ISOMETRICALLY REMOVABLE SETS FOR FUNCTIONS IN THE HARDY SPACE ARE POLAR

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For a domain D in $\mathbf{C}^n$ and $0<p<\infty$, $H^p(D)$ denotes the Hardy space of analytic functions $f: D \to \mathbf{C}$ for which $|f|^p$ has a harmonic majorant. If E is a relatively closed subset of D, then E is said to be a set of *removable singularities* (or E is said to be *removable*) for $H^p(D \setminus E)$ provided that $D \setminus E$ is connected and each f in $H^p(D \setminus E)$ has an analytic extension to a function in $H^p(D)$. This can be phrased in functional analysis terms by saying that E is a set of removable singularities for $H^p(D \setminus E)$ precisely when the restriction map $H^p(D) \to H^p(D \setminus E)$ is surjective. With this observation, say that E is *isometrically removable* if the restriction map $H^p(D) \to H^p(D \setminus E)$ is a surjective isometry. The main result of this paper is the characterization of isometrically removable sets as the polar sets (provided $D \setminus E$ supports a nonconstant function in H^p). In particular, this shows that isometric removability is independent of p.

The study of removable singularities for functions in a Hardy space does not originate with this paper. One of the first papers on Hardy spaces for arbitrary domains in $\mathbf{C}$ is [11], where (among other things) it is shown that if E has logarithmic capacity 0 and $E \subseteq D$, then E is removable for $H^p(D \setminus E)$. In [9] (compare [8]), as an extension of results of [1], it was shown that a relatively closed polar subset E of a domain D in $\mathbf{C}^n$ is a removable set of singularities for $H^p(D \setminus E)$. Järvi's proof [9] that polar sets are removable sets of singularities for the Hardy spaces actually shows that they are isometrically removable. The main contribution of this note, therefore, is that the converse holds. Indeed, the key to this converse is the first lemma below, which is a purely potential theoretic one. An application of this lemma will also be given to the study of removable singularities for the Hardy spaces h^p of harmonic functions, where the results do not exactly parallel those for the spaces H^p.

Before stating and proving the main results of this paper, it is advisable to collect some of the more crucial definitions as well as some relevant background information. For any p, if $f \in H^p(D)$, let u_f denote the least harmonic majorant of $|f|^p$. Fix a point a in D. For $1 \le p < \infty$, $\|f\|_p \equiv u_f(a)^{1/p}$ defines a norm on $H^p(D)$; for $p<1$, $d(f,g) \equiv \|f-g\|_p^p = u_{f-g}(a)$ defines a metric on $H^p(D)$. The connectedness of D is necessary in order for $\|f\|_p$ to define a norm. (Though

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inaccurate, $\|f\|_p$ will be called the norm of f even when $p<1$.) The definition of the norm in $H^p(D)$ depends on the choice of the point a, but an application of Harnack's Inequality shows that a change in this point produces an equivalent norm. Call the point a in D used to define the norm the *norming point.*

For the theory of Hardy spaces on domains in $\mathbf{C}^n$ with "nice" boundary, the reader can see [10]; the elements of this theory for arbitrary domains in $\mathbf{C}$ can be found in [4]. The arguments of [4], relevant for the rest of this paper, easily carry over to domains in $\mathbf{C}^n$. In particular, Hardy spaces are Banach spaces for $p\geq 1$ and Fréchet spaces for $p<1$. An easy way to demonstrate these facts as well as many other results about Hardy spaces is to first show that the norm of a function f in $H^p(D)$ is the supremum of the L^p norms $[\int |f|^p\, d\omega]^{1/p}$, where ω is the harmonic measure evaluated at the norming point a of a subdomain G of D such that $\operatorname{cl} G$, the closure of G, is contained in D. In taking this supremum, it suffices to restrict one's attention to subregions with nice boundary or even a sequence of nice subregions that form an exhaustion of D.

In order that the relevant definitions be satisfied, it is necessary to restrict attention to relatively closed subsets E of a domain D such that $D\backslash E$ is connected. Moreover, it must be assumed that the same point in $D\backslash E$ is used as a norming point for $H^p(D)$ and $H^p(D\backslash E)$. Otherwise a discussion of isometric removability becomes absurd.

A set E in $\mathbf{C}^n$ is *polar* if there exists a superharmonic function on $\mathbf{C}^n$ that is identically $+\infty$ on E. For properties of polar sets, see Chapter 1.V of [3]. If E is a compact subset of the domain D and there is a superharmonic function on D (not constantly equal to $+\infty$) that is identically $+\infty$ on E, then E is polar. The sets E considered here are relatively closed subsets of a domain D, and hence can be written as the countable union of compact sets. In the complex plane $\mathbf{C}$, such a set is polar if and only if it has logarithmic capacity zero ([5, §3 of VII] and [3, 1.XIII.18]).

