ON A GEOMETRIC LOCALIZATION OF THE CAUCHY POTENTIALS

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1. Introductory remarks. Let $u \in D'(\mathbf{C})$ be a distribution in $\mathbf{C} = \mathbf{R}^2$. If h is an arbitrary C_0^∞-function in $\mathbf{C}$ (i.e., a C^∞-function with a compact support), then it is well known that the Leibniz differentiation rule still holds for the product uh (see, e.g., [12, Ch. VI]).

In particular,

$$\frac{\partial}{\partial \bar{z}}(uh) = \left(\frac{\partial}{\partial \bar{z}} u\right) \cdot h + u\left(\frac{\partial}{\partial \bar{z}} h\right),$$

where, as usual,

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and the equality is understood in the sense of distributions.

Let μ be a finite Borel measure in $\mathbf{C}$. The Cauchy potential (transform) $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int_{\mathbf{C}} \frac{d\mu}{\zeta - z}.$$

It is well known (see [8, Ch. II]) that $\hat{\mu}(z)$ is defined almost everywhere with respect to the area and that $\hat{\mu}(z) \in L^1_{\mathrm{loc}}(dx\,dy)$; that is, for any compact set $K \subset \mathbf{C}$,

$$\int_K |\hat{\mu}|\, dx\, dy < +\infty.$$

So $\hat{\mu} \in D'(C)$ and, as is known,

$$\frac{\partial \hat{\mu}}{\partial \bar{z}} = -\pi\mu$$

(see [7, Ch. II]; [8, Ch. II]). Thus, for all $h \in C_0^\infty$, we have

$$\frac{\partial}{\partial \bar{z}}(\hat{\mu}h) = -\pi\mu h + \hat{\mu}\frac{\partial h}{\partial \bar{z}}. \tag{1}$$

In other words,

$$\hat{\mu}h = \widehat{\mu \cdot h - \frac{1}{\pi}\hat{\mu}\frac{\partial h}{\partial \bar{z}}\, dx\, dy}.$$

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For $h \equiv 1$ on a small disk Δ_0 and $\equiv 0$ outside of a little larger disk Δ_1, equation (1) allows us to "localize" the Cauchy transform $\hat{\mu}$ in Δ_0. This leads to various applications of (1) to the problems in rational approximation (see [2]; [7, Ch. VIII]; [13]).

However, for a general discontinuous function h, (1) does not make any sense (even any "distributional sense") unless some additional assumptions are made. In §2 we show that (1) can still be meaningful in the case when h is a characteristic function of a smoothly bounded domain. (Speaking in terms of geometric measure theory, h is a 2-dimensional current of finite perimeter—cf. [4, Ch. IV].)

More precisely, we prove (Theorem 1) that for an arbitrary μ, any $\zeta \in \mathbf{C}$, and almost all $r > 0$, equation (1) holds for $h = \chi_{\Delta_r}$, where $\Delta_r = \{z : |z - \zeta| < r\}$. (Here, χ_{Δ_r} is a characteristic function of Δ_r.)

As a direct corollary of this result we obtain the "splitting" theorem for measures orthogonal to rational functions due to E. Bishop (for polynomials) and L. Kodama (cf. [3], [11]). Moreover, we establish an explicit formula for the resulting measures. This has not been done in [3] or [11].

As we show in §3, Corollary 1 (the particular form of Theorem 1) is especially useful since it allows us to localize the Cauchy potentials in small disks explicitly, and not by means of specially chosen test functions h.

In §3 we develop certain techniques of working with such "localized" Cauchy transforms. As an illustration of those methods we obtain a series of local estimates concerning the "thickness" of the measure μ provided by the "nice" properties of its Cauchy potentials such as boundedness, BMO, continuity, etc.

We want to mention that the ideas behind Theorem 1 and the results in §3 are, in spirit, very close to E. De Giorgi's localization of the perimeter measure contained in [11], or (more generally) to the theory of slices introduced in geometric measure theory by H. Federer (cf. [5], [6]).

