GENERAL ELEMENTS AND JOINT REDUCTIONS

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Introduction. Throughout this paper we will be concerned with a local ring (Q, m, k, d), where this implies that the local ring Q has maximal ideal m, residue field $k = Q/m$, and Krull dimension d. The general extension Q_g of Q will play an important role, and is defined as follows. We suppose that $X_1, X_2, \ldots$ is a countable set of indeterminates over Q. Then Q_g is the localization of the ring $Q[X_1, X_2, \ldots]$ at the prime ideal $m[X_1, X_2, \ldots]$. Q_g is a local Noetherian ring, the fact that it is Noetherian following from Proposition 1 of [1, Ch. 9]. It is a flat extension of Q and has maximal ideal $m_g = mQ_g$, its residue field is $k(X_1, X_2, \ldots)$ and it has Krull dimension d. It is also the union of the local rings Q_N, defined as the localization of $Q[X_1, \ldots, X_N]$ at $m[X_1, \ldots, X_N]$.

Now we come to the definition of general elements. Let $\underline{I} = (I_1, \ldots, I_s)$ be a set of ideals of Q, not necessarily distinct. We first define a standard independent set of general elements of $\underline{I}$ as follows. Let $a(i,1), \ldots, a(i, n_i)$ be a set of generators of I_i for $i = 1, \ldots, s$. Write $X(i,j)$ for X_h, where $h = n_1 + \cdots + n_{i-1} + j$ with $0 < j \leq n_i$. Finally, let $x_i = \sum X(i,j)a(i,j)$, the sum being from $j = 1$ to n_i. Then we term the elements $x_1, \ldots, x_s$ a standard independent set of general elements of $\underline{I}$.

We now define an independent set of general elements of $\underline{I}$ to be a set of elements $x_1, \ldots, x_s$ of Q_g such that there exists an automorphism θ of Q_g, which fixes the elements of Q and the elements X_i for all sufficiently large i, such that the set of elements $\theta(x_i)$ is a standard set of general elements of $\underline{I}$. We shall prove below that this definition is independent of the choice of the sets of generators of the ideals I_i used in the definition of standard sets of independent general elements, by proving that any one set of independent general elements of $\underline{I}$ can be taken into any other such set by applying a suitable automorphism of Q_g of the type indicated. This implies that the ideal $X(\underline{I}) = (x_1, \ldots, x_s) \cap Q$ of Q is independent of the choice of the independent set of general elements $x_1, \ldots, x_s$ and that the Q-algebra $Q_g/(x_1, \ldots, x_s)$ is independent of the choice of $x_1, \ldots, x_s$ to within isomorphism as a Q-algebra.

Now we turn to the second term in our title, joint reductions. We recall that if I and $J \subseteq I$ are two ideals of a Noetherian ring then J is termed a reduction of I if $I^{r+1} = I^r J$ for some r (and hence all large r). Now suppose that $\underline{I} = (I_1, \ldots, I_d)$ is a set of d m-primary ideals of Q. Then we term a set of elements $y_1, \ldots, y_d$ of Q (with y_i in I_i) a joint reduction of $\underline{I}$ if, for some r,

$$(I_1 \cdots I_d)^{r+1} = \sum_{i=1}^{d} y_i (I_1 \cdots I_d)^r I_1 \cdots I_{i-1} I_{i+1} \cdots I_d.$$

In general, joint reductions need not exist, although they do if k is infinite. In particular, if $\underline{I}_g = (I_1 Q_g, \ldots, I_d Q_g)$ then any set of independent general elements

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$(x_1, \ldots, x_d)$ of $\underline{I}$ is a joint reduction of $\underline{I}_g$. We will, in fact, prove this directly, and use this result to prove that $y_1, \ldots, y_d$ is a joint reduction of $\underline{I}$ for "almost all" choices of y_i in I_i $(i = 1, \ldots, d)$. We will also prove that $X(\underline{I})$ is contained in $(y_1, \ldots, y_d)$ for "almost all" joint reductions $y_1, \ldots, y_d$ of $\underline{I}$.

This last statement leads almost immediately to the main purpose of this paper. In general it is not true that $X(\underline{I})$ is contained in the ideal generated by the elements of a joint reduction of $\underline{I}$. We will give a counterexample with $d = 1$. However, it is true if we restrict Q to be Cohen–Macaulay, and the proof of this is the main objective of the second section of this paper (the first section being devoted to the proofs of some of the general results indicated above).

In the third section we limit Q still further, in that Q is assumed to be not only Cohen–Macaulay but also analytically unramified. We also make a slight change of notation, in that $\underline{I}$ is assumed to be a set $(I_1, \ldots, I_{d-1})$ of $d-1$ m-primary ideals. We then consider an independent set of general elements $(x_1, \ldots, x_{d-1})$ of $\underline{I}$ and the Q-algebra $L = Q_g/(x_1, \ldots, x_{d-1})$, which we show is a 1-dimensional analytically unramified Cohen–Macaulay local ring. It follows that the integral closure L^* of L in its complete ring of fractions F is a finite L-module, and hence the conductor $C(L^*/L)$ is defined. We define $C(\underline{I})$ to be the intersection of $C(L^*/L)$ with Q and show that $X(\underline{I}, J)$ contains $(C(\underline{I})J)^*$ for any m-primary ideal J. Using the main ingredient of the proof of a theorem of Skoda and Briançon (due to Lipman and Sathaye [2]), we show that if Q is a regular local ring then $C(\underline{I})$ contains $(I_1 \cdots I_{d-1})^*$, and hence obtain the following minor extension of the theorem of Skoda and Briançon:

If R is a regular local ring of Krull dimension d and if $\underline{I} = (I_1, \ldots, I_d)$ is a set of m-primary ideals of R, then $(I_1 \cdots I_d)^*$ is contained in every joint reduction of $\underline{I}$.

1. General results. In this section we prove a number of general results needed in the sequel. First we show that if $(x_1, \ldots, x_s)$ and $(y_1, \ldots, y_s)$ are independent sets of general elements of a set of ideals $\underline{I} = (I_1, \ldots, I_s)$ of Q then there is an automorphism θ of Q_g fixing the elements of Q and the elements X_i for large i, such that $y_i = \theta(x_i)$ for $i = 1, \ldots, s$. From the definitions given in the introduction, it is clearly sufficient to restrict attention to the case where both sets are standard independent sets of general elements, and for this case we will state our restriction on θ in a slightly different (although equivalent) form. We will term θ acceptable if it is the extension of an automorphism θ_N of Q_N for some N, which fixes the elements of Q and is defined by taking $\theta(X_i) = X_i$ for $i > N$. We first deal with the case $s = 1$.