When $n>1$, it is easy to see that nonpolar sets can be removable for H^p for *all* p, $0<p<\infty$. (The reader is invited to contrast this with harmonic Hardy space, h^p, by comparing this with Theorem 2 below.) Indeed, if E is a compact subset of a domain D such that $D\backslash E$ is connected, then every function f in $H^p(D\backslash E)$ will extend to an analytic function $\tilde{f}$ on D by the Hartogs extension phenomenon [10]. To produce a harmonic majorant for $|\tilde{f}|^p$, consider a smooth majorant v which is harmonic near ∂D (for the moment, assume that D is bounded). If u is the solution of the Dirichlet problem $\Delta u=\Delta v$ with zero boundary data, then $h=v-u$ will be a harmonic majorant of $|\tilde{f}|^p$. (The relevant boundary regularity results may be found in Chapter 1.VIII of [3].) When $n=1$, the situation is far from obvious. Nevertheless, it is known that there are sets which are removable for H^p functions for all p, $0<p<\infty$, but which are not polar sets. See, for example, Corollary 1.2 of [6] and the remark following it. For a simple example, which is valid for $1\leq p<\infty$, see page 50 of [7]. Also, for $n=1$, there is a canonical decomposition for functions in $H^p(D\backslash E)$ as the sum of a function in $H^p(D)$ and a function in $H^p(\mathbf{C}\backslash E)$. See Theorem 2.1 of [2], for example. In general, removability depends on the value of p (see [6] and the references there).

We now come to the key lemma mentioned earlier. This lemma is purely potential theoretic and is valid in $\mathbf{R}^n$ $(n \geq 2)$.

LEMMA. *Let D be a domain in $\mathbf{R}^n$ $(n \geq 2)$ and assume that E is a relatively closed subset of D such that $D \setminus E$ is connected. Assume that there is a subharmonic function u on $D \setminus E$ that is not harmonic, but which has a least harmonic majorant h. If u admits a subharmonic continuation to D which is dominated by a superharmonic continuation of h to D, then E is polar.*

Proof. Let $\tilde{u}$ be the subharmonic extension of u to D. From the hypothesis, $\tilde{u}$ has a superharmonic majorant on D. Let $\tilde{h}$ be the least superharmonic majorant of $\tilde{u}$ on D; thus $\tilde{h}$ is harmonic. It is a straightforward consequence of the hypothesis that $\tilde{h} = h$ on $D \setminus E$. Thus $\tilde{h}$ is a harmonic extension of h to D.

Let $\{E_k\}$ be a sequence of compact subsets of E whose union is E. Note that the hypothesis of the lemma is satisfied by $\tilde{u} \mid D \setminus E_k$, $\tilde{h} \mid D \setminus E_k$, and E_k. Therefore if the lemma is proved under the additional assumption that E is compact, it will follow in full generality since E is the union of a countable number of polar sets and, hence, must itself be polar.

So assume that E is compact with $\tilde{u}$ and $\tilde{h}$ defined as above. Since u is not harmonic, u and h are not identical and so

$$\tilde{h}(x) > \tilde{u}(x) \quad \text{for all } x \text{ in } D. \tag{1}$$

Because $\tilde{h} - \tilde{u}$ is lower semicontinuous and E is compact, (1) implies that there is a $\delta > 0$ such that $\tilde{h}(x) - \tilde{u}(x) \geq \delta$ for all x in E. Equivalently,

$$[\tilde{h}(x) - \tilde{u}(x)]/\delta \geq 1 \quad \text{for all } x \text{ in } E. \tag{2}$$

Denote by H the upper Perron solution (or the PWB = Perron–Wiener–Brelot solution, in the terminology of [3]) on $D \setminus E$ for the boundary function that vanishes identically on ∂D and is identically 1 on ∂E. (Note that the set $D \setminus E$ is Greenian in Dobb's terminology, since it supports the positive nonconstant superharmonic function $h - u$. See 1.II.13 in [3].) The inequality (2) shows that the class of superharmonic functions v on $D \setminus E$ satisfying

$$\liminf_{\substack{x \to x_0 \\ x \in D \setminus E}} v(x) \geq 0 \tag{3}$$

for x_0 in ∂D and

$$\liminf_{\substack{x \to x_0 \\ x \in D \setminus E}} v(x) \geq 1 \tag{4}$$

for x_0 in ∂E is nonempty and, in fact, contains the function $(\tilde{h} - \tilde{u})/\delta$. Therefore, H, which is the infimum over this class, is harmonic; that is, H is not identically equal to $+\infty$. Also, H satisfies

$$[\tilde{h}(x) - \tilde{u}(x)]/\delta \geq H(x) \geq 0 \quad \text{for } x \text{ in } D \setminus E.$$

Equivalently,

$$h(x) \geq h(x) - \delta H(x) \geq u(x) \quad \text{for } x \text{ in } D \backslash E. \tag{5}$$

But h is the least harmonic majorant of u on $D \backslash E$, so (5) implies that $H \equiv 0$.

