Coarsely Quasi-Homogeneous Circle Packings in the Hyperbolic Plane

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1. Introduction

Let $\mathfrak{J}$ be a triangulation of an open topological disk. By [HS1, Cor. 1.5] (and [Sc]), there is a circle packing $P_{\mathfrak{J}^{(1)}}$ in the complex plane $\mathbf{C}$, unique up to Möbius transformations, whose graph is combinatorially equivalent to the 1-skeleton $\mathfrak{J}^{(1)}$ of $\mathfrak{J}$ and whose carrier is either the unit disk U or the whole plane $\mathbf{C}$. We call the graph $\mathfrak{J}^{(1)}$ *hyperbolic* if the carrier of $P_{\mathfrak{J}^{(1)}}$ is U, and *parabolic* otherwise. For any vertex v in $\mathfrak{J}^{(1)}$, its *valence* is defined to be the number of edges of $\mathfrak{J}^{(1)}$ with an endpoint at v. The graph $\mathfrak{J}^{(1)}$ is said to have *bounded valence* if there is a uniform bound on the valences of its vertices.

The 2-manifold $|\mathfrak{J}|$ is naturally endowed with the unique (singular Riemannian) metric so that every 2-simplex is isometric to a unit equilateral triangle in the Euclidean plane. With this metric, there is also a well-defined conformal structure in the manifold. By Koebe's uniformization theorem, $|\mathfrak{J}|$ is conformally equivalent to either U or $\mathbf{C}$. When $\mathfrak{J}^{(1)}$ has bounded valence, the ring lemma of [RS] (see Lemma 2.2 below) implies that $|\mathfrak{J}|$ is conformally equivalent to U if and only if $\mathfrak{J}^{(1)}$ is hyperbolic. In the following, we will consider $|\mathfrak{J}|$ as both a metric space and as a Riemann surface.

Let $K \geq 1$ be a constant. A (not necessarily continuous) map $f: X \to Y$ between two metric spaces is called a *coarse K-quasi-isometry* if for each pair of points u and v in X,

$$\frac{d(u,v)}{K} - K \leq d(h(u),h(v)) \leq Kd(u,v)+K, \tag{1.1}$$

and if for each point w in Y there is some u in X such that $d(w,f(u)) \leq K$, where (by an abuse of notation) we have used d to denote the metrics in both X and Y. A metric space X will be called *coarsely quasi-homogeneous* if there is some $K \geq 1$ such that for each pair of points u and v in X there is a coarse K-quasi-isometry $h: X \to X$ with $h(u) = v$. Our main theorem is as follows.

THEOREM 1.1. *Let $\mathfrak{J}$ be a triangulation of an open topological disk such that $\mathfrak{J}^{(1)}$ is hyperbolic and of bounded valence, and let $P_{\mathfrak{J}^{(1)}}$ be a circle packing whose carrier is U and whose graph is combinatorially equivalent to $\mathfrak{J}^{(1)}$.*

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If $|\mathfrak{J}|$ is coarsely quasi-homogeneous, then there is a positive uniform lower bound for the hyperbolic radii of the circles in $P_{\mathfrak{J}^{(1)}}$.

It is easy to prove that the hyperbolic radii of the circles of the packing $P_{\mathfrak{J}^{(1)}}$ in the above theorem have a uniform upper bound (see Lemma 2.1). In particular, Theorem 1.1 implies that $|\mathfrak{J}|$ is quasi-isometric to the Poincaré disk U by some homeomorphism which maps the vertices in $\mathfrak{J}$ to the centers of the corresponding circles of $P_{\mathfrak{J}^{(1)}}$ and maps the edges to the corresponding geodesic segments.

Let G be a finitely generated group. Then there is a word metric on G which is well-defined up to quasi-isometries. Moreover, with this metric G is coarsely quasi-homogeneous. In particular, any metric space that is coarsely quasi-isometric to G is also coarsely quasi-homogeneous. As a corollary to Theorem 1.1 we obtain the following result, which is equivalent to a theorem of G. Mess stating that: If a finitely generated group is coarsely quasi-isometric to a complete Riemannian manifold of "bounded geometry" and if the Riemannian manifold is conformally equivalent to U, then the group is coarsely quasi-isometric to U. See [Me] for details.