LEMMA 1.1. *Let I be an ideal of Q and let $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ be two sets of generators of I. Then there is an automorphism θ of Q_{m+n} fixing the elements of Q so that, if $x = \sum a_i X_i$ and $y = \sum b_j X_j$, then $\theta(x) = y$.*

Proof. Let $z = \sum a_i X_i + \sum b_j X_{m+j}$, and suppose also that $b_j = \sum c_{ji} a_i$ and $a_i = \sum d_{ij} b_j$. Define automorphisms α, β of Q_{m+n}, fixing the elements of Q by taking

$$\alpha(X_i) = \begin{cases} X_i + \sum c_{ji} X_{m+j} & \text{if } i \le m, \\ X_i & \text{if } i > m; \end{cases}$$

$$\beta(X_j) = \begin{cases} X_{j+m} + \sum d_{ij} X_i & \text{if } j \le n, \\ X_j & \text{if } j > n; \end{cases}$$

so that $\alpha(x) = z$ and $\beta(y) = z$. Then, if $\theta = \beta^{-1}\alpha$, $\theta(x) = y$ as required. □

THEOREM 1.2. *If $\underline{I} = (I_1, \dots, I_s)$ is a set of s ideals of Q, and if $(x_1, \dots, x_s)$ and $(y_1, \dots, y_s)$ are two standard independent sets of general elements of $\underline{I}$, then there exists an acceptable automorphism θ of Q_g such that*

$$\theta(x_i) = y_i \quad (i = 1, \dots, s).$$

Proof. The case $s = 1$ follows from the lemma above. We use induction on s and therefore we can assume that there exists an acceptable automorphism θ' of Q_g such that $\theta'(x_i) = y_i$ $(i = 1, \dots, s-1)$. Let N be such that θ' extends an automorphism of Q_N and fixes X_i if $i > N$. Then we can find automorphisms α and β of Q_g arising from a permutation of a finite number of the elements X_i such that $\alpha(x_s) = \sum a_i X_{N+i}$ and $\beta(y_s) = \sum b_j X_{N+j}$, where $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$ are two sets of generators of I_s. It follows that $\alpha(x_s)$ and $\beta(y_s)$ can be considered as standard general elements of $I_s Q_N$. Hence, applying Lemma 1.1, there is an acceptable automorphism ϕ of Q_g fixing the elements of Q_N such that

$$\phi(\alpha(x_s)) = \beta(y_s).$$

Hence we can take $\theta = \beta^{-1}\phi\theta'\alpha$. □

Now we turn to our second general result, namely: if $\underline{I} = (I_1, \dots, I_d)$ is a set of m-primary ideals of Q and $(x_1, \dots, x_d)$ is an independent set of general elements of $\underline{I}$, then $(x_1, \dots, x_d)$ is a joint reduction of $\underline{I}$. Note that we do not need the restriction that the ideals I_j are m-primary.

First we require some notation. Let $R = (r(1), \dots, r(s))$ denote a set of s integers and let $\underline{I} = (I_1, \dots, I_s)$ be a set of s ideals of Q. Then $\underline{I}^R$ will denote the product $I_1^{r(1)} \cdots I_s^{r(s)}$, with the convention that if $r(j) \le 0$ then $I_j^{r(j)} = Q$. If

$$R = (r(1), \dots, r(s)) \quad \text{and} \quad r(j) > 0$$

then $R\{j\}$ will denote the set of integers $(r(1), \dots, r(j)-1, \dots, r(s))$. If $\underline{x} = (x_1, \dots, x_s)$ is a set of elements of Q such that x_j belongs to I_j, then $\underline{x}(\underline{I}^R)$ will denote the ideal

$$x_1 \underline{I}^{R\{1\}} + \cdots + x_s \underline{I}^{R\{s\}}.$$

We will extend the definition of a joint reduction and say that $\underline{x} = (x_1, \dots, x_s)$ is a joint reduction of $\underline{I}$ if $\underline{I}^R = \underline{x}(\underline{I}^R)$ for some R; that is, we do not insist that $s = d$. Note that if $(x_1, \dots, x_s)$ is a joint reduction of $\underline{I}$ and if $\underline{I}' = (I_1, \dots, I_s, \dots, I_{s+t})$ is a set of ideals of Q containing $\underline{I}$, then $(x_1, \dots, x_{s+t})$ is a joint reduction of $\underline{I}'$ for any choice of the elements x_{s+j} in I_{s+j} for $j = 1, \dots, t$.

As before, we will denote by $\underline{I}_g$ the set of ideals $(I_1 Q_g, \dots, I_s Q_g)$.

Next let $T = (t_1, \dots, t_d)$ be a set of indeterminates over Q. Then we will write U for the set $(t_1^{-1}, \dots, t_d^{-1})$, and u_i for t_i^{-1}. We will write T^R for $t_1^{r(1)} \cdots t_d^{r(d)}$, negative or zero exponents being allowed. Finally we define $R(\underline{I})$ to be the graded subring of $Q[T, U]$ consisting of all finite sums $\sum a(R) T^R$ with $a(R)$ in $\underline{I}^R$, the exponents $r(i)$ being allowed to be positive, negative, or zero.

LEMMA 1.3. *Let x_j be a general element of I_j. Then there exists an integer $r_0(j)>0$ such that if R satisfies $r(j)\geq r_0(j), r(i)$, with $i\neq j$ being unrestricted, then*

$$\underline{I}_g^R \cap x_j Q_g = x_j \underline{I}_g^{R\{j\}}.$$

Proof. We divide the proof of this lemma into two parts, the concluding statement of part (1) being required to complete the proof in part (2).

(1) Since $u=u_1u_2\cdots u_d$ is a nonzero divisor of $R(\underline{I})$, the set of prime ideals associated with $u^nR(\underline{I})$ is the same for all n. We divide this set of prime ideals into two subsets: S_1 consisting of those that contain $I_jt_jR(\underline{I})$, and S_2 those that do not.

