathbf{Z})$ as shown in Table (3.13) (see next page). In general,

$$\operatorname{rank} H^{2\ell}(H; \mathbf{Z}) = \operatorname{rank} H^{2\ell-1}(H; \mathbf{Z}) = \ell \quad \text{for } \ell > 0.$$

n	rank $H^n(H;\mathbf{Z})$	rank $H^n(H;\mathbf{F}_2)$
1	1	1
2	1	2
3	2	3
4	2	4
5	3	5
6	3	6

Table (3.13)

LEMMA (3.14). *There is a direct summand isomorphic to* $\mathbf{Z}/8$ *in* $H^{4\ell}(H;\mathbf{Z})$.

To see this, we examine the LHS spectral sequence associated to the central extension

$$1 \longrightarrow \mathbf{Z}[AB]^2 \longrightarrow H \longrightarrow D_{16} \longrightarrow 1$$

which appeared in (3.8). The cohomology of $\mathbf{Z}$ is an exterior algebra on a 1-dimensional class, say ρ. On the other hand, the integral cohomology of the dihedral group can be found in [Th, p. 47, Thm. 4.6]: $H^*(D_{16};\mathbf{Z})$ is generated by classes α, β, ζ, $|\alpha|=|\beta|=2$, $|\gamma|=3$, $|\zeta|=4$, subject to the relations

$$8\zeta = 2\gamma = 2\alpha = 2g = 0,$$

$$\gamma^2 = \beta\zeta, \qquad \alpha^2 = \alpha\beta.$$

Hence we have the following spectral sequence diagram, which must collapse at E_3.

$$\begin{array}{c|cccccc}
\rho & \cdot 0 & \begin{matrix} \cdot\rho\alpha \\ \rho\beta \end{matrix} & \cdot\rho\gamma & \begin{matrix} \rho\alpha\beta \\ \cdot\rho\beta^2 \\ \rho\zeta \end{matrix} & \begin{matrix} \cdot\rho\alpha\gamma \\ \rho\beta\gamma \end{matrix} & \cdots \\
\hline
 & 0 & \begin{matrix} \alpha \\ \beta \end{matrix} & \gamma & \begin{matrix} \alpha\beta \\ \beta^2 \\ \zeta \end{matrix} & \begin{matrix} \alpha\gamma \\ \beta\gamma \end{matrix} & \begin{matrix} \alpha\beta^2 \\ \beta^3 \\ \alpha\zeta \\ \beta\zeta = \gamma^2 \end{matrix}
\end{array}$$

Diagram (3.15)

The differentials are determined by $d_2(\rho) = c_1\alpha + c_2\beta$, $c_1, c_2 \in \{0,1\}$, and cup product. Abelianizing the presentation of H shows that the torsion piece of $H^2(H;\mathbf{Z})$ is $\mathbf{Z}/2$, so this differential is nonzero. It is clear that the class ζ is not in the image of $d_2^{2,2}$ because ζ is not in the ideal generated by α, β, and that $d_2(\rho\gamma)$ has a nontrivial image. Hence at E_3 the term $\rho\gamma$ disappears, and since there exist no other terms in $E_3^{1,3}$ there is no extension problem and the class ζ survives to a $\mathbf{Z}/8$ summand in $H^4(H,\mathbf{Z})$. By cup product, ζ^ℓ gives a corresponding $\mathbf{Z}/8$ summand in degree divisible by 4, proving (3.14).

PROPOSITION (3.16). *The d_2 differentials in the spectral sequence* (3.12) *are determined by:*

$$d_2(h_1)=0, \quad d_2(h_2)=h_1h_0, \quad d_2(h_3)=0,$$
$$d_2(h_4)=0, \quad d_5(h_5)=h_1h_4h_0.$$

From the E_2 diagram, we see that there is a single class, $[h_1]$, in degree 1. Abelianizing the presentation of H, it is clear that this class must be a permanent cocycle. Further, the reduction modulo 2 of this class is δ_1. Since the extension in (3.2) is split, $\mathbf{Z}/2[h_0]$ generates a direct summand of the cohomology, so the terms in the bottom row of the spectral sequence are permanent cocycles. So we see that $d_2(h_1)=0$.

We now consider degree 2. It is clear that h_0 must reduce to $(\delta_0)^2$ modulo 2. There is, however, a second class—that is, $\mathbf{Z}/2[h_2]$. From the rank calculation above, the only possibility is that $d_2(h_2)=h_1h_0$.

There are three classes of degree 3: $\mathbf{Z}/2[h_2h_1]$, $\mathbf{Z}/2[h_4]$, and $\mathbf{Z}/2[h_1h_0]$. It follows from the preceding paragraphs that h_2h_1 and h_1h_0 are in the kernel of d_2, and that h_1h_0 is in the image of d_2. Thus we have $d_2(h_4)=0$ from rank considerations, or more simply from its position.

We now consider degree 4, which is the direct sum of the classes $\mathbf{Z}/4[h_3]$, $\mathbf{Z}/2[h_4h_1]$, $\mathbf{Z}/2[h_2h_0]$, and $\mathbf{Z}/2[(h_0)^2]$. The term h_2h_0 does not survive to E_3, because its image under d_2 is nonzero. We know that $H^4(H;\mathbf{Z})$ is rank 2 and contains a summand isomorphic to $\mathbf{Z}/8$ by Lemma (3.14), and that $(h_0)^2$ is a $\mathbf{Z}/2$ direct summand. The only possibility is that there is an extension between h_3 and h_4h_1. Thus $d_2(h_3)=0$.

The degree-5 classes in the spectral sequence are given by

$$\mathbf{Z}/4[h_1h_3]\oplus\mathbf{Z}/2[h_5]\oplus\mathbf{Z}/2[h_2h_1h_0]\oplus\mathbf{Z}/2[h_4h_0]\oplus\mathbf{Z}/2[h_1h_0^2].$$

From our previous computations we know that h_1h_3, $h_2h_1h_0$, and h_4h_0 are in the kernel of d_2, and that $h_1h_0^2$ has a nonzero image. As for the remaining class h_5, its image under d_2 is determined by naturality. Comparing with the spectral sequence for G, we have $g_5\mapsto h_4h_1$ and $g_6\mapsto h_5$ under φ_1^*. Because $d_2(g_6)=g_5g_0$, $d_2(h_5)=h_2h_1h_0$.

An examination of the E_3 term now shows that all the generators are permanent cocycles. This concludes (3.16).