COROLLARY 1.2. *Let G be a finitely generated group equipped with a word metric. Suppose that there is a triangulation $\mathfrak{J}$ of an open topological disk such that $|\mathfrak{J}|$ is coarsely quasi-isometric to G. If $\mathfrak{J}^{(1)}$ has bounded valence and $|\mathfrak{J}|$ is conformally equivalent to the hyperbolic plane U, then G is coarsely quasi-isometric to U.*

It follows by the works of P. Tukia, A. Casson, and D. Gabai that a finitely generated group G which is coarsely quasi-isometric to U is actually a Fuchsian group. Based on some results of N. Varapoulos on recurrent random walks on groups, Mess [Me] also showed separately that if $|\mathfrak{J}|$ in Corollary 1.2 is conformally equivalent to $\mathbf{C}$, then G is virtually abelian and hence coarsely quasi-isometric to $\mathbf{C}$. We do not know if this result can also be proved using circle packings.

The proof of our main theorem starts with the following simple observation.

LEMMA 1.3. *Let c be a circle in U. Let $m(c)$ be the conformal modulus of the annulus bounded by c and ∂U, and let $r_{\text{hyp}}(c)$ be the hyperbolic radius of c. Then*

$$r_{\text{hyp}}(c) = \log\left(\frac{e^{m(c)}+1}{e^{m(c)}-1}\right). \tag{1.2}$$

In particular, $r_{\text{hyp}}(c)$ is bounded from below if and only if $m(c)$ is bounded from above.

Thus, in order to prove the theorem, it is enough to show that for c in $P_{\mathfrak{J}^{(1)}}$ the moduli $m(c)$ are uniformly bounded from above. This will be achieved using the argument of transboundary extremal length of [Sc].

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2. Some Lemmas

This section contains some elementary results on circle packings.

LEMMA 2.1. *Let $\mathfrak{T}$ be a triangulation of an open topological disk, and let P be a circle packing in U whose graph is combinatorially equivalent to $\mathfrak{T}^{(1)}$. If the graph $\mathfrak{T}^{(1)}$ has bounded valence then there is some constant C_1, depending only on the maximal valence of $\mathfrak{T}^{(1)}$, such that the hyperbolic radius of any circle in P is bounded by C_1.*

Proof. Let c be a circle in P. Since $\mathfrak{T}$ is a triangulation of a 2-manifold, the circles of P that are tangent to c form a closed chain of length, say, k. As $\mathfrak{T}^{(1)}$ has bounded valence, k is bounded from above.

By some Möbius transformation of U (which preserves the Poincaré metric), we may assume that c is centered at 0. Now it is elementary to see that the Euclidean distance between c and ∂U should not be too small in order for a closed chain of k circles in U surrounding c to exist. This implies that the hyperbolic radius of c is bounded. □

REMARK. Under the hypotheses of Lemma 2.1, let the valences of the vertices of $\mathfrak{T}^{(1)}$ be bounded by k_0. Then $k_0 \geq 7$ (see e.g. [BS] or [HS2]). There is an obvious circle packing P_{k_0} in U in which all circles have the same hyperbolic radius and each circle is tangent to a closed chain of k_0 other circles. Let δ_{k_0} denote the hyperbolic radius of the circles in P_{k_0}. Then the hyperbolic radius of any circle of P in Lemma 2.1 is bounded by δ_{k_0}. Here is a proof due to the referee and based on the maximum principle. Suppose that there is a circle of P, say c_0, whose hyperbolic radius is strictly bigger than δ_{k_0}. Let $0 < a < 1$ be a constant such that the map $f(z) = az$ takes c_0 to a circle whose hyperbolic radius is still strictly bigger than δ_{k_0}. Since $a < 1$, $f(P)$ is contained in some compact subset of the hyperbolic plane. It follows that there is some circle $f(c)$ in $f(P)$ whose hyperbolic radius $\mathrm{rad}_{\mathrm{hyp}}(f(c))$ is maximal. In particular, $\mathrm{rad}_{\mathrm{hyp}}(f(c)) \geq \mathrm{rad}_{\mathrm{hyp}}(f(c_0)) > \mathrm{rad}_{\mathrm{hyp}}(c^*)$, where c^* is some circle of P_{k_0}. We now compare the flower of $f(c)$ in $f(P)$ to the flower of c^* in P_{k_0}. Since the valence of $f(c)$ in $f(P)$ is bounded by k_0, which is the valence of c^* in P_{k_0}, and since $\mathrm{rad}_{\mathrm{hyp}}(f(c)) > \mathrm{rad}_{\mathrm{hyp}}(c^*)$, it is elementary to see that there is some circle in the flower (i.e. the first generation) of $f(c)$ whose hyperbolic radius is strictly bigger than $\mathrm{rad}_{\mathrm{hyp}}(f(c))$. This contradicts the maximality of $\mathrm{rad}_{\mathrm{hyp}}(f(c))$.