We now replace Q by Q_g and $\underline{I}$ by $\underline{I}_g=(I_1Q_g,\dots,I_dQ_g)$, and make a similar division of the set of prime ideals associated with $u^nR(\underline{I}_g)$ into sets S_{g1} and S_{g2}. Then S_{gi} will consist of the prime ideals $p_g=pR(\underline{I}_g)$, where $p\in S_i$. It follows that the element x_jt_j does not belong to any prime ideal p_g in S_{g2}.

Now denote by M_n the $R(\underline{I}_g)$-module $(u^nR(\underline{I}_g):x_jt_j)/u^nR(\underline{I}_g)$. Then M_n is annihilated by $(I_jt_j)^N$ for some integer N depending on n. But M_n is a finitely generated $R(\underline{I}_g)$-module, and hence the last sentence implies that any homogeneous element of M_n whose jth degree $r(j)$ is large enough must be zero.

(2) Now let J denote the graded ideal of $R(\underline{I}_g)$ consisting of all finite sums $\sum a(R)T^R$ with $a(R)$ in $x_jQ_g\cap \underline{I}_g^R$. Then J will have a finite set of generators $(y_1,\dots,y_m)$ which will be of the form $y_j=x_jt_jb_j$, with b_j a homogeneous element of $Q_g[T,U]$. It follows that we can find an integer q such that u^qb_j belongs to $R(\underline{I}_g)$ for $j=1,\dots,m$. Hence, if zT^R is a homogeneous element of J then

$$u^qzT^R=x_jt_jw,$$

where w is a homogeneous element of $R(\underline{I}_g)$ whose jth degree is $r(j)-q-1$. Now w belongs to $u^qR(\underline{I}_g):x_jt_j$, and so by the conclusion of (1) it follows that w belongs to $u^qR(\underline{I}_g)$ if $r(j)-q-1$ is sufficiently large. Hence we can find $r_0(j)$ such that, if $r(j)\geq r_0(j)$, w belongs to $u^qR(\underline{I}_g)$ and hence $zT^R\in x_jt_jR(\underline{I}_g)$. Hence if $r(j)\geq r_0(j)$ then $x_jQ_g\cap \underline{I}_g^R=x_j\underline{I}_g^{R\{j\}}$. □

THEOREM 1.4. *If $(x_1,\dots,x_d)$ is an independent set of general elements of $\underline{I}=(I_1,\dots,I_d)$, then $x_1,\dots,x_d$ is a joint reduction of $\underline{I}_g$.*

Proof. We will eventually prove this result by induction on d. We first note that if the ideal $I_1I_2\cdots I_d$ is nilpotent then, for N sufficiently large,

$$(I_1\cdots I_d)^N=(0)=(I_1\cdots I_d)^{N+1}.$$

This implies that if $y_j\in I_j$ $(j=1,\dots,d)$ then $(y_1,\dots,y_d)$ is a joint reduction of $\underline{I}$. It also implies the case $d=0$.

Before proceeding to the induction, we first show that we can impose the restriction that $(0:I_1I_2\cdots I_d)=(0)$, which will imply that each of the ideals I_j contains a nonzero divisor, and hence that the elements x_j are nonzero divisors.

Suppose therefore that this theorem has been proved for all local rings Q' of dimension $d'\leq d$ and all sets of ideals $\underline{I}'=(I_1,\dots,I_{d'})$ of Q' satisfying $(0:I_1\cdots I_{d'})=(0)$. Then we will prove that it holds for all local rings Q' of dimension $d'\leq d$ and all sets of ideals $\underline{I}'$ without restriction. To this end, suppose that $I_1I_2\cdots I_{d'}$ is not nilpotent; let $J_q=(0\colon (I_1\cdots I_{d'})^q)$ and let J denote the stable value of J_q (i.e., the value of J_q when q is large); and suppose that q is such that $J_q=J$. Consider the ring $Q''=Q'/J$ and the set of ideals $I''_j=I_j+J/J$ of Q''. Since $\dim Q''\le d$ and $(0\colon I''_1\cdots I''_d)=(0)$, it follows that, for a suitable R,

$$(\underline{I}'_g)^R=\underline{x}((\underline{I}'_g)^R)+JQ_g\cap(\underline{I}'_g)^R.$$

Now multiplying both sides by $(I_1\cdots I_{d'})^q$ yields the required result that $\underline{x}$ is a joint reduction of $\underline{I}'$. Note that R is replaced by $(r(1)+q,\dots,r(d')+q)$.

We now come to the inductive step. We suppose that the required result has been proved for all local rings Q of dimension $<d$. We further suppose that Q has dimension d, that the ideals I_j are subject to the restriction described above, and that the general elements x_j are given in standard form. It follows that we can find N such that x_1 belongs to Q_N. We now take

$$Q'=Q_N/x_1Q_N,\qquad I'_j=I_jQ_N+x_1Q_N/x_1Q_N,$$

and let x'_j denote the image of x_j in $(Q')_g$. Then $x'_2,\dots,x'_d$ is an independent set of general elements of the set of ideals $\underline{I}'=(I'_2,\dots,I'_d)$. Hence, applying our inductive assumption and lifting to Q_g we have, for a suitable R,

$$\underline{I}_g^R\subseteq x_1Q_g+x_2\underline{I}_g^{R\{2\}}+\cdots+x_d\underline{I}_g^{R\{d\}},$$

and this can be written

$$\underline{I}_g^R=x_1Q_g\cap\underline{I}_g^R+x_2\underline{I}_g^{R\{2\}}+\cdots+x_d\underline{I}_g^{R\{d\}}.$$