Finally, we note that it seems possible that those ideas can be applied to potentials related to elliptic operators other than $\partial/\partial\bar{z}$.

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2. The localization formula. At first, we recall the well-known generalization of the Cauchy formula due to Pompeiu (see [7, Ch. II, §1]). If G is an arbitrary finitely connected smoothly bounded domain and ϕ is a differentiable function in $\bar{G}$, then

$$\frac{1}{2\pi i}\int_{\partial G}\frac{\phi(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{\pi}\iint_G \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z}\,dx\,dy = \begin{cases} \phi(z), & z \in G, \\ 0, & z \notin \bar{G}. \end{cases} \tag{2}$$

In particular, for any differentiable function ϕ with a compact support, we have

$$\phi(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z}\,dx\,dy.$$

THEOREM 1. *Let μ be a complex finite Borel measure in $\mathbf{C}$ with a compact support. Fix $\zeta_0 \in \mathbf{C}$ and let $\Delta_r = \{z : |z-\zeta_0| < r\}$. Then, for almost all $r > 0$, the following holds:*

$$\frac{\partial}{\partial \bar{z}}(\hat{\mu}\chi_{\Delta_r}) = \left(\frac{\partial}{\partial \bar{z}}\hat{\mu}\right)\chi_{\Delta_r} + \hat{\mu}\cdot\left(\frac{\partial}{\partial \bar{z}}\chi_{\Delta_r}\right) = -\pi\mu\,|_{\Delta_r} - \frac{1}{2i}\hat{\mu}(\zeta)\,d\zeta\,|_{\partial\Delta_r}. \tag{3}$$

REMARK. Equation (3) is understood in the distribution sense. $\mu\,|_{\Delta_r}$ denotes the restriction of μ on Δ_r.

Proof. Since

$$U_\mu(z) = \int_{\mathbf{C}} \frac{d|\mu|(\zeta)}{|\zeta - z|} \in L^1_{\mathrm{loc}}$$

and μ is a finite measure, the following conditions are satisfied for almost all $r > 0$:

$$U_\mu(z)\,|_{\partial\Delta_r} \in L^1(\partial\Delta_r, d|\zeta|); \tag{4}$$

$$|\mu|(\partial\Delta_r) = 0. \tag{5}$$

Choose $r > 0$ such that (4) and (5) hold. Take an arbitrary $\phi \in C_0^\infty$. Applying Fubini's theorem and using (5) we obtain

$$\begin{aligned}\left\langle\frac{\partial}{\partial \bar{z}}(\hat{\mu}\cdot\chi_{\Delta_r}), \phi\right\rangle &= -\left\langle\hat{\mu}\cdot\chi_{\Delta_r}, \frac{\partial\phi}{\partial\bar{z}}\right\rangle = -\int_{\Delta_r}\hat{\mu}\cdot\frac{\partial\phi}{\partial\bar{z}}\,dx\,dy \\ &= -\int_{\mathbf{C}} d\mu(\zeta)\int_{\Delta_r}\frac{\partial\phi}{\partial\bar{z}}\cdot\frac{1}{\zeta - z}\,dx\,dy \\ &= -\int_{\mathbf{C}\setminus\bar{\Delta}_r} d\mu(\zeta)\left\{\int_{\Delta_r}\frac{\partial\phi}{\partial\bar{z}}\cdot\frac{1}{\zeta-z}\,dx\,dy\right\} \\ &\quad - \int_{\Delta_r} d\mu(\zeta)\left\{\int_{\Delta_r}\frac{\partial\phi}{\partial\bar{z}}\cdot\frac{1}{\zeta-z}\,dx\,dy\right\}.\end{aligned} \tag{6}$$