We now determine the induced map φ_1^* in cohomology. As mentioned above, $g_5\mapsto h_4h_1$ and $g_6\mapsto h_5$. Furthermore, it is clear from the definition that $\varphi_1^*(g_0)=h_0$. The images of the remaining classes can be determined by examining the induced homomorphism

$$\varphi_1^*\colon H^*(\mathbf{Z}/4\times\mathbf{Z}/4;\mathbf{Z})\longrightarrow H^*(\mathbf{Z}\times\mathbf{Z}/4;\mathbf{Z}).$$

Since the homomorphism φ_1 sends $1\times\mathbf{Z}/4$ to the antidiagonal subgroup $\mathbf{Z}/4(1,-1)$ in $\mathbf{Z}/4\times\mathbf{Z}/4$, the induced homomorphism φ_1^* sends β_1 to δ and β_2 to $-\delta$. By using the Künneth formula, it is not difficult to see that φ_1^*:

$H^3(\mathbf{Z}/4 \times \mathbf{Z}/4; \mathbf{Z}) \to H^3(\mathbf{Z} \times \mathbf{Z}/4; \mathbf{Z})$ is a surjection, and so $\varphi_1^*(\chi) = \pm \rho\delta$. In terms of the $(\mathbf{Z}/2)$-invariant cohomology classes, we have $g_1 \mapsto 0$, $g_2 \mapsto h_1 h_2$, $g_3 \mapsto h_3$, and $g_4 \mapsto 2h_1 h_2$. Thus we have the following.

Proposition (3.17). *The induced map in cohomology*

$$\varphi_1^*: H^*(G; R) \longrightarrow H^*(H; R)$$

is given by

$$\gamma_0 \mapsto \delta_0, \quad \gamma_1 \mapsto \delta_1, \quad \gamma_2 \mapsto \delta_2,$$
$$\gamma_3 \mapsto 0, \quad \gamma_4 \mapsto \delta_1\delta_3, \quad \gamma_5 \mapsto \delta_3^2$$

for $R = \mathbf{F}_2$, *and by*

$$g_0 \mapsto h_0, \quad g_1 \mapsto 0, \quad g_2 \mapsto h_1 h_2,$$
$$g_3 \mapsto h_3, \quad g_4 \mapsto 2h_1 h_3$$

for $R = \mathbf{Z}$.

In the remaining sections, we find it convenient to express the cohomology classes of $H^*(H; \mathbf{Z})$ in a different form. In the spectral sequence (3.15), $d_2(\rho) \neq 0$, because $H^2(H; \mathbf{Z})$ is isomorphic to the torsion summand of $H_1(H; \mathbf{Z})$ and this is easily seen to be cyclic of order 2. Furthermore, $d_2(\rho) \neq \beta$, since β generates a direct summand in the cohomology of H. This follows from the fact that there is a split surjection of H onto a cyclic group of order 2 corresponding to the generator C. Thus, $d_2(\rho)$ is either α or $\alpha + \beta$. These two choices differ by an automorphism of $H^*(D_{16}; \mathbf{Z})$, and we take $d_2(\rho) = \alpha$.

Defining $\sigma = \rho(\alpha + \beta)$ and $\rho' = 2\rho$, we have the following E_3 terms, at which the spectral sequence collapses.

$$\begin{array}{c|ccccc}
 & & & & \beta\sigma & \\
\rho' & \cdot 0 & \cdot\sigma & \cdot 0 & \cdot\rho'\zeta & \cdots \\
\hline
 & 0 & \beta & \gamma & \beta^2 & \beta\gamma \\
 & & & & \zeta &
\end{array}$$

Diagram (3.18)

We note that ρ' is a class of infinite order, which we denote by ρ in the sequel. Further, we note that $\rho\zeta \in H^5(H; \mathbf{Z})$ has order 4.

The correspondence between the notation here and that in the previous calculation of $H^*(H; \mathbf{Z})$ follows. It is not hard to see that $h_0 = \beta$, $h_1 = \rho$, and $h_3 = \zeta$. The classes $h_4, h_2 h_1 \in H^3(H; \mathbf{Z})$ must correspond to classes in $\langle \gamma, \sigma \rangle$. However, by examining the reduction modulo 2 of the relation $\gamma^2 = \beta\zeta$, we see that $h_4 = \gamma$. Thus $h_2 h_1 = \sigma +$ lower filtration terms.

For the convenience of the reader we include the following table, which summarizes information about the integral cohomology classes of H.

class	degree	alias	reduction	order
ρ	1	h_1	δ_1	∞
β	2	h_0	δ_0^2	2
γ	3	h_4	$\delta_0\delta_3$	2
σ	3	h_2h_1	$\delta_0\delta_2$	2
ζ	4	h_3	δ_3^2	8

Table (3.19)

COROLLARY (3.20). *The image of the induced homomorphism*

$$\varphi_1^*: H^*(G;\mathbf{Z}) \longrightarrow H^*(H;\mathbf{Z})$$

is the subalgebra of $H^*(H;\mathbf{Z})$ *generated by* ζ, σ, *and* $2\rho\zeta$.

It follows immediately that the cokernel of φ_1^* has the following additive basis:

$$\rho,\ \rho\zeta \bmod 2,\ \rho\zeta^2 \bmod 2, \ldots, \rho\zeta^i \bmod 2, \gamma, \beta\gamma, \zeta\gamma, \ldots, \beta^j\zeta^k\gamma, \ldots . \tag{3.21}$$

Note that, with the exception of the infinite cyclic summand $\mathbf{Z}[\rho]$, the cokernel of φ_1^* consists entirely of $\mathbf{Z}/2$ summands.

4. The Image of φ_2^*

In [BC; Co], Benson and Cohen computed the mod 2 integral cohomology of the mapping class groups Γ_0^6 and Γ_2^0. Their method involves examining the group extension

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow \Gamma_2^0 \longrightarrow \Gamma_0^6 \longrightarrow 1. \tag{4.1}$$

In this section we apply their results to get a description of φ_2^*. We first recall the presentations of the relevant groups. The braid group B_n has a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$[\sigma_i, \sigma_j] = 1, \quad |i-j| > 1; \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{4.2}$$

To get a presentation of the mapping class group Γ_0^n, we add the relations

$$(\sigma_1\sigma_2\cdots\sigma_{\nu-1})^n = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 = 1$$

to the relations (4.2). If $n=6$ and we add the relations

$$(\sigma_1\cdots\sigma_4\sigma_5^2\sigma_4\cdots\sigma_1)^2 = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^6 = 1 \quad \text{and} \quad [\sigma_1\cdots\sigma_4\sigma_5^2\sigma_4\cdots\sigma_1, \sigma_i] = 1$$

to the braid relations (4.2), we get a presentation of Γ_2^0. We denote braid group generators by σ_i, the generators of Γ_0^6 by $\hat{\sigma}_i$, and the generators of Γ_2^0 by ζ_i. Further, we define the words

$$\mathfrak{Z} = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_1\zeta_2\zeta_3\zeta_4\zeta_1\zeta_2\zeta_3\zeta_1\zeta_2\zeta_1;$$

$$\zeta=\zeta_1\cdots\zeta_{n-2}\zeta_{n-1}^2\zeta_{n-2}\cdots\zeta_1.$$

The image of the half-twist $\mathfrak{Z}$ has order 2 in Γ_2^0 since its square, considered as an element of B_6, generates the center. Furthermore, we have

$$\mathfrak{Z}\sigma_1\mathfrak{Z}=\sigma_5,\quad \mathfrak{Z}\sigma_2\mathfrak{Z}=\sigma_4,\quad \mathfrak{Z}\sigma_3\mathfrak{Z}=\sigma_3.$$

The image of ζ in Γ_2^0 is the hyperellpitic involution, so the central extension (4.1) has the form

$$1\longrightarrow \mathbf{Z}/2[\zeta]\longrightarrow \Gamma_2^0\xrightarrow{\psi}\Gamma_0^6\longrightarrow 1.$$

In order to proceed, we need a more explicit description of the inclusion $\varphi_2: H\to\Gamma_2^0$.