Now, by suitably increasing $r(1)$, we can apply Lemma 1.3 and obtain

$$\underline{I}_g^R=x_1\underline{I}_g^{R\{1\}}+x_2\underline{I}_g^{R\{2\}}+\cdots+x_d\underline{I}_g^{R\{d\}},$$

and the result is proved. □

We conclude this section with two general results. One shows the existence of joint reductions if the field k is infinite, and the other concerns the ideal $X(\underline{I})=(x_1,\dots,x_s)\cap Q$ defined in the introduction. The proofs are similar and depend on a general lemma, which requires some additional notation. Let M be a finitely generated Q-module. Then M_g will denote the Q_g-module $M\otimes_Q Q_g$. Next let $\underline{I}=(I_1,\dots,I_s)$ be a set of ideals of Q and let $\underline{x}=(x_1,\dots,x_s)$ be an independent set of general elements of $\underline{I}$ given in standard form, and suppose that N is such that the elements x_j all belong to Q_N. Then we term a set of elements $\underline{y}=(y_1,\dots,y_s)$ a specialization of $\underline{x}$ if the elements y_i are derived from the set of elements x_i by replacing the indeterminates X_i by elements a_i of Q. Note that $y_i\in I_i$ for each i. We say that a statement is true for almost all specializations of $\underline{x}$ if there is a polynomial $f(X_1,\dots,X_N)$ with coefficients in Q but not all in m, such that the statement is true whenever $f(a_1,\dots,a_N)$ is a unit of Q. Note that if k is infinite, this would imply that the statement is true for an infinite set of specializations.

LEMMA 1.5. *Let M be a finitely generated Q-module, let $\underline{I},\underline{x}$ be as above, and let $M_1,\dots,M_s$ be a finite set of submodules of M. Let $z_1,\dots,z_n$ be elements of the submodule $(x_1M_{1g}+\cdots+x_sM_{sg})\cap M$ of M. Then, for almost all specializations $\underline{y}$ of $\underline{x}$, $z_1,\dots,z_n$ belong to the module $y_1M_1+\cdots+y_sM_s$.*

Proof. A simple induction reduces the proof to the case $n=1$, and we will write z in place of z_1. Now we can write

$$zf(X_1,\dots,X_N)=x_1g_1(X_1,\dots,X_N)+\cdots+x_sg_s(X_1,\dots,X_N),$$

where $g_i(X_1,\dots,X_N)$ is a polynomial in $X_1,\dots,X_N$ with coefficients in M_i, and the coefficients of f belong to Q but do not all belong to m. We can take this f as the polynomial f for the above definition of "almost all".

THEOREM 1.6. *Let (Q,m,k,d) be a local ring, with k infinite. Then, if $\underline{I}=(I_1,\dots,I_d)$ is a set of d ideals of Q and $\underline{x}=(x_1,\dots,x_d)$ is a set of general elements of $\underline{I}$ in standard form, then*

(i) *almost all specializations $\underline{y}$ of $\underline{x}$ are joint reductions of $\underline{I}$;*
(ii) *$X(\underline{I})=(x_1Q_g+\cdots+x_dQ_g)\cap Q$ is contained in $y_1Q+\cdots+y_dQ$ for almost all specializations $\underline{y}$ of $\underline{x}$.*

Proof. (i) We start with the equation

$$\underline{I}_g^R=x_1\underline{I}_g^{R\{1\}}+\cdots+x_d\underline{I}_g^{R\{d\}},$$

which holds for some R. We take $M=Q$ and $z_1,\dots,z_n$ to be a set of generators of $\underline{I}^R$. We take M_j to be $\underline{I}^{R\{j\}}$ and apply the above lemma to obtain

$$\underline{I}^R=y_1\underline{I}^{R\{1\}}+\cdots+y_d\underline{I}^{R\{d\}}$$

for almost all specializations $\underline{y}$.

(ii) We again take $M=Q$ and $z_1,\dots,z_n$ to be a set of generators of $X(\underline{I})$. Take $M_1,\dots,M_s$ all equal to Q, and again apply the lemma. $\square$

2. The main theorem. The two statements of Theorem 1.6 can be stated loosely as follows. For almost all choices of $y_j\in I_j$ with $j=1,\dots,d$, $(y_1,\dots,y_d)$ is a joint reduction of $\underline{I}=(I_1,\dots,I_d)$; further, for almost all choices of $y_j\in I_j$ with $j=1,\dots,d$, the ideal $y_1Q+\cdots+y_dQ$ contains $X(\underline{I})$. A natural question to ask is whether for *all* joint reductions $(y_1,\dots,y_d)$ the ideal $y_1Q+\cdots+y_dQ$ contains $X(\underline{I})$. (We will also refer to the ideal $y_1Q+\cdots+y_dQ$ as a joint reduction.) This is false even in the case $d=1$, as the following example shows.

EXAMPLE. Let $Q=k[[X,Y]]/(X^3,XY)$, and denote the images of X,Y in Q by u,v. Take I to be the maximal ideal $m=uQ+vQ$. Then, as $(uX_1+vX_2)u=u^2X_1$, $u^2\in X(I)$. But $m^3=(v^3)=vm^2$, implying that vQ is a reduction of m. But u^2 does not belong to vQ.

Because in this example m consists entirely of zero divisors, Q is not Cohen–Macaulay. The primary purpose of this section is to show that, if we restrict (Q,m,k,d) to be Cohen–Macaulay, then all joint reductions of a set $\underline{I}$ consisting of d m-primary ideals contain the ideal $X(\underline{I})$. We consider first the case $d=1$.

For the proof of this case we need the following well-known result, which we state without proof. (A proof can be found in, e.g., [4, Thm. 17, p. 271], noting that Northcott uses semi-regular for Cohen–Macaulay.)

Let Q be a Cohen–Macaulay local ring, X an indeterminate over Q, and Q' the localization of $Q[X]$ at a prime ideal p. Then Q' is also Cohen–Macaulay.

We now collect together the basic ingredients of the proof of the case $d=1$ in the following lemma.

LEMMA 2.1. *Let (Q, m, k) be a 1-dimensional Cohen–Macaulay local ring with complete ring of fractions F. Let a, b be elements of m such that $I=(a, b)$ is m-primary and aQ is a reduction of I. Let $S=Q[X]$, let $Q(X)$ denote the localization of S at $m[X]$, and let $x=aX+b$. Then*

(i) *$xS=J\cap J'$, where $J=xF[X]\cap S$ and $J'=xQ(X)\cap S$;*

(ii) *J is the kernel of the homomorphism $S\to F$ in which Q maps identically and X maps into $-b/a$; and*

(iii) *$xQ(X)\cap Q\subseteq aQ$.*

Proof. (i) The result quoted above implies that xS is unmixed of height 1. Now suppose that p is a height-1 prime ideal of S containing x. If it contains a then it contains mS, which is a prime ideal of height 1 and so must equal it. If it does not contain a then, as the ring of fractions z/a^n is equal to F, it is the intersection of a prime ideal P of $F[X]$ associated with $xF[X]$ with S. Hence $xS=J\cap J'$, where J and J' are as defined.