This implies that for every positive integer n there is a nonnegative superharmonic function v_n on $D \backslash E$ satisfying (3) and (4) as well as $v_n(a) < 2^{-n}$. Therefore $v \equiv \sum v_n$ is superharmonic on $D \backslash E$, $v(a) < +\infty$, and

$$\liminf_{\substack{x \to x_0 \\ x \in D \backslash E}} v(x) = +\infty$$

for every point x_0 in E. If v is extended to D by letting it be identically $+\infty$ on E, then v is superharmonic. This implies that E is a polar set. □

The following theorem is the main result of this paper.

THEOREM 1. *Let D be a domain in $\mathbf{C}^n$ and let E be a relatively closed subset of D such that $D \backslash E$ is connected. Using the same point a in $D \backslash E$ as the norming point for both $H^p(D)$ and $H^p(D \backslash E)$, consider the following statements.*

(a) *For $0 < p < \infty$, the restriction map $H^p(D) \to H^p(D \backslash E)$ is a surjective isometry.*

(b) *For $0 < p < \infty$, the restriction map $H^p(D) \to H^p(D \backslash E)$ is an isometry.*

(c) *There is a value of p, $0 < p < \infty$, such that $H^p(D \backslash E)$ contains a nonconstant function and for which the restriction map $H^p(D) \to H^p(D \backslash E)$ is an isometry.*

(d) *There is a value of p, $0 < p < \infty$, and a nonconstant function f in $H^p(D \backslash E)$ that has an analytic extension to a function $\tilde{f}$ in $H^p(D)$ with*

$$\|\tilde{f}\|_{H^p(D)} = \|f\|_{H^p(D \backslash E)}. \tag{6}$$

(e) *E is polar.*

Then (c) ⇒ (d) ⇒ (e) ⇒ (a) ⇒ (b).

The first thing that should be noted in the statement of this theorem is that condition (e), that E is a polar set, is independent of p as well as independent of the domain D that contains E. Therefore one would not expect that all the conditions are equivalent. Indeed, it is easy to find a counterexample. Let $D = \mathbf{C}^n$ and let E be a nonpolar set such that, for all p, $H^p(\mathbf{C}^n \backslash E)$ contains only constant functions. As discussed earlier, it is easy to find such a set E when $n > 1$ and it is known that such a set exists when $n = 1$ [6]. Then conditions (a) and (b) of the theorem are clearly satisfied and the remaining conditions just as clearly fail.

The key to obtaining the above conditions to be equivalent is to require $H^p(D)$ to have some nonconstant functions. This is done in the following corollary.

COROLLARY. *If D is biholomorphically equivalent to a bounded domain, then the conditions* (a) *through* (e) *in Theorem* 1 *are equivalent.*

Proof. Under the hypothesis, $H^p(D)$ and $H^p(D \backslash E)$ both contain a nonconstant function for every value of p, so it is trivial that (b) implies (c). □

Proof of Theorem 1. It is trivial that (c) implies (d) and that (a) implies (b). As mentioned previously, it is implicit in [9] that (e) implies (a). Indeed, it is shown in [9] that the least harmonic majorant of $|f|^p$ on $D\setminus E$ extends harmonically across E and from this the extendability of f follows as does the fact that the extended function has the same norm as f. (See [8] for an elegant deduction of the extendability of the majorant.)

It remains to show that (d) implies that E is polar. Let f and $\tilde{f}$ be as in (d) and let h and $\tilde{h}$ be the least harmonic majorants of f and $\tilde{f}$, respectively. Note that $\tilde{h} \geq h$ on $D\setminus E$. Equation (6) implies that h and $\tilde{h}$ agree at the norming point a; by the Maximum Principle, they must be identical on $D\setminus E$. That is, $\tilde{h}$ is a harmonic extension of h to D. Also, since f is not a constant function, $|f|^p$ is not harmonic on $D\setminus E$. It is now an immediate consequence of the preceding lemma that E is polar. This concludes the proof of the theorem. $\square$

There are analogous results for the spaces of harmonic functions $h^p(D)$ and $h^p(D\setminus E)$ for $1 \leq p < \infty$. The space $h^p(D)$ is defined as the set of all harmonic functions $u: D \to \mathbf{C}$ for which $|u|^p$ has a harmonic majorant. The norm on $h^p(D)$ is defined as for $H^p(D)$ and $h^p(D)$ is a Banach space. Here, however, it may be assumed that $D \subseteq \mathbf{R}^n$, as no underlying complex structure is necessary.