Proposition (4.1). *Let* $\mathfrak{B}$ *be defined as in Section 1. The map* $\varphi_2:\mathfrak{B}\rtimes\mathbf{Z}/2\to\Gamma_2^0$ *maps* $\mathfrak{B}\rtimes\mathbf{Z}/2$ *isomorphically onto the subgroup generated by* ζ_1, ζ_2, ζ_4, ζ_5, *and* $\mathfrak{Z}$.

Proposition (4.2). *The map* $\varphi_2|_H: H\to\Gamma_2^0$ *is given by*

$$A\mapsto\zeta_1\zeta_2\zeta_1(\zeta_4\zeta_5\zeta_4)^{-1},\quad B\mapsto\zeta_4\zeta_5\zeta_4,\quad C\mapsto\mathfrak{Z},$$

where $\mathfrak{Z}$ *denotes the image of the half-twist braid. Furthermore, composing* φ_2 *with the surjection* $\Gamma_2^0\to\Sigma_6$, *we obtain a map* $\Phi: H\to\Sigma_6$ *whose image is isomorphic to* D_8, *the dihedral group of order* 8.

We note that the image of A^2 is the hyperelliptic involution. Consider the group extension

$$1\longrightarrow\mathbf{Z}/2\longrightarrow\Gamma_2^0\xrightarrow{\psi}\Gamma_0^6\longrightarrow 1.$$

The element $(\zeta_1\zeta_2\zeta_1)^2(\zeta_4\zeta_5\zeta_4)^{-2}$ is mapped to the identity element by ψ. In particular, this element can be taken to be one of the generators of the kernel of the map $B_6\mapsto\Gamma_0^6$ (cf. [MKS]). Thus $(\zeta_1\zeta_2\zeta_1)^2(\zeta_4\zeta_5\zeta_4)^{-2}$ must be either the identity in Γ_2^0 or the hyperelliptic involution. However, it is easy to see that it is not the identity, since its image in Sp(4, **Z**) under ι_2 is $-I$.

Given the map $H\to\Gamma_2^0$, its image is easily computed:

$$A\mapsto(13)(46):=a,\quad B\mapsto(46):=b,\quad C\mapsto(16)(25)(43):=c.$$

Adding the relations $a^2=b^2=1$ to the relations that come from H, we have the presentation of the dihedral group of order 8. The subgroup of the image in A_6 is cyclic of order 4, generated by cb.

An examination of the relations following (4.2) shows that there is a natural homomorphism from Γ_0^6 to the symmetric group Σ_6. We call the kernel of this map the *projective symplectic group*, $\mathrm{PSp}\,\Gamma_0^6$,

$$1\longrightarrow\mathrm{PSp}\,\Gamma_0^6\longrightarrow\Gamma_0^6\longrightarrow\Sigma_6\longrightarrow 1. \tag{4.5}$$

A set of generators and relations for H' can be derived in the same manner as for H (cf. Section 3). In fact, we call the corresponding generators A', B', and C'. There is a map $\Phi'\colon H'\to\Sigma_6$, defined analogously to Φ. This leads us to the following commutative diagram of group extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}[(B')^2] & \longrightarrow & H' & \xrightarrow{\Phi'|_{H'}} & D_8 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_2} & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{PSp}\,\Gamma_0^6 & \longrightarrow & \Gamma_0^6 & \xrightarrow{\Phi'} & \Sigma_6 & \longrightarrow & 1. \end{array} \tag{4.6}$$

The generators of H' have the same images in Σ_6 as their counterparts in H.

Let D_8 be the semidirect product of $\mathbf{Z}/4[d]$ and $\mathbf{Z}/2[c]$, and let $x_i\colon D_8\to\mathbf{Z}/2$ be the homomorphism given by

$$x_1(c)=1,\quad x_1(d)=0,$$
$$x_2(c)=0,\quad x_2(d)=1.$$

In our notation above, $d=bc$; thus the maps are given by

	b	c	bc
x_1	1	0	1
x_1+x_2	0	1	1
x_2	1	0	1.

We now compute the map $H^*(\Sigma_6;\mathbf{F}_2)\to H^*(D_8;\mathbf{F}_2)$ arising from the commutative diagram (4.6). The cohomology ring $H^*(\Sigma_6;\mathbf{F}_2)$ is given in [BC, Lemma 5.6]:

$$H^*(\Sigma_6;\mathbf{F}_2)=\mathbf{F}_2[\delta,\alpha,\beta,\gamma]/\langle\beta\gamma\rangle\cong H^*(A_6;\mathbf{F}_2)\otimes\mathbf{F}_2[\delta],$$
$$|\delta|=1,\ |\alpha|=2,\ |\beta|=|\gamma|=3.$$

The action of the Steenrod algebra is given by

	Sq^1	Sq^2
α	$\beta+\gamma$	α^2
β	0	$\alpha\beta$
γ	0	$\alpha\gamma$.

Here δ is the sign homomorphism, and hence $\delta\mapsto x_1$.

PROPOSITION (4.7). *Under the induced homomorphism $H^*(\Sigma_6;\mathbf{F}_2)\to H^*(D_8;\mathbf{F}_2)$, the element α is sent to $\omega+x_1^2$.*

Under our explicit embedding of D_8 in Σ_6, we have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}/4 & \longrightarrow & D_8 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \vartheta} & & \downarrow & & \\ 1 & \longrightarrow & A_6 & \longrightarrow & \Sigma_6 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 1. \end{array}$$

By restriction to A_6, we see that $\vartheta^*(\alpha)=\omega+c_1x_1^2+c_2x_1x_2$ for $c_1,c_2\in\mathbf{F}_2$.