(ii) The map $S\to F$ extends to a map $F[X]\to F$ and, since b/a belongs to F, the kernel of this restriction is $xF[X]$. Hence the kernel of the map $S\to F$ is $xF[X]\cap S=J$.

(iii) Let z be an element of $xQ(X)\cap Q=J'\cap Q$. If f is any element of J then $zf\in xS$. Now, as aQ is a reduction of I, $-b/a$ is integrally dependent on Q and so satisfies an equation $f(X)=X^m+a_1X^{m-1}+\cdots+a_m=0$. Then $f(X)\in J$ so that $zf(X)$ belongs to xS; say, $zf(X)=(aX+b)g(X)$, where $g(X)$ is a polynomial over Q of degree $m-1$, since a is not a zero divisor. Equating coefficients of X^m, we see that $z\in aQ$.

The case $d=1$ is now easily dealt with. □

LEMMA 2.2. *Let (Q, m, k) be a 1-dimensional Cohen–Macaulay local ring, and let I be an m-primary ideal of Q and $a\in m$ such that aQ is a reduction of I. Then $aQ\supseteq X(I)=xQ_g\cap Q$ for any general element x of I.*

Proof. Since $X(I)$ is independent of the choice of x, we can choose x as follows. Let $(a_1, a_2, \ldots, a_m=a)$ be a minimal basis of I. This choice is possible since aQ is a reduction of I and hence a does not belong to Im. Take $x=\sum a_iX_i$. Now $xQ_g\cap Q_m=xQ_m$ since $x\in Q_m$. In (iii) above, replace Q by Q_{m-1}, X by X_m, and b by $a_1X_1+\cdots+a_{m-1}X_{m-1}$. Then $xQ_m\cap Q_{m-1}\subseteq aQ_{m-1}$, and taking intersections with Q yields the required result. □

We now return to the proof for $d>1$; in order to set up the induction, we first require three lemmas. We also require some further notation. Let (Q, m, k, d) be a Cohen–Macaulay local ring with $d\geq 2$, let $\underline{I}=(I_1, \ldots, I_d)$ be a set of m-primary ideals of Q, and let $\underline{b}=(b_1, \ldots, b_d)$ be a joint reduction of $\underline{I}$. We will denote by $Q_{(r)}$ the ring $Q/b_1Q+\cdots+b_rQ$ and denote the image of an element or ideal of Q

under the canonical map $Q \to Q_{(r)}$ by adding a suffix (r). If $R=(r(1),\ldots,r(d))$ is any set of nonnegative integers, then $\underline{b}^R$ will denote $b_1^{r(1)}\cdots b_d^{r(d)}$.

LEMMA 2.3. *The set of elements $b_{r+1,(r)},\ldots,b_{d,(r)}$ is a joint reduction of the set of ideals $\underline{I}_{(r)}=(I_{r+1,(r)},\ldots,I_{d,(r)})$.*

Proof. A straight reduction reduces the proof to the case $r=1$. Now let

$$J=\sum_{j=1}^{d} b_j I_1\cdots I_{j-1}I_{j+1}\cdots I_d.$$

Then the condition that $b_1,\ldots,b_d$ is a joint reduction of $\underline{I}$ is equivalent to the statement that J is an ordinary reduction of $I=I_1\cdots I_d$. This follows from the well-known result that $J\subseteq I$ is a reduction of I if there exists a finitely generated module M with zero annihilator such that $IM=JM$. Since we are assuming that Q is Cohen–Macaulay, the ideals I_j do not consist entirely of zero divisors. The definition of joint reduction supplies such a module which is a product of powers of the ideals I_j.

Now write $\underline{J}_{(1)}$ for the ideal $\sum_{j=1}^{d} b_{j,(1)}I_{2,(1)}\cdots I_{j-1,(1)}I_{j+1,(1)}\cdots I_{d,(1)}$ and $I_{(1)}$ for $I_{2,(1)}\cdots I_{d,(1)}$. Then as J is a reduction of I, $I_{1,(1)}J_{(1)}$ is a reduction of $I_{1,(1)}I_{(1)}$. But, as $d\geq 2$, $I_{1,(1)}$ does not consist of zero divisors, and hence $J_{(1)}$ is a reduction of $I_{(1)}$. This implies that $b_{2,(1)},\ldots,b_{d,(1)}$ is a joint reduction of $\underline{I}_{(1)}$, as required. □

In the following lemma, (Q,m,k,d) is not assumed to be Cohen–Macaulay.

LEMMA 2.4. *Let M be a finitely generated Q-module, let I be an ideal of Q, and suppose there exists an element b of I such that $bM=IM$. Then, if x is a general element of I, $xM_g=IM_g$.*

Proof. Let $a_1,\ldots,a_r$ be a minimal basis of I and let $b=\sum b_ja_j$. Without loss of generality, we can assume that $x=\sum X_ja_j$. $bM=IM$ if and only if $bM+mIM=IM$. Let U, V, W denote the vector spaces $I/mI, M/mM, IM/mIM$ over k. Then we have a bilinear map $U\times V\to W$ and each element of U therefore induces a linear map $V\to W$. Let V, W have dimensions m, n and let β_j be the image of b_j in U. Now consider the map $V_g\to W_g$ induced by multiplication by x. Relative to bases of V, W this map will have matrix $A(X)$ whose entries are linear forms over k in $X_1,\ldots,X_r$. Since the map induced by multiplication by b is onto, $m\geq n$ and $A(\beta)$ has rank n. Hence $A(X)$ has rank n and the map induced by multiplication by x is onto; that is, $xM_g=IM_g$.