The following lemma is proved by Rodin and Sullivan [RS].

LEMMA 2.2 (Ring Lemma). *Let $\mathfrak{T}$ be a triangulation of an open topological disk, and let P be a circle packing in $\mathbf{C}$ whose graph is combinatorially equivalent to $\mathfrak{T}^{(1)}$. If the graph $\mathfrak{T}^{(1)}$ has bounded valence then there is some* *constant C_2, depending only on the maximal valence of $\mathfrak{J}^{(1)}$, such that the ratio of the Euclidean radii of each pair of tangent circles in P is bounded by C_2.*

LEMMA 2.3. *Let $\mathfrak{J}$ and P satisfy the conditions in Lemma 2.2. Suppose there is a circle c_0 in P which is centered at 0. Then there exists a constant C_3, depending only on the maximal valence of $\mathfrak{J}^{(1)}$, such that for each circle $c \neq c_0$ in P,*

$$r_{\mathrm{Euc}}(c) \leq C_3 d_{\mathrm{Euc}}(0, c), \tag{2.1}$$

where $r_{\mathrm{Euc}}(c)$ denotes the Euclidean radius of c and $d_{\mathrm{Euc}}(0, c)$ is the Euclidean distance between 0 and c.

Proof. Transforming P by a similarity (fixing 0 if necessary), we may assume that $r_{\mathrm{Euc}}(c) = 1$. Then we need to show that $d_{\mathrm{Euc}}(0, c)$ is bounded from below by some constant. Let c' be a circle of P which is tangent to c and which intersects the line segment joining 0 and the center of c. By Lemma 2.2, we have

$$r_{\mathrm{Euc}}(c') \geq \frac{r_{\mathrm{Euc}}(c)}{C_2} = \frac{1}{C_2}. \tag{2.2}$$

If $c' = c_0$, then $d_{\mathrm{Euc}}(0, c) = r_{\mathrm{Euc}}(c_0) = r_{\mathrm{Euc}}(c')$ and (2.1) follows by (2.2). So we may assume that $c' \neq c_0$ and as a consequence 0 is not contained in the disk bounded by c'. Then it is elementary to see that the radius of c' tends to 0 if $d_{\mathrm{Euc}}(0, c)$ tends to 0. By (2.2), it follows that $d_{\mathrm{Euc}}(0, c)$ is bounded from below. □

Let M be a Riemannian 2-manifold and let K be positive. A connected subset $D \subseteq M$ is called *K-nondegenerate* if

$$(\mathrm{diam}(D))^2 \leq K \cdot \mathrm{area}(D). \tag{2.3}$$

Let $S^1 = \mathbf{R}/(2\pi\mathbf{Z})$ and let $\mathbf{R} \times S^1$ be endowed with the product metric. Let $J = \log\colon \mathbf{C}^* = \mathbf{C} - \{0\} \to \mathbf{C}/(2\pi i\mathbf{Z}) = \mathbf{R} \times S^1$ be the logarithm function.