Below is the result for $h^p(D)$ that is analogous to Theorem 1. The reader will notice, however, that there is an additional benefit in that removability and isometric removability are equivalent for $p>1$. On the other hand, something is lost in that the result fails for $p=1$. The full statement of the valid implications for all possible values of p becomes rather cumbersome to state, and so, for convenience, it will be assumed throughout the statement of the next theorem that $1 < p < \infty$. The reader will be left to his own devices and inclinations to sort out the implications for $p=1$.

THEOREM 2. *Let D be a domain in $\mathbf{R}^n$ and let E be a relatively closed subset of D such that $D\setminus E$ is connected. Using the same point a in $D\setminus E$ as the norming point for both $h^p(D)$ and $h^p(D\setminus E)$, consider the following statements.*

(a) *For $1<p<\infty$, the restriction map $h^p(D) \to h^p(D\setminus E)$ is a surjective isometry.*
(b) *There is a value of p, $1<p<\infty$, such that the restriction map $h^p(D) \to h^p(D\setminus E)$ is surjective.*
(c) *There is a value of p, $1<p<\infty$, such that the restriction map $h^p(D) \to h^p(D\setminus E)$ is an isometry.*
(d) *There is a value of p, $1<p<\infty$, and a nonconstant function u in $h^p(D\setminus E)$ that has a harmonic extension to a function $\tilde{u}$ in $h^p(D)$ with $\|\tilde{u}\|_{h^p(D)} = \|u\|_{h^p(D\setminus E)}$.*
(e) *E is polar.*

Then conditions (a), (b), (c), *and* (e) *are equivalent and* (d) *implies* (e).

Proof. The proof that (d) implies (e) follows the lines of the proof of the corresponding implication in Theorem 1. The proof that (e) implies (a) can be found in Theorem 2.5 of [8]. (Once again, though the result only states that polar sets are removable, the proof given in [8] shows that they are isometrically removable.) Clearly (a) implies (b) and (c). It will now be shown that (b) implies that E is polar.

If $n=2$, then the proof that (b) implies that E is polar can be found on page 34 of [4]. The proof of this implication when $n \geq 3$ is probably known to some, but seems to be written down nowhere; thus a proof for this case will be sketched. This same proof can be adapted to the case when $n=2$.

So assume that $n \geq 3$ and that the restriction map $h^p(D) \to h^p(D\setminus E)$ is surjective. For convenience, also assume that E is compact (the proof of the general case following by writing E as the union of a sequence of compact subsets). Let h be the solution of the Dirichlet Problem for the domain $\Omega \equiv \mathbf{R}^n \setminus E$ with boundary function that is 1 on ∂E and 0 at ∞. Since h is bounded, $h \mid (D\setminus E) \in h^p(D\setminus E)$. By (b), h has a harmonic extension $\tilde{h}$ to $\mathbf{R}^n$. Since h is positive, so is $\tilde{h}$. Thus $\tilde{h}$, and hence h, is constant. But since $h(x) \to 0$ as $x \to \infty$, $h \equiv 0$. On the other hand, the set of points in ∂E where h does not assume the value 1 is a polar set ([3, p. 106]). Therefore ∂E, and hence E, is a polar set.

It remains to show that (c) implies (e). If D is unbounded, then an argument similar to that of the preceding paragraph shows that E must be polar. On the other hand, if D is bounded, then the assumptions of (d) are satisfied and so (e) follows. □

COROLLARY. *If D is a bounded domain in $\mathbf{R}^n$ or if $n=2$ and D is holomorphically equivalent to a bounded domain in $\mathbf{R}^2$, then the conditions* (a) *through* (e) *in Theorem* 2 *are equivalent.*

Here is an example that illustrates that Theorem 2 fails for $p=1$. Let $n \geq 3$, let $D=$ the unit ball in $\mathbf{R}^n$, and let $E=\{0\}$. If $h(x)=|x|^{2-n}$, then $h \in h^1(D\setminus E)$ and h has no harmonic extension to D, even though E is polar. Similarly, $\log|z|^{-1}$ belongs to $h^1(\mathbf{D}\setminus\{0\})$.

Since the Nevanlinna class $N(D)=\{f$ analytic in $D\colon \log_+|f|$ has a harmonic majorant$\}$ is the limiting space of H^p as $p \to 0$, the reader might wonder what happens to Theorem 1 in the case of the Nevanlinna class. In fact, the theorem is false here. Indeed, $z^{-1} \in N(\mathbf{D}\setminus\{0\})$ for $n=1$, and $z_1^{-1} \in N(B_n \setminus [\{0\} \times \mathbf{C}^{n-1}])$ for $n \geq 2$. Also, see Theorem 3.4 of [8], where the possibility of meromorphic extensions is discussed for functions in the Nevanlinna class.

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