LEMMA 2.5. *Let (Q,m,k,d) be a Cohen–Macaulay local ring with $d\geq 2$, let $\underline{I}=(I_1,\ldots,I_d)$ be a set of m-primary ideals of Q, and let $\underline{b}=(b_1,\ldots,b_d)$ be a joint reduction of $\underline{I}$. Then, if $\underline{x}=(x_1,\ldots,x_d)$ is an independent set of general elements of $\underline{I}$ and if $r<d$, then $(b_1,\ldots,b_r,x_{r+1},\ldots,x_d)$ is a joint reduction of $\underline{I}_g$.*

Proof. We proceed by induction on $d-r$. Hence we first consider the case $r=d-1$. By hypothesis, there exists a set of integers $R=(r(1),\ldots,r(d))$ such that

$$\underline{I}^R=b_1\underline{I}^{R\{1\}}+\cdots+b_{d-1}\underline{I}^{R\{d-1\}}+b_d\underline{I}^{R\{d\}}.$$

Let J be the ideal $b_1\underline{I}^{R\{1\}}+\cdots+b_{d-1}\underline{I}^{R\{d-1\}}$ of Q, and let M be the Q-module $\underline{I}^{R\{d\}}+J/J$. Then, by hypothesis, $I_dM=b_dM$. Hence, by the last lemma, $x_dM_g=I_dM_g$ and therefore

$$\underline{I}_g^R=b_1\underline{I}^{R\{1\}}+\cdots+b_{d-1}\underline{I}^{R\{d-1\}}+x_d\underline{I}^{R\{d\}},$$

whence $(b_1,\ldots,b_{d-1},x_d)$ is a joint reduction of $\underline{I}_g$. □

Now we turn to the induction. Suppose we have proved that $(b_1,\ldots,b_{r+1},x_{r+2},\ldots,x_d)$ is a joint reduction of $\underline{I}_g$, and suppose that N is an integer such that $x_{r+2},\ldots,x_d$ belong to Q_N. Then $(x_{r+2},\ldots,x_d,\ldots,b_{r+1})$ is a joint reduction of $\underline{I}_N$, and we can apply the case $r=d-1$ to replace b_{r+1} by a general element x_{r+1} of I_{r+1}.

Now we come to our main theorem.

THEOREM 2.6. *Let (Q,m,k,d) be a Cohen–Macaulay local ring and let $\underline{I}=(I_1,\ldots,I_d)$ be a set of m-primary ideals of $\underline{I}$. Further, let $\underline{x}=x_1,\ldots,x_d$ be an independent set of general elements of $\underline{I}$, and let B be an ideal of Q generated by a joint reduction $(b_1,\ldots,b_d)$ of $\underline{I}$. Then*

$$X(\underline{I})=(x_1Q_g+\cdots+x_dQ_g)\cap Q\subseteq B.$$

Proof. We will assume $x_1,\ldots,x_d$ are chosen in standard form. This will imply that we can choose N so that $x_1,\ldots,x_{d-1}$ are in Q_N but x_d is general over Q_N.

The case $d=1$ has been proved in Lemma 2.2. Suppose the result has been proved for Cohen–Macaulay local rings of dimension $<d$. Then the result is true for the ring $Q'=Q/b_dQ$ and the ideals $I_j'=I_j+b_dQ/b_dQ$. Since (by Lemma 2.3) the images of $b_1,\ldots,b_{d-1}$ in Q' form a joint reduction of $(I_1',\ldots,I_{d-1}')$ and the images of $x_1,\ldots,x_{d-1}$ in Q_g' form an independent set of general elements of the same set of ideals, we have

$$B\supseteq(x_1,\ldots,x_{d-1},b_d)\cap Q.$$

If we choose N as indicated above then, by Lemma 2.5, $(x_1,\ldots,x_{d-1},b_d)$ is a joint reduction of $\underline{I}_N$. Replacing Q_N by $Q''=Q_N/x_1Q_N+\cdots+x_{d-1}Q_N$ (which has dimension 1), we can apply Lemma 2.3 to see that b_d is a reduction of I_dQ'' and that x_d is a general element of the same ideal. Hence $b_dQ''\supseteq x_dQ_g''\cap Q''$, which can be written

$$(x_1Q_N+\cdots+x_{d-1}Q_N+b_dQ_N)\supseteq(x_1Q_g+\cdots+x_dQ_g)\cap Q_N,$$

and taking intersections with Q yields the final result. □

We conclude this section with an application of the above theorem.

THEOREM 2.7. *Let (Q,m,k,d), $\underline{I}$, and $\underline{x}$ be as in the last theorem, and let $\underline{I}'=(I_1',\ldots,I_d')$ be a second set of ideals of Q such that I_j' is a reduction of I_j for $j=1,\ldots,d$. Then, if $\underline{x}'=(x_1',\ldots,x_d')$ is an independent set of general elements of $\underline{I}'$,*

$$X(\underline{I})=(x_1Q_g+\cdots+x_dQ_g)\cap Q\subseteq X(\underline{I}')=(x_1'Q_g+\cdots+x_d'Q_g)\cap Q.$$

Proof. We can suppose that x_j, x_j' $(j=1,\dots,d)$ and N are so chosen that $x_j' \in Q_N$ for $j=1,\dots,d$, and that x_j $(j=1,\dots,d)$ form an independent set of general elements of $\underline{I}_N$.

A straightforward calculation shows that, as x_j' $(j=1,\dots,d)$ is a joint reduction of $\underline{I}'$, it is a joint reduction of $\underline{I}$. Hence

$$X(\underline{I}_N) \subseteq x_1' Q_N + \cdots + x_d' Q_N,$$

and taking intersections with Q we obtain the desired result. □

3. An extension of the Skoda–Briançon theorem. We commence this section with some general observations and a definition. Let A be a Noetherian ring and let $I, J \subseteq I$ be two ideals of A.

DEFINITION. An ideal C of A is termed a conductor of J in I if

(i) $(0:C)$ is nilpotent; and
(ii) $JC = IC$.

We first observe that the existence of a conductor of J in I implies that J is a reduction of I. This follows in view of (i), and we leave the proof to the reader. If $(0:I^r)$ is nilpotent for all r then the converse is true, since then (by the definition of reduction) some power of I is a conductor of J in I. It is also clear that if $I \supseteq J' \supseteq J$ then C is a conductor of J in I if and only if it is a conductor of J' in I and of J in J'. Further, if C, C' are conductors of J in I then so is their sum. Hence, if J is a reduction of I then there is a unique maximal conductor of J in I.