Examining the diagram

$$\begin{array}{ccc}
\alpha & \xrightarrow{\vartheta^*} & \omega + c_1 x_1^2 + c_2 x_1 x_2 \\
\big\downarrow \mathrm{Sq}^1 & & \big\downarrow \\
\beta+\gamma & \longrightarrow & x_1\omega \\
\big\downarrow \mathrm{Sq}^2 & & \big\downarrow \\
\alpha(\beta+\gamma) & \longrightarrow & (\omega + x_1^2)x_1\omega,
\end{array}$$

we see that $\vartheta^*(\beta+\gamma) = x_1\omega$, but since multiplication by $x_1\omega$ is injective in $H^*(D_8; \mathbf{F}_2)$, $\vartheta^*(\alpha) = \omega + x_1^2$. By a similar computation, we see that

$$\vartheta^*(\beta) = x_2\omega, \qquad \vartheta^*(\gamma) = (x_1+x_2)\omega,$$

or vice versa. To be precise, there is a choice involved in here; however, because the two possibilities are related by an automorphism of the cohomology algebra, it does not matter in our calculation. To be consistent, we make sure that our choice agrees with that of [BC].

The characteristic class of the upper central extension in (4.6) is given by x_2^2, since x_2 is the reduction modolu 2 of the integral class in $H^*(H'; \mathbf{Z})$. From [BC, p. 70], we see that $(\delta^3 + \alpha\delta + \beta)^2 = 0$ is a relation in $H^*(\Gamma_0^6; \mathbf{F}_2)$; hence, by naturality, the pullback of this relation must be 0 in $H^*(H'; \mathbf{F}_2)$. We have

$$\begin{aligned}
\vartheta^*(\delta^3 + \alpha\delta + \beta) &= x_1^3 + x_1(x_1^2 + \omega) + \psi^*(\beta) \\
&= x_1\omega + \psi^*(\beta),
\end{aligned}$$

but $\vartheta^*(\beta)$ cannot be $x_2\omega$ because $(x_1+x_2)\omega$ is not nilpotent. Thus $\vartheta^*(\beta) = (x_1+x_2)\omega = (x_1+x_2)\omega$. Summarizing the computations above, we have the following proposition.

PROPOSITION (4.8). *The map*

$$\vartheta^*: H^*(\Sigma_6; \mathbf{F}_2) \longrightarrow H^*(D_8; \mathbf{F}_2)$$

is given by

$$\delta \mapsto x_1, \quad \alpha \mapsto x_1^2 + \omega,$$

$$\beta \mapsto (x_1+x_2)\omega, \quad \gamma \mapsto x_2\omega.$$

Computing the spectral sequence associated to the top exact sequence in (4.6), we have $E_2^{*,*} = \Lambda[\rho] \otimes H^*(D_8; \mathbf{F}_2)$. Thus, the E_2 terms are

$$\begin{array}{c|ccc}
 & & \rho x_1^2 & \\
\rho & \cdot\rho x_1 & \cdot\rho x_1 x_2 & \cdots \\
 & \rho x_2 & \rho\omega & \\
\hline
 & x_1 & x_1^2 & x_1^3 \\
 & x_2 & x_1 x_2 & x_1^2 x_2 \\
 & & \omega & x_1\omega \\
 & & & x_2\omega
\end{array}$$

and all differentials are determined by $d_2(\rho) = x_1 x_2$. As no other differentials are possible, the spectral sequence collapses at E_3:

$$\begin{array}{c|ccc} 0 & \cdot\sigma & \cdot\sigma x_1 & \cdots \\ \hline & x_1 & x_1^2 & x_1^3 \\ & x_2 & x_1 x_2 & x_1\omega \\ & & \omega & x_2\omega \end{array}$$

Here $\sigma = \rho(x_1 + x_2)$ and satisfies the cohomology relations

$$x_2^2 = x_2\sigma = \sigma^2 = 0.$$

PROPOSITION (4.9). *The map*

$$(\phi_2)^*: H^*(\Gamma_0^6; \mathbf{F}_2) \longrightarrow H^*(H'; \mathbf{F}_2)$$

is given by Proposition (4.8) and $u \mapsto \sigma$.

The proposition follows from the preceding computations and also from determining the induced map

$$H^1(A_6; H^1(\mathrm{PSp}\,\Gamma_0^6; \mathbf{F}_2)) \longrightarrow H^1(D_8; H^1(\mathbf{Z}; \mathbf{F}_2)).$$

In [BC], a change of notation was made at this point, and to avoid confusion we write $U = u + \delta^2$. The characteristic class of the central extension

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow \Gamma_2^0 \longrightarrow \Gamma_0^6 \longrightarrow 1 \tag{4.10}$$

is given by $U + \alpha$. In the third part of [BC], it is determined that

$$\begin{aligned} \mathrm{Sq}^1(U) &= \delta U + \alpha\delta + \beta \\ &= \delta^3 + \alpha\delta + \beta + \delta U. \end{aligned}$$

The central extension (4.10) leads us to another commutative diagram of group extensions,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \Gamma_2^0 & \longrightarrow & \Gamma_0^6 & \longrightarrow & 1, \end{array} \tag{4.11}$$

and to a corresponding morphism of spectral sequences. For the bottom spectral sequence of (4.11), we have

$$E_2^{*,*} = \mathbf{F}_2[z] \otimes H^*(\Gamma_0^6; \mathbf{F}_2),$$

with

$$d_2(z) = U + \alpha, \qquad d_3(z^2) = \gamma,$$

and $d_n \equiv 0$ for $n \geq 4$. Proposition (4.9) and the computation of $\mathrm{Sq}^1(U)$ determine the cohomology algebra structure of $H^*(H'; \mathbf{F}_2)$ by naturality. Explicitly, we have the following.

PROPOSITION (4.12). *The cohomology algebra of H' is generated by x_1, x_2, ω, σ, together with the relations $x_1^2 = x_1x_2 = x_2\sigma = \sigma^2 = 0$; that is,*

$$H^*(H'; \mathbf{F}_2) \cong \mathbf{F}_2[x_1, x_2, \omega, \sigma]/\langle x_1^2, x_1x_2, x_2\sigma, \sigma^2\rangle$$

and

$$\mathrm{Sq}^1(\omega) = x_1\omega, \qquad \mathrm{Sq}^1(\sigma) = x_1\sigma + x_2\omega.$$

Now, for the top spectral sequence of the extension (4.11), we have

$$E_2^{*,*} = \mathbf{F}_2[\hat{z}] \otimes H^*(H'; \mathbf{F}_2),$$

$$d_2(\hat{z}) = \sigma + \omega, \qquad d_3(\hat{z}^2) = x_2\omega.$$

However, $d_2(\hat{z}x_2) = x_2(\sigma+\omega) = x_2\omega$, so the spectral sequence collapses at E_3. Following is a diagram of the E_2 terms of this spectral sequence.