In view of (2), again applying Fubini's theorem we transform the first integral in (6) into

$$\int_{\mathbf{C}\setminus\bar{\Delta}_r} d\mu(\zeta)\left\{\frac{1}{2i}\int_{\partial\Delta_r}\frac{\phi(z)}{\zeta-z}\,dz\right\} = \frac{1}{2i}\int_{\partial\Delta_r}\phi(z)\left\{\int_{\mathbf{C}\setminus\bar{\Delta}_r}\frac{d\mu(\zeta)}{\zeta-z}\right\}dz. \tag{7}$$

According to (2), the second integral in (6) can be written as

$$\begin{aligned}&\int_{\Delta_r}\left\{\pi\cdot\phi(\zeta) + \frac{1}{2i}\int_{\partial\Delta_r}\frac{\phi(z)\,dz}{\zeta-z}\right\}d\mu(\zeta) \\ &= \int_{\Delta_r}\pi\phi(\zeta)\,d\mu(\zeta) + \frac{1}{2i}\int_{\partial\Delta_r}\phi(z)\left\{\int_{\Delta_r}\frac{d\mu(\zeta)}{\zeta-z}\right\}dz.\end{aligned} \tag{8}$$

(The use of Fubini's theorem here is justified by (4).) Combining (6), (7), and (8) we obtain

$$\left\langle\frac{\partial}{\partial\bar{z}}(\hat{\mu}\cdot\chi_{\Delta_r}), \phi\right\rangle = -\int_{\Delta_r}\pi\cdot\phi(\zeta)\,d\mu(\zeta) - \frac{1}{2i}\int_{\partial\Delta_r}\phi(z)\hat{\mu}(z)\,dz,$$

and the proof is complete. □

NOTE. We have never really used the fact that Δ_r is a disk. So, the disks Δ_r in Theorem 1 can be replaced by any regions Δ with rectifiable boundaries (i.e., $(\partial/\partial\bar{z})\chi_\Delta = -(1/2i)d\zeta\,|_{\partial\Delta}$) provided that (4) and (5) hold on $\partial\Delta$.

The following corollary follows immediately from Theorem 1 and from the fact that the measure μ is uniquely defined by its Cauchy potential (see [8, Ch. II, Thm. 1.4]).

COROLLARY 1. *Let μ, r, Δ_r be the same as in Theorem* 1. *Then $\hat{\mu}\,|_{\Delta_r} = \hat{\mu}\cdot\chi_{\Delta_r}$ is the Cauchy transform of the measure*

$$\mu_1 \overset{\text{def}}{=} \mu\,|_{\Delta_r} + \frac{1}{2\pi i}\hat{\mu}(\zeta)\,d\zeta\,|_{\partial\Delta_r}. \tag{9}$$

COROLLARY 2 (Bishop–Kodama; cf. [3], [11]). *Let X be a compact set in* $\mathbf{C}$ *and $R(X)$ denote the uniform closure on X of the algebra of rational functions with poles outside of X. Let μ be a measure orthogonal to $R(X)$. Then the following statements holds:*

(i) *Fix an arbitrary $\zeta_0 \in \mathbf{C}$. For almost all $r > 0$ the measure μ_1 defined by* (9) *is orthogonal to $R(X \cap \bar{\Delta}_r)$.*

(ii) *For almost all real numbers x_0 the measure*

$$\mu_2 \overset{\text{def}}{=} \mu\,|_{\{\operatorname{Re}\zeta > x_0\}} - \frac{1}{2\pi i}\hat{\mu}(\zeta)\,d\eta\,|_{\operatorname{Re}\zeta = x_0}, \quad \zeta = \xi + i\eta,$$

is orthogonal to $R(X \cap \{\zeta : \xi = \operatorname{Re}\zeta \geq x_0\})$.