In the present paper we need only the case where A is a 1-dimensional local ring (Q, m, k) which is (in the first instance) Cohen–Macaulay and (later) will be assumed to be, in addition, analytically unramified. We will also restrict J to be a principal ideal aQ of Q which is m-primary (i.e., a is not a zero divisor). I will be an ideal of which aQ is a reduction. For the moment we only suppose that Q is Cohen–Macaulay, and state our results for this case in the following lemma. In this lemma $\lambda(\cdot)$ denotes length and $e(I)$ denotes the multiplicity of I. Note that in this case C is assumed to be m-primary.

LEMMA 3.1. *The following properties of an m-primary ideal C of Q are equivalent:*

(i) $\lambda(C/IC) = e(I)$;
(ii) *for all a such that aQ is a reduction of I, C is a conductor of aQ in I;*
(iii) *for some reduction aQ of I, C is a conductor of aQ in I.*

Proof. (i) ⇒ (ii). For any a such that aQ is m-primary,

$$\lambda(C/aC) = \lambda(Q/aQ) + \lambda(aQ/aC) - \lambda(Q/C) = \lambda(Q/aQ),$$

and if aQ is a reduction of I then $e(I) = \lambda(Q/aQ)$. Hence, if (i) holds then $aC = IC$ and therefore (ii) is true.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). If (iii) holds, $IC = aC$ and hence $\lambda(C/IC) = \lambda(C/aC) = e(I)$. □

COROLLARY. *If y is a general element of I and C is a conductor of aQ in I, then*

$$IC \subseteq yQ_g \cap Q.$$

Proof. Since $\lambda(CQ_g/ICQ_g) = \lambda(C/IC) = e(I) = e(IQ_g)$, it follows that CQ_g is a conductor of yQ_g in IQ_g. Hence $IC \subseteq yCQ_g \cap Q \subseteq yQ_g \cap Q$. □

Now we impose the condition that Q is analytically unramified. This implies that the integral closure Q^* of Q in the complete ring of fractions F of Q is a finite Q-module, and hence we can define the conductor $C(Q^*/Q)$ of Q^* in Q.

LEMMA 3.2. *If Q is analytically unramified, then $C = C(Q^*/Q)$ is a conductor of J in I for any m-primary ideal I and any reduction J of I.*

Proof. We can take $J = aQ$. Then aC is an ideal of Q^* and hence

$$aC = aCQ^* \cap Q = (aC)^*,$$

where $(aC)^*$ is the integral closure of aC considered as an ideal of Q. But this is also the integral closure of IC and so contains IC. Hence $aC \subseteq IC \subseteq aC$, and hence C is a conductor of J in I. □

We now return to a local ring (Q, m, k, d), with the restriction that Q is Cohen–Macaulay and also analytically unramified, but do not assume $d = 1$. We will consider a set $\underline{I} = (I_1, \ldots, I_{d-1})$ of $d-1$ m-primary ideals of Q and an independent set of general elements $(x_1, \ldots, x_{d-1})$ of $\underline{I}$. We will also consider a further m-primary ideal J and a general element y of J such that $(x_1, \ldots, x_{d-1}, y)$ is an independent set of general elements of the set $\underline{I}$, J.

Our main concern is with the sequence of local rings

$$L_r = Q_g/x_1Q_g + \cdots + x_rQ_g \quad (r = 1, \ldots, d-1),$$

particularly in the case $r = d-1$, and we will write L for L_{d-1}.

LEMMA 3.3. *If Q is analytically unramified then so is Q_g.*

Proof. If Q is analytically unramified then there exists a constant c such that $(m^{n+c})^* \subseteq m^n$ for all n. Now consider $(m^{n+c})^*Q_g$. This is integrally closed, because there exists a finite set of valuations $v_1, \ldots, v_s$, taking as values nonnegative integers or ∞ on Q and positive values on m, such that $x \in (m^{n+c})^*$ if and only if

$$v_i(x) \geq (n+c)v_i(m) \quad (i = 1, \ldots, s),$$

where $v_i(m) = \operatorname{Min} v_i(y)$ for y in m. Now we can extend v_i to a valuation V_i on Q_g as follows. Let $f = \sum a_\mu \mu$ be an element of $Q[X_1, X_2, \ldots]$, where μ denotes a monomial in $X_1, X_2, \ldots$. We define $V_i(f) = \operatorname{Min}(v_i(a_\mu))$, and then extend V_i to Q_g in the usual way. It is then clear that $x \in (m^{n+c})^*Q_g$ if and only if

$$V_i(x) \geq (n+c)v_i(m) \quad (i = 1, \ldots, s),$$

proving that $(m^{n+c})^*Q_g$ is integrally closed and hence equal to $(m^{n+c}Q_g)^*$. Hence

$$(m^{n+c}Q_g)^* \subseteq m^nQ_g$$

for all n, and this implies that Q_g is analytically unramified. □

LEMMA 3.4. *If Q is analytically unramified then so are the rings L_r for $r = 1, \ldots, d-1$.*

Proof. We prove this by induction on r. First suppose that $r=1$. Let I_1 have minimal basis $(a_1, \ldots, a_n)$, with a_1 a nonzero divisor. Since the ring L_r is (to within isomorphism as a Q-algebra) independent of the choice of the element x_1, we can take $x_1 = \sum a_i X_i$.

Now we can define a Q-algebra isomorphism of Q_g into itself by mapping $X_i \to X_{n+i}$ $(i = 1, 2, \ldots)$. In this way we can identify Q_g with the localization of $Q_g[X_1, \ldots, X_n]$ at the ideal $m_g[X_1, \ldots, X_n]$. Now let

$$x' = -(a_2 X_2 + \cdots + a_n X_n).$$

Then, by Micali's theorem (see [3]), the homomorphism

$$Q_g[X_1, \ldots, X_n] \to S_1 = Q_g[x'/a_1, X_2, \ldots, X_n],$$

where Q_g and the elements X_i $(i \neq 1)$ are mapped identically and where X_1 is mapped to x'/a_1, has kernel the principal ideal generated by x_1. Hence L_1 is the localization of S_1 at mS_1. As a localization of a finitely generated extension of an analytically unramified local ring, L_1 is itself analytically unramified. Since $L_r = (L_1)_g/(x_2, \ldots, x_r)$, the general case follows by induction. □

We now take $r = d-1$ and consider the ring $L = Q_g/(x_1, \ldots, x_{d-1})$, which we have seen is a 1-dimensional, Cohen-Macaulay, analytically unramified local ring. It follows that the integral closure L^* of L in its complete ring of fractions is a finite L-module, and hence the conductor $C(L^*/L)$ is defined. But L is a Q-algebra and so we can define $Q \cap C(L^*/L)$. We denote this ideal by $C(\underline{I})$.