$$\begin{array}{c|cccc}
 & & \vdots & & \\
\hat{z}^2 & \begin{array}{c} \cdot\hat{z}^2x_1 \\ \hat{z}^2x_2 \end{array} & \begin{array}{c} \hat{z}^2x_1^2 \\ \cdot\hat{z}^2\omega \\ \hat{z}^2\sigma \end{array} & \cdots & \\
 & & & & \\
\hat{z} & \begin{array}{c} \cdot\hat{z}x_1 \\ \hat{z}x_2 \end{array} & \begin{array}{c} \hat{z}x_1^2 \\ \cdot\hat{z}\omega \\ \hat{z}\sigma \end{array} & \cdots & \\
\hline
 & \begin{array}{c} x_1 \\ x_2 \\ \\ \\ \\ \end{array} & \begin{array}{c} x_1^2 \\ \omega \\ \sigma \\ \\ \\ \end{array} & \begin{array}{c} x_1^3 \\ x_1\omega \\ x_2\omega \\ x_1\sigma \\ \\ \end{array} & \begin{array}{c} x_1^4 \\ x_1^2\omega \\ \omega^2 \\ x_1^2\sigma \\ \omega\sigma \end{array}
\end{array}$$

Diagram (4.13)

The resulting $E_3 = E_\infty$ is the tensor product of $\mathbf{F}_2[\hat{z}^2]$ and the algebra on the bottom line of the spectral sequence. We denote the inflation of $\omega \in H^2(H'; \mathbf{F}_2)$ by $\hat{\omega}$, and note that this is also the inflation of σ. Hence this algebra is given by

$$\mathbf{F}_2[\hat{x}_1, \hat{x}_2, \hat{x}_3]/\langle \hat{x}_2^2 = \hat{x}_1\hat{x}_2 = \hat{x}_2\hat{\omega} = \hat{\omega}^2 = 0\rangle,$$

with $\mathrm{Sq}^1(\hat{\omega}) = \hat{x}_1\hat{\omega}$. Comparing with our previous calculation of $H^*(H; \mathbf{F}_2)$ we have a correspondence between classes given by $\hat{x}_1, \hat{x}_2\hat{\omega}, \hat{z}^2$ and $\delta_0, \delta_1, \delta_2, \delta_3$.

Recall that the mod 2 cohomology of Γ_2^0 is generated as an algebra by the classes $\alpha, \beta, \delta, z^2\delta, z^2\alpha, z^2\beta$, and z^4 [BC, p. 102]. The map $\varphi_2^*\colon H^*(\Gamma_2^0; \mathbf{F}_2) \to H^*(H; \mathbf{F}_2)$ is determined by the morphism of group extensions (4.6) and the fact that all the classes above the bottom line split as products in the E_2 terms of the spectral sequence for $H^*(H; \mathbf{F}_2)$. From (4.9), we have that $\delta \mapsto \hat{x}_1$, $\alpha \mapsto \hat{x}_1^2 + \hat{\omega}$, and $\beta \mapsto \hat{x}_1\hat{\omega}$.

We now determine the map φ_2^* from $H^*(\Gamma_2^0;\mathbf{Z})$ to $H^*(H;\mathbf{Z})$. To avoid confusion, for the remainder of this section *we will denote some classes of $H^*(H;\mathbf{Z})$ with the subscript H.* Hopefully, the reader will excuse this temporary overlap of notation. Refering to [C, p. 35], we have the following table.

i	$\dim_{\mathbf{F}_2} H^*(\Gamma_2^0;\mathbf{Z})\otimes\mathbf{F}_2$
1	0
2	1
3	1
4	3
5	3

Table (4.14)

From the preceding computations and results of Cohen [Co, Lemmas 7.4 and 7.5], the cohomology of Γ_0^2 is generated as an abelian group by the integral classes whose reduction modulo 2 is given by Table (4.15).

dimension	reduction of generator
2	δ^2
3	β
4	$z^4,\ \beta\delta+\delta^2\alpha,\ \delta^4$
5	$\delta^2\beta,\ z^2\beta,\ z^2\delta^3$

Table (4.15)

We assign the following names to the algebra generators of the integral cohomology:

$$\begin{gathered}\nu_1=\delta^2,\quad \nu_2=\beta,\quad \nu_3=z^4,\\ \nu_4=\beta\delta+\delta^2\alpha,\quad \nu_5=z^2\beta,\quad \nu^6=z^2\delta^3.\end{gathered}\tag{4.16}$$

Since $\delta^4=\nu_2^2$ and $\nu_1\nu_2=\delta^2\beta$, it follows that all of the classes in Table (4.15) are accounted for by this assignment. The reductions modulo 2 of ν_1,ν_2,ν_4 map to $\hat{x}_1^2,\hat{x}_1\hat{\omega},\hat{x}_1^4$ and come from the bottom row of the spectral sequence (4.13). However, these are the reductions of integral classes in $H^*(H;\mathbf{Z})$. Hence we see that $\nu_1\mapsto\beta_H$, $\nu_2\mapsto\sigma$, and $\nu_4\mapsto\beta_H^2$.

Since its reduction is z^4, the class ν_3 is a class of order 8 in $H^*(\Gamma_2^0;\mathbf{Z})$. From (4.15) we have $z^4\mapsto\hat{z}^4$, and so $\nu_3\mapsto\zeta_H$. The reduction of ν_6 is $z^2\delta^3$ and this maps to $\hat{z}^2\hat{x}_1^3$. Referring to Table (3.19), we see that this $\mathbf{F}_2$ class corresponds to $\delta_3\delta_0^3$ and this is the reduction of $\beta_H\gamma_H$. This gives $\nu_6\mapsto\beta_H\gamma_H$. Finally, the reduction of ν_5 is $z^2\beta$, and this maps to $\hat{z}^2\hat{x}_1\hat{\omega}\cong\delta_0\delta_3\delta_2$. This is not the reduction of an integral class, because it has a nontrivial image under Sq^1. Thus, $\nu_5\mapsto 0$.

Summarizing the results of the preceding paragraphs yields the next proposition.

PROPOSITION (4.17). *The induced map in cohomology*

$$\varphi_2^*: H^*(\Gamma_2^0; \mathbf{Z}) \longrightarrow H^*(H; \mathbf{Z})$$

is given by

$$\nu_1 \mapsto \beta_H, \quad \nu_2 \mapsto \sigma, \quad \nu_3 \mapsto \zeta,$$
$$\nu_4 \mapsto \beta_H^2, \quad \nu_5 \mapsto 0, \quad \nu_6 \mapsto \beta_H \gamma_H.$$

In (3.21) we showed that the cokernel of φ_1^* is isomorphic to

$$\mathbf{Z}_{(2)}\rho \oplus \mathbf{Z}/2\rho\zeta \oplus \mathbf{Z}/2\rho\zeta^2 \oplus \cdots \oplus \mathbf{Z}/2\rho\zeta^i \oplus \cdots$$
$$\oplus \mathbf{Z}/2\gamma \oplus \mathbf{Z}/2\beta_H\gamma_H \oplus \mathbf{Z}/2\zeta\gamma_H \oplus \cdots \oplus \mathbf{Z}/2\beta_H^i\zeta_H^k\gamma_H \oplus \cdots.$$

On the other hand, from (4.13), the image of φ_2^* contains the subalgebra generated by β_H, ζ, $\beta_H\gamma_H$, and in particular, all elements of the form $\beta_H^i\zeta_H^k\gamma_H$, $i > 0$. Since other elements in $\mathrm{Im}(\varphi_2^*)$ have already appeared in $\mathrm{Im}(\varphi_1^*)$, we have the following.