Proof. (i) According to Corollary 1, $\hat{\mu}_1 = \hat{\mu}\cdot\chi_{\Delta_r} \equiv 0$ on $\mathbf{C}\setminus\Delta_r$. Also, $\hat{\mu} \equiv 0$ on $\Delta_r\setminus X$, since $\mu \perp R(X)$. Hence, $\hat{\mu}_1 \equiv 0$ on $\Delta_r\setminus X$. Therefore, $\hat{\mu}_1 \equiv 0$ on $\mathbf{C}\setminus(X \cap \bar{\Delta}_r)$ which is equivalent to μ_1 being orthogonal to $R(X \cap \bar{\Delta}_r)$ (see [7, Ch. II]).

In view of the note following Theorem 1 the proof of (ii) is the same and we shall omit it. □

3. Estimates of measures induced by the local properties of their Cauchy potentials. Let μ be a finite compactly supported Borel measure in $\mathbf{C}$ and let, as usual, $|\mu|$ denote the total variation of μ.

DEFINITION. We call $\zeta_0 \in \operatorname{supp}\mu$ a regular point for the measure μ (a "μ-regular point") if there exists a real number $\theta : 0 \leq \theta \leq 2\pi$ such that

$$\lim_{r\to 0+} \frac{\mu(\Delta_r)}{|\mu|(\Delta_r)} = e^{i\theta},$$

where $\Delta_r = \{z : |z - \zeta_0| < r\}$.

Let $E_\mu = \{\zeta_0 \in \mathbf{C} : \zeta_0 \text{ is } \mu\text{-regular}\}$. From standard results in measure theory it follows that μ-almost all points ζ_0 belong to E_μ. So, for any Borel subset A of $\mathbf{C}$, $\mu(A) = \mu(A \cap E_\mu)$. (See [4, Ch. II, 2.9].)

We recall that for a continuous function f (i.e. $f \in C(\mathbf{C})$), the *modulus of continuity* of f $\omega(\delta, f)$ is defined to be

$$\omega(\delta, f) = \sup|f(z) - f(w)|, \quad z, w : |z - w| < \delta.$$

Also, we recall that a function $f \in L^1_{\mathrm{loc}}(dx\,dy)$ is said to belong to BMO (bounded mean oscillation) if for every disk Δ there is a constant $c = c(\Delta)$ such that

$$\iint_\Delta |f(z) - c(\Delta)|\,dx\,dy \le A_f \operatorname{area}(\Delta),$$

where A_f is a constant depending only on f.

Finally, $f \in L^1(dx\,dy)$ is said to belong to VMO (vanishing mean oscillation) if for any disk Δ there is a constant $c(\Delta)$ such that

$$\iint_\Delta |f(z) - c(\Delta)|\,dx\,dy \le \epsilon(\operatorname{area}(\Delta)) \operatorname{area}(\Delta),$$

where $\epsilon(\operatorname{area}(\Delta)) = \epsilon_f(t)$ depends only on f and $\epsilon(0+) = 0$.

THEOREM 2. *Let μ be a Borel measure in* $\mathbf{C}$ *with a compact support. Fix $\zeta_0 \in E_\mu$. Then the following hold.*

(i) *If $\|\hat{\mu}\|_{L^\infty(dx\,dy)} \le M < +\infty$, then*

$$\varlimsup_{r \to 0+} \frac{|\mu|(\Delta_r)}{r} \le C < +\infty,$$

where C is a constant which depends only on M.

(ii) *If $\hat{\mu}(z) \in C(\mathbf{C})$ and $\omega(\delta, \hat{\mu}) = \omega(\delta)$ is its modulus of continuity, then*

$$\varlimsup_{r \to 0+} \frac{|\mu|(\Delta_r)}{\omega(r) r} \le C < +\infty,$$

where C is a constant which does not depend on ζ_0.

(iii) *If $\hat{\mu}(z) \in$ BMO, then*

$$\varlimsup_{r \to 0+} \frac{|\mu|(\Delta_r)}{r} \le C < +\infty,$$

where C is a constant which does not depend on ζ_0.