LEMMA 2.4. *Let $\mathfrak{J}$, P, c_0, and c satisfy the conditions in Lemma 2.3. Let $D(c)$ be the disk bounded by c. Then $J(D(c)) \subseteq \mathbf{R} \times S^1$ is C_4-nondegenerate for some positive constant C_4 which depends only on the maximal valence of $\mathfrak{J}^{(1)}$.*

Proof. For p and q in $D(c)$, the ratio of the norms of the derivatives of J at these two points is $|J'(p)|/|J'(q)| = |q|/|p|$. By Lemma 2.3, this ratio is bounded by $2C_3 + 1$. Then it is easy to see that $J(D(c))$ is C_4-nondegenerate for $C_4 = 4(2C_3 + 1)^2/\pi^2$. □

3. Proof of Main Theorem

We will prove the following lemma, which implies Theorem 1.1 in virtue of Lemma 1.3.

LEMMA 3.1. *Let $\mathfrak{I}$ and $P_{\mathfrak{I}^{(1)}}$ satisfy the conditions in Theorem 1.1. Then there is a constant C such that, for each pair of circles c_0 and c_0^* in $P_{\mathfrak{I}^{(1)}}$,*

$$m(c_0^*) \leq Cm(c_0).$$

Proof. For each pair of points u and v in $|\mathfrak{I}^{(1)}|$, let $d(u, v)$ denote the minimal length of paths in $|\mathfrak{I}^{(1)}|$ joining u and v. Since $\mathfrak{I}^{(1)}$ has bounded valence, it is easy to see that the metric spaces $(|\mathfrak{I}^{(1)}|, d)$ and $(\mathfrak{I}^{(0)}, d)$ are both coarsely quasi-isometric to $|\mathfrak{I}|$. In particular, the hypotheses of Theorem 1.1 imply that $(\mathfrak{I}^{(0)}, d)$ is coarsely quasi-homogeneous. Let v_0 and v_0^* be the vertices of $\mathfrak{I}^{(1)}$ which correspond to the circles c_0 and c_0^*, respectively. Let $h: \mathfrak{I}^{(0)} \to \mathfrak{I}^{(0)}$ be a coarse K-quasi-isometry such that $h(v_0) = v_0^*$, where $K \geq 1$ is some constant independent of v_0 and v_0^*. We may assume that for $v \neq v_0$, $h(v) \neq v_0^*$.

For c in $P_{\mathfrak{I}^{(1)}}$, we will denote by v_c the vertex of $\mathfrak{I}^{(1)}$ corresponding to c, and denote by $h(c)$ the circle such that $v_{h(c)} = h(v_c)$. We also write $d(c, c') = d(v_c, v_{c'})$. For each positive k, let $\mathfrak{N}_k(c)$ denote the set of all circles c' in $P_{\mathfrak{I}^{(1)}}$ such that $d(c, c') \leq k$. Then, for each pair of tangent circles c_1 and c_2, we have $d(h(c_1), h(c_2)) \leq Kd(c_1, c_2) + K = 2K$; therefore the union of circles in $\mathfrak{N}_{2K}(h(c_1))$ is a continuum joining $h(c_1)$ and $h(c_2)$. For each circle $c' \neq c_0^*$ of $P_{\mathfrak{I}^{(1)}}$, let $E(c^*)$ denote the union of all circles in $\mathfrak{N}_{2K+1}(c^*)$ except possibly c_0^*. We then have the following lemma.

LEMMA 3.2. *Let c_1 and c_2 be two tangent circles in $P_{\mathfrak{I}^{(1)}}$ with $c_2 \neq c_0$. Then $E(h(c_2))$ contains a continuum joining $h(c_1)$ and $h(c_2)$.*

Proof of Lemma 3.1 (continued). Changing $P_{\mathfrak{I}^{(1)}}$ by a Möbius transformation of U, we may assume that c_0 is centered at 0. Let $T: U \to U$ be the Möbius transformation which maps c_0^* to a circle centered at 0. Let $A(c_0, \partial U)$ be the annulus bounded by c_0 and ∂U, and let $A(c_0^*, \partial U)$ be the annulus bounded by c_0^* and ∂U. Let $G: A(c_0, \partial U) \to (-m(c_0), 0) \times S^1 \subseteq \mathbf{R} \times S^1$ be the mapping defined by $G(z) = J(z)$, and let $G^*: A(c_0^*, \partial U) \to (-m(c_0^*), 0) \times S^1 \subseteq \mathbf{R} \times S^1$ be defined by $G^*(z^*) = JT(z^*)$, where J is the map in Lemma 2.4. By an abuse of notation, for each closed curve Γ which bounds a Jordan domain in $\mathbf{R} \times S^1$, let us denote by area(Γ) the area of the Jordan domain bounded by Γ. Then, by Lemma 2.4, for $c^* \neq c_0^*$ we have

$$\operatorname{diam}^2(G^*(c^*)) \leq C_4 \cdot \operatorname{area}(G^*(c^*)). \tag{3.1}$$