PROPOSITION (4.18). *The cokernel of* $\varphi_1^* \oplus \varphi_2^*$ *is isomorphic to*

$$\mathbf{Z}_{(2)}\rho \oplus \mathbf{Z}/2\rho\zeta \oplus \mathbf{Z}/2\rho\zeta^2 \oplus \cdots \oplus \mathbf{Z}/2\rho\zeta^i \oplus \cdots$$
$$\oplus \mathbf{Z}/2\gamma \oplus \mathbf{Z}/2\gamma\zeta \oplus \mathbf{Z}/2\gamma\zeta^2 \oplus \cdots \oplus \mathbf{Z}/2\gamma\zeta^i \oplus \cdots,$$

where $\deg(\rho\zeta^i) = 4i+1$ *and* $\deg(\gamma\zeta^i) = 4i+3$.

5. The Final Computation

From the Mayer–Vietoris sequence of the triad $(\mathrm{Sp}(4,\mathbf{Z}); G, \Gamma_0^2; H)$, we have

$$0 \longrightarrow \mathrm{coker}(\varphi_1^{q-1} \oplus \varphi_2^{q-1}) \longrightarrow H^q(\mathrm{Sp}(4,\mathbf{Z}); \mathbf{Z}_{(2)}) \longrightarrow \ker(\varphi_1^q \oplus \varphi_2^q) \longrightarrow 0, \quad (5.1)$$

where $\varphi_1^* \oplus \varphi_2^*$ are the induced homomorphisms

$$\varphi_1^* \oplus \varphi_2^*: H^*(G; \mathbf{Z}_{(2)}) \oplus H^*(\Gamma_2^0; \mathbf{Z}_{(2)}) \longrightarrow H^*(H; \mathbf{Z}_{(2)})$$

between cohomology groups. Thus, it remains only to determine the kernel and cokernel of $\varphi_1^* \oplus \varphi_2^*$ and to solve the extension problem in (5.1).

We determined the cokernel of $\varphi_1^* \oplus \varphi_2^*$ at the end of the preceding section. An immediate consequence is that for q odd, there exists an isomorphism

$$H^q(\mathrm{Sp}(4,\mathbf{Z}); \mathbf{Z}_{(2)}) \cong \ker(\varphi_1^q \oplus \varphi_2^q).$$

On the other hand, for q even, there are three group extensions:

$$0 \longrightarrow \mathbf{Z}_{(2)}[\rho] \xrightarrow{\partial} H^2(\mathrm{Sp}(4,\mathbf{Z}); \mathbf{Z}_{(2)}) \longrightarrow \ker(\varphi_1^2 \oplus \varphi_2^2) \longrightarrow 0, \quad (5.2)$$

$$0 \longrightarrow \mathbf{Z}/2[\rho\zeta^i] \xrightarrow{\partial} H^{4i+2}(\mathrm{Sp}(4,\mathbf{Z}); \mathbf{Z}_{(2)}) \longrightarrow \ker(\varphi_1^{4i+2} \oplus \varphi_2^{4i+2}) \longrightarrow 0, \quad (5.3)$$

$$0 \longrightarrow \mathbf{Z}/2[\gamma\zeta^i] \xrightarrow{\partial} H^{4i+4}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)}) \longrightarrow \ker(\varphi_1^{4i+4} \oplus \varphi_2^{4i+4}) \longrightarrow 0, \qquad (5.4)$$

where $i > 0$.

PROPOSITION (5.5). *The cohomology $H^2(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)})$ is isomorphic to $\mathbf{Z}_{(2)} \oplus \mathbf{Z}/2$, and the group extension in* (5.2) *takes the form*

$$0 \longrightarrow \mathbf{Z}_{(2)}[\rho] \xrightarrow{\partial} \mathbf{Z}/2 \oplus \mathbf{Z}_{(2)} \longrightarrow \mathbf{Z}/4 \oplus \mathbf{Z}/2 \longrightarrow 0.$$

As is well known [LW, p. 206], the homology $H_1(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}/2)$ is a cyclic group of order 2, and so the torsion subgroup in $H^2(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)})$ is isomorphic to $\mathbf{Z}/2$ by the universal coefficient theorem. Because the rank of $H^2(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)})$ is known to be 1 [BL], we must have $H^2(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)}) \cong \mathbf{Z}_{(2)} \oplus \mathbf{Z}/2$. As with the groups G, H, and Γ_2^0, there is a $\mathbf{Z}/2$ direct summand of $H^*(\mathrm{Sp}(4,\mathbf{Z}),\mathbf{Z}_{(2)})$ with a generator $\hat{\beta}$ in $H^2(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)})$. This comes from the sign homomorphism $\mathrm{Sp}(4,\mathbf{Z}) \to \mathbf{Z}/2$, which factors through $\mathrm{Sp}(4,\mathbf{Z}/2) \cong \Sigma_6$. Explicitly,

$$H^2(G;\mathbf{Z}_{(2)}) \cong \mathbf{Z}/2[g_0] \oplus \mathbf{Z}/4[g_1] \quad \text{and} \quad H^2(\Gamma_0^2;\mathbf{Z}_{(2)}) \cong \mathbf{Z}/2[\nu_1].$$

By our previous computations, g_0 and ν_1 map to the class β in $H^2(H;\mathbf{Z}_{(2)})$ and g_1 maps to 0. Cancelling off the terms corresponding to the sign homomorphism in (5.2), we have

$$0 \longrightarrow \mathbf{Z}_{(2)}[\rho] \xrightarrow{\partial} \mathbf{Z}_{(2)} \longrightarrow \mathbf{Z}/4[g_1] \longrightarrow 0;$$

clearly, ∂ must be multiplication by 4 in this short exact sequence.

PROPOSITION (5.6). *The group extensions* (5.3) *and* (5.4) *do not split.*

To prove that the sequences (5.3) and (5.4) are not split extensions, it suffices to show that the mod 2 reductions of $\partial(\rho\zeta^i)$ and $\partial(\gamma\zeta^i)$, $i > 0$, are zero in $H^{4i+2}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{F}_2)$ and $H^{4i+4}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{F}_2)$, respectively. Using the Mayer–Vietoris sequence (1.1) with $\mathbf{F}_2$ coefficients, this is equivalent to showing that the mod 2 reductions of $\rho\zeta^i$ and $\gamma\zeta^i$ lie in the image of $\varphi_1^* \oplus \varphi_2^*$.