THEOREM 3.5. *Let J be any m-primary ideal of Q, and let y be a general element of J such that $(x_1, \ldots, x_{d-1}, y)$ is an independent set of general elements of $(\underline{I}, J)$. Then*

$$X(\underline{I}, J) = (x_1 Q_g + \cdots + x_{d-1} Q_g + y Q_g) \cap Q \supseteq (JC(\underline{I}))^*.$$

Proof. Both $C(L^*/L)$ and $yC(L^*/L)$ are ideals of L^* contained in L and so are integrally closed ideals of L. Further, yL is a reduction of JL, whence $yC(L^*/L) = JC(L^*/L)$. Now $X(\underline{I}, J) = yL \cap Q$. Hence $X(\underline{I}, J)$ contains $JC(L^*/L) \cap Q$, which is the intersection of the integrally closed ideal $JC(L^*/L)$ of L with Q and so is integrally closed. But $JC(L^*/L) \cap Q$ contains $JC(\underline{I})$ and therefore its integral closure. Hence $X(\underline{I}, J) \supseteq (JC(\underline{I}))^*$. □

We now apply Theorem 3.5 to obtain an extension of the theorem of Skoda and Briançon. We must first recall a result of Lipman and Sathaye [2] which was the principal ingredient in the proof of Skoda–Briançon.

Let Q be a regular local ring of dimension d with field of fractions F, and let S be a finitely generated domain over Q whose field of fractions E is a separable algebraic extension of F, so that the integral closure S^* of S in E is a finite S-module. Suppose further that S is a homomorphic image of $Q[X_1, \ldots, X_n]$ and denote by

Δ the ideal of S generated by the Jacobians of $f_1, \dots, f_n$ with respect to $X_1, \dots, X_n$, where $f_1, \dots, f_n$ ranges over all sets of n elements of the kernel of the map

$$Q[X_1, \dots, X_n] \to S.$$

Let Δ^* be the ideal of S^* defined in a similar manner for the finitely generated extension S^* of Q. Let $(S : \Delta)$ denote the set of elements x of E such that $x\Delta \subseteq S$ and $(S^* : \Delta^*)$, the set of elements x of E such that $x\Delta^* \subseteq S^*$. Then $\Delta \cdot (S^* : \Delta^*) \subseteq C(S^*/S)$, implying that $\Delta \subseteq C(S^*/S)$.

Next we refer back to the proof of Lemma 3.4 to describe an alternative characterization of the rings L_r.

We define a ring $S_{r(\phi)}$ as follows. Suppose that the ideal I_j has basis nonzero divisors $a(N_{j-1}+h)$ $(h=1, \dots, n_j)$, where $N_0 = 0$ and $N_j = n_1 + \cdots + n_{j-1}$ if $j > 0$. Let $X(i)$ $(i = 1, \dots, N_r)$ be a set of indeterminates over Q_g and adjoin these to Q_g with the exception of indeterminates $X(\phi(j))$ $(j=1, \dots, r)$, where $N_{j-1} < \phi(j) \leq N_j$. Instead of the omitted indeterminates we adjoin $z_j(\phi) = x'(\phi(j))/a(\phi(j))$ $(j = 1, \dots, r)$, where

$$x'(\phi(j)) = a(\phi(j))X(\phi(j)) - \sum a(N_{j-1}+h)X(N_{j-1}+h),$$

the last sum being over $h = 1, \dots, n_j$.

LEMMA 3.6. *L_r is isomorphic to the localization of $S_r(\phi)$ at $mS_r(\phi)$.*

The case $r=1$ is proved in Lemma 3.4. The general case then follows by an induction similar to that used at the end of the proof of Lemma 3.4.

We make one further point. After the localization at $mS_{r(\phi)}$, we can absorb the additional indeterminates adjoined to Q_g to obtain a larger ring still isomorphic to Q_g. Hence we can consider the elements z_j as contained in the field of fractions of Q_g.

THEOREM 3.7. *Let (Q, m, k, d) be a regular local ring, and let $\underline{I} = (I_1, \dots, I_d)$ be a set of d m-primary ideals of Q. Then $X(\underline{I}) \supseteq (I_1 \cdots I_d)^*$.*

We consider the ring $S_{d-1}(\phi) = Q_g[z_1, \dots, z_{d-1}]$ and apply the theorem of Lipman and Sathaye. The kernel of the map $Q_g[X_1, \dots, X_{d-1}] \to S_{d-1}(\phi)$ contains the elements $a(\phi(j))X_j - x'(\phi(j))$, and Δ therefore contains the product $a(\phi(1)) \cdots a(\phi(d-1))$. Hence, if $\underline{I}' = (I_1, \dots, I_{d-1})$ then

$$a(\phi(1)) \cdots a(\phi(d-1)) \in C((L^*/L) \cap Q = C(\underline{I}').$$

But this is true for all choices of the function ϕ, and hence $C(\underline{I}')$ contains the product $I_1 \cdots I_{d-1}$. Hence, by Theorem 3.5, $X(\underline{I}) \supseteq (I_1 \cdots I_d)^*$.

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REFERENCES

1. N. Bourbaki, *Algebre commutative,* Masson, Paris, 1983.
2. J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon–Skoda,* Michigan Math. J. 28 (1981), 199–222.
3. A. Micali, *Sur les algebres universelles,* Ann. Inst. Fourier Grenoble 14 (1964), 1–89.
4. D. G. Northcott, *Lectures on rings, modules, and multiplicities,* Cambridge University Press, Cambridge, 1968.
5. D. Rees, *Generalisations of reductions and mixed multiplicities,* J. London Math. Soc. (2) 29 (1984), 397–414.
6. ———, *The general extension of a local ring and mixed multiplicities,* Lecture Notes in Math., 1184, Springer, Berlin, 1986.

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