A similar inequality holds for $G(c)$ where $c \neq c_0$.

For any θ in S^1, let $L_\theta = \{\eta e^{i\theta}; \eta > 0\}$. Let $c_1, c_2, \ldots$ be the sequence of circles of $P_{\mathfrak{I}^{(1)}} - \{c_0\}$ which intersect L_θ so that c_1 is tangent to c_0, c_2 is tangent to c_1, and so on. Then these circles form a continuum in $A(c_0, \partial U)$ joining c_0 to the circle at infinity ∂U. By Lemma 3.2, $\bigcup_{i \geq 1} E(h(c_i))$ contains a continuum in $A(c_0^*, \partial U)$ joining $h(c_0) = c_0^*$ to ∂U. Hence, $\bigcup_{i \geq 1} G^*(E(h(c_i)))$ contains a continuum joining the two ends of $(-m(c_0^*), 0) \times S^1$. It follows that

$$m(c_0^*) \leq \sum_{c \neq c_0,\, c \cap L_\theta \neq \emptyset} \operatorname{diam}(G^*(E(h(c)))). \tag{3.2}$$

Integrating this with respect to $\theta \in S^1$, we obtain

$$2\pi m(c^*) \leq \sum_{c \neq c_0} \operatorname{diam}(G^*(E(h(c)))) \operatorname{diam}(G(c)).$$

By the Schwarz inequality, we deduce that

$$4\pi^2 m(c^*)^2 \leq \sum_{c \neq c_0} \operatorname{diam}^2(G^*(E(h(c)))) \cdot \sum_{c \neq c_0} \operatorname{diam}^2(G(c)). \tag{3.3}$$

Since $\mathfrak{J}^{(1)}$ has bounded valence, the number of circles of $P_{\mathfrak{J}^{(1)}}$ in $E(h(c))$ is uniformly bounded. And because h is a coarse K-quasi-isometry, for each circle $c^* \neq c_0^*$, the number of $E(h(c))$s for which $c^* \subseteq E(h(c))$ is also uniformly bounded. Hence there is some constant $C_5 > 0$ such that

$$\begin{aligned}\sum_{c \neq c_0} \operatorname{diam}^2(G^*(E(h(c)))) &\leq \sqrt{C_5} \sum_{c \neq c_0} \sum_{c^* \subseteq E(h(c))} \operatorname{diam}^2(G^*(c^*)) \\ &\leq C_5 \sum_{c^* \neq c_0^*} \operatorname{diam}^2(G^*(c^*)).\end{aligned}$$

By (3.1), this implies that

$$\begin{aligned}\sum_{c \neq c_0} \operatorname{diam}^2(G^*(E(h(c)))) &\leq C_4 C_5 \sum_{c^* \neq c_0^*} \operatorname{area}(G^*(c^*)) \\ &\leq C_4 C_5 \cdot \operatorname{area}((-m(c_0^*), 0) \times S^1) \\ &= C_4 C_5 \cdot 2\pi m(c^*). \end{aligned} \tag{3.4}$$

Similarly,

$$\sum_{c \neq c_0} \operatorname{diam}^2(G(c)) \leq C_4 \cdot \operatorname{area}((-m(c_0), 0) \times S^1) \leq C_4 \cdot 2\pi m(c). \tag{3.5}$$

Combining (3.3) with (3.4) and (3.5), we obtain

$$4\pi^2 m(c^*)^2 \leq 4\pi^2 (C_4)^2 C_5 m(c^*) m(c).$$

Thus $m(c^*) \leq (C_4)^2 C_5 m(c)$. □

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