From (3.7) we see that the reductions of the classes $\rho\zeta^i$ and $\gamma\zeta^i$ are $\delta_1\delta_3^{2i}$ and $\delta_0\delta_3^{2i+1}$, respectively. Hence it follows from (3.17) that $\varphi_1^*(\gamma_1\gamma_5) = \delta_1\delta_3^{2i}$. For the other extension, we have that $\gamma\zeta^i$ reduces to $\delta_0\delta_3^{2i+1}$. Furthermore, this class is the image of $\gamma_1\gamma_3\gamma_5^i \oplus z^{4i+2}\delta$.

PROPOSITION (5.7). *The element $g_1 g_3^i \oplus 0$ of*

$$H^{4i+2}(G;\mathbf{Z}_{(2)}) \oplus H^{4i+2}(\Gamma_2^0;\mathbf{Z}_{(2)})$$

lies in the kernel of $\varphi_1^{4i+2} \oplus \varphi_2^{4i+2}$. The restriction of the extension (5.3) *to the summand $\mathbf{Z}/4[g_1 g_3^i \oplus 0]$ gives a $\mathbf{Z}/8$ summand in $H^{4i+2}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)})$; that is,*

$$\begin{array}{ccccccc}
0 \longrightarrow \mathbf{Z}/2[\rho\zeta^i] \longrightarrow & & \mathbf{Z}/8 & \longrightarrow & \mathbf{Z}/4[g_1 g_3^i \oplus 0] & \longrightarrow 0 \\
\downarrow = & & \downarrow & & \downarrow & \\
0 \longrightarrow \mathbf{Z}/2[\rho\zeta^i] \longrightarrow & & H^{4i+2}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)}) & \longrightarrow & \ker(\varphi_1^{4i+2} \oplus \varphi_2^{4i+2}) & \longrightarrow 0.
\end{array}$$

PROPOSITION (5.8). *Let $i>0$. The element $g_3^i - \nu_3^i$ lies in the kernel of $\varphi_1^{4i}\oplus\varphi_2^{4i}$. The restriction of the extension* (5.4) *to the summand $\mathbf{Z}/8[g_3^i-\nu_3^i]$ gives rise to a $\mathbf{Z}/16$ summand in $H^{4i}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)})$; that is,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/2[\gamma\zeta^i] & \longrightarrow & \mathbf{Z}/16 & \longrightarrow & \mathbf{Z}/8[g_3^i-\nu_3^i] & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}/2[\gamma\zeta^i] & \longrightarrow & H^{4i}(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z}_{(2)}) & \longrightarrow & \ker(\varphi_1^{4i}\oplus\varphi_2^{4i}) & \longrightarrow & 0. \end{array}$$

In order to prove (5.7) and (5.8), we first observe that the $\mathbf{Z}/4$ class $g_1g_3^i$ is mapped to zero by φ_1^{4i+2} and so $\mathbf{Z}/4[g_1g_3^i\oplus 0]$ represents a direct summand in $\ker(\varphi_1^{4i+2}\oplus\varphi_2^{4i+2})$. On the other hand, the $\mathbf{Z}/8$ classes $g_3\in H^4(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z})$ and $\nu_3\in H^4(\Gamma_0^2;\mathbf{Z})$ both map to $\zeta\in H^4(H;\mathbf{Z})$, so their difference $g_3^i-\nu_3^i$ lies in the kernel of $\varphi_1^{4i}\oplus\varphi_2^{4i}$.

To show (5.7) and (5.8) we will demonstrate that, by restricting the group extensions (5.3) and (5.4) to a summand in $\ker(\varphi_1^*\oplus\varphi_2^*)$ different from $\mathbf{Z}/4[g_1g_3^i\oplus 0]$ and $\mathbf{Z}/8[g_3^i-\nu_3^i]$, we obtain a split extension. Since the group extensions do not split by (5.6), the only possibility is that the restrictions to the summands $\mathbf{Z}/4[g_1g_3^i\oplus 0]$ and $\mathbf{Z}/8[g_3^i-\nu_3^i]$ form a nonsplit extension.

We can view our setting as that of a cofibration $A\xrightarrow{f}X\xrightarrow{p}X/A$, with $A=BH$, $X=B\Gamma_2^0\vee BG$, and $X/A=BSP(4,\mathbf{Z})$. Suppose we are given a direct summand $\mathbf{Z}/2^m[b]$ in $H^*(X;\mathbf{Z})$ which lies in the kernel of f^*. A necessary and sufficient condition for lifting $\mathbf{Z}/2^m[b]$ to a summand in $H^*(X/A;\mathbf{Z})$ by p^* is that there exists a class c in $H^*(X,\mathbf{F}_2)$ such that the mth-order Bockstein operation β_m takes c to b mod 2 and $f^*(c)=0$. For example, consider the $\mathbf{Z}/4$ summand generated by $g_1^2\oplus 0$ in $H^4(G;\mathbf{Z})\oplus H^4(\Gamma_2^0;\mathbf{Z})$. The mod 2 reduction of $g_1^2\oplus 0$ is the class $\gamma_3^2\oplus 0$ which in turn lies in the image of second-order Bockstein $\gamma_3^2\oplus 0=\beta_2[\gamma_1\gamma_2\oplus 0]$. Since $\varphi_1^*(\gamma_1\gamma_3)=0$, it follows that the summand $\mathbf{Z}/4[g_1^2\oplus 0]$ in $\ker(\varphi_1^4\oplus\varphi_2^4)$ can be lifted to a $\mathbf{Z}/4$ summand in $H^4(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z})$.

In (5.12), we work out a set of algebraic generators in $\ker(\varphi_1^*\oplus\varphi_2^*)$ which include $[g_1g_3^i\oplus 0]$ and $[g_3^i-\nu_3^i]$. Applying the above criterion to these generators, it can be shown that all the summands but $[g_1g_3^i\oplus 0]$ and $[g_3^i-\nu_3^i]$ can be lifted. Note that if a summand $\mathbf{Z}/2^i[a]$ can be lifted then its tensor product with $\mathbf{Z}/2^j[b]$, $j\geq i$, can also be lifted. For example, $\ker\varphi_1^*$ is spanned by the following: $g_1, g_1^2, \ldots, g_1g_3, g_1^2g_3, \ldots, g_1g_3^i, g_1^2g_3^i, \ldots$. Since in the above we have shown that $\mathbf{Z}/4[g_1^2\oplus 0]$ can be lifted, it follows that all the summands $\mathbf{Z}/4[g_1^ig_3^j\oplus 0]$ ($i\geq 2$, $j\geq 0$) can be lifted to $H^*(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z})$. Applying this technique to the aforementioned generators of $\ker(\varphi_1^*\oplus\varphi_2^*)$, we obtain the proof of (5.7) and (5.8).

To determine $\ker(\varphi_1^*\oplus\varphi_2^*)$, we observe that from the homomorphism

$$\varphi_1^*\oplus\varphi_2^*\colon H^*(G;\mathbf{Z}_{(2)})\oplus H^*(\Gamma_2^0;\mathbf{Z}_{(2)})\longrightarrow H^*(H;\mathbf{Z}_{(2)})$$

we can factor out $H^*(G;\mathbf{Z}_{(2)})$ in the domain as well as the image of φ_1^* in the range. On the corresponding quotients this gives rise to an induced homomorphism

$$\tilde{\varphi}_2^*\colon H^*(\Gamma_2^0;\mathbf{Z}_{(2)}) \longrightarrow H^*(H;\mathbf{Z}_{(2)})/\operatorname{Im}\varphi_1^*.$$

Similarly, by interchanging the role of φ_1^* and φ_2^*, there is an induced homomorphism

$$\tilde{\varphi}_1^*\colon H^*(G;\mathbf{Z}_{(2)}) \longrightarrow H^*(H;\mathbf{Z}_{(2)})/\operatorname{Im}\varphi_1^*.$$

From the definition of these homomorphisms it is easy to see that there are exact sequences

$$0 \longrightarrow \ker\varphi_1^* \longrightarrow \ker(\varphi_1^*\oplus\varphi_2^*) \longrightarrow \ker\tilde{\varphi}_2^* \longrightarrow 0, \tag{5.9}$$

$$0 \longrightarrow \ker\varphi_2^* \longrightarrow \ker(\varphi_1^*\oplus\varphi_2^*) \longrightarrow \ker\tilde{\varphi}_1^* \longrightarrow 0. \tag{5.10}$$

Since $\ker\varphi_1^*$ has been determined in (3.21), we can study $\ker(\varphi_1^*\oplus\varphi_2^*)$ by calculating $\ker\tilde{\varphi}_2^*$.

Recall that $\operatorname{coker}\varphi_1^* \cong H^*(H;\mathbf{Z}_{(2)})Z/\operatorname{Im}\varphi_1^*$ is isomorphic to the direct sum

$$\begin{gathered}\mathbf{Z}_{(2)}\rho\oplus\mathbf{Z}/2\rho\zeta\oplus\mathbf{Z}/2\rho\zeta^2\oplus\cdots\oplus\mathbf{Z}/2\rho\zeta^i\oplus\cdots\\ \oplus\mathbf{Z}/2\gamma\oplus\mathbf{Z}/2\zeta\gamma\oplus\cdots\oplus\mathbf{Z}/2\beta^j\zeta^k\gamma\oplus\cdots,\end{gathered}$$

and the image of $\tilde{\varphi}_2^*$ consists of the summands

$$\mathbf{Z}/2\beta\gamma\oplus\mathbf{Z}/2\beta^2\gamma\oplus\cdots\oplus\mathbf{Z}/2\beta^j\zeta^k\gamma\oplus\cdots.$$

An elementary calculation shows that the rank polynomial $\sum\operatorname{rank}(\operatorname{Im}\tilde{\varphi}_2^q)t^q$ of $\operatorname{Im}\tilde{\varphi}_2^*$ is of the form $t^5/(1-t^2)(1-t^4)$. Combining this with the known structure of $H^*(\Gamma_2^0;\mathbf{Z}_{(2)})$, we have the following proposition.

PROPOSITION (5.11). *As a graded abelian group,* $\ker\tilde{\varphi}_2^*$ *is isomorphic to*

$$\ker\tilde{\varphi}_2^* \cong \begin{cases}\mathbf{Z}/8\oplus\bigoplus_{g_q-1}(\mathbf{Z}/2) & q\equiv 0 \bmod 4,\\ \bigoplus_{g_q}\mathbf{Z}/2 & \text{otherwise},\end{cases}$$

where g_q are the coefficients in the polynomial

$$\sum g_q t^q = \frac{1+t^3+t^4+t^5+t^6}{(1-t^2)(1-t^4)}.$$

To complete our analysis of $\ker(\varphi_1^*\oplus\varphi_2^*)$, we observe that the group extension in (5.9) is a split extension. For there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker\varphi_2^* \arrow[r] & \ker(\varphi_1^*\oplus\varphi_2^*) \arrow[r] & \ker\tilde{\varphi}_1^* \arrow[r] & 0 \\ & & \ker\varphi_1^*. \arrow[u] \arrow[ur] & & \end{tikzcd}$$

Using the explicit description of $\ker\varphi_1^*$ in (3.21), it is not difficult to see that $\ker\varphi_1^*$ is a direct summand in $\ker\tilde{\varphi}_1^*$ and hence a direct summand in $\ker(\varphi_1^*\oplus\varphi_2^*)$.

PROPOSITION (5.12). *The kernel* $\ker(\varphi_1^*\oplus\varphi_2^*)$ *is isomorphic to* $\ker\varphi_1^*\oplus\ker\tilde{\varphi}_2^*$, *and as a graded abelian group it has the following decomposition:*

$$\ker(\varphi_1^* \oplus \varphi_2^*) = \begin{cases} \mathbf{Z}/8 \oplus \bigoplus_{e_q} \mathbf{Z}/4 \oplus \bigoplus_{g_q - 1} \mathbf{Z}/2 & q \equiv 0 \bmod 4, \\ \bigoplus_{e_q} \mathbf{Z}/4 \oplus \bigoplus_{g_q} \mathbf{Z}/2 & q \equiv 2 \bmod 4 \ \textit{and} \ q > 2, \\ \bigoplus_{f_q} \mathbf{Z}/2 \oplus \bigoplus_{g_q} \mathbf{Z}/2 & q \equiv 1 \bmod 4, \end{cases}$$

where e_q, f_q, g_q *are respectively the coefficients in the polynomial*

$$\sum g_q t^q = \frac{1 + t^3 + t^4 + t^5 + t^6}{(1 - t^2)(1 - t^4)}, \quad \sum e_q t^q = \frac{t^2}{(1 - t^2)(1 - t^4)},$$

$$\sum f_q t^q = \frac{t^5}{(1 - t^2)(1 - t^4)}.$$

Finally, the proof of Theorem (0.1) is immediate from (5.7), (5.8), and (5.12). To get the answer, all that is required is to replace the $\mathbf{Z}/8$ factor in $\ker(\varphi_1^{4i} \oplus \varphi_2^{4i})$ by $\mathbf{Z}/16$ and a $\mathbf{Z}/4$ factor in $\ker(\varphi_1^{4i+2} \oplus \varphi_2^{4i+2})$ by $\mathbf{Z}/8$.

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