(iv) *If $\hat{\mu}(z) \in$ VMO, then*

$$\varlimsup_{r \to 0+} \frac{|\mu|(\Delta_r)}{r \cdot \epsilon(r)} \le C < +\infty.$$

Here, $\epsilon(r) = \epsilon_{\hat{\mu}}$ is the same as in the definition of VMO *and C is a constant which does not depend on ζ_0.*

Proof. (i) As $\zeta_0 \in E_\mu$, we can find $r_0 = r_0(\zeta_0)$ such that for all $r < r_0$

$$|\mu|(\Delta_r) < 2|\mu(\Delta_r)|. \tag{10}$$

Since $\|\hat{\mu}\|_{L^\infty} \le M$, then for almost all $r < r_0$ the following holds:

$$\|\hat{\mu}\|_{L^\infty(\partial\Delta_r, d|\zeta|)} \le M. \tag{11}$$

Also, according to Theorem 1, for almost all $r < r_0$

$$\frac{\partial}{\partial \bar{z}}(\hat{\mu} \cdot \chi_{\Delta_r}) = -\pi\mu\,|_{\Delta_r} - \frac{1}{2i}\hat{\mu}(\zeta)\,d\zeta\,|_{\partial\Delta_r}. \tag{12}$$

From now on we assume that $r>0$ is such that (10), (11), and (12) are satisfied. Take $\phi\in C_0^\infty(C)$ such that $\phi\equiv 1$ on Δ_r and $\operatorname{supp}\phi\subset\Delta_{2r}$. From (12) we obtain

$$0=\int_{\Delta_r}\hat{\mu}\frac{\partial\phi}{\partial\bar{z}}\,dx\,dy=\pi\int_{\Delta_r}\phi\,d\mu+\frac{1}{2i}\int_{\partial\Delta_r}\phi\hat{\mu}(\zeta)\,d\zeta=\pi\mu(\Delta_r)+\frac{1}{2i}\int_{\partial\Delta_r}\hat{\mu}(\zeta)\,d\zeta.$$

From this, using (10) and (11), we obtain that

$$|\mu|(\Delta_r)\le\frac{1}{\pi}\left|\int_{\partial\Delta_r}\hat{\mu}(\zeta)\,d\zeta\right|\le 2M\cdot r. \tag{13}$$

Hence, for almost all $r<r_0$, we have $|\mu|(\Delta_r)/r\le 2M$. At the same time, for any $r<r_0/2$, $\exists r'\colon r<r'<2r<r_0$ and such that (10)–(12) hold for r'. Then we have

$$\frac{|\mu|(\Delta_r)}{r}\le\frac{|\mu|(\Delta_{r'})}{r'}\cdot\frac{r'}{r}\le 4M. \tag{*}$$

Thus,

$$\varlimsup_{r\to 0+}\frac{|\mu|(\Delta_r)}{r}\le 4M,$$

and (i) is proved.

The proof of (ii) can be obtained in a similar way, if in the estimate (13) we observe that

$$|\mu|(\Delta_r)\le\frac{1}{\pi}\left|\int_{\partial\Delta_r}\hat{\mu}(\zeta)\,d\zeta\right|=\frac{1}{\pi}\left|\int_{\partial\Delta_r}\{\hat{\mu}(\zeta)-\hat{\mu}(\zeta_0)\}\,d\zeta\right|\le 2r\omega(r).$$

(iii) Consider $r>0$ such that (10) and (12) are satisfied. Fix $\rho>0$; then, for almost all $r<\rho$ we have

$$\begin{aligned}|\mu|(\Delta_r)&\le\frac{1}{\pi}\left|\int_{\partial\Delta_r}\hat{\mu}(\zeta)\,d\zeta\right|=\frac{1}{\pi}\left|\int_{\partial\Delta_r}(\hat{\mu}(\zeta)-c(\Delta_\rho))\,d\zeta\right|\\&\le\frac{1}{\pi}\int_0^{2\pi}|\hat{\mu}(re^{i\theta})-c(\Delta_\rho)|r\,d\theta.\end{aligned}$$

Therefore, since $\hat{\mu}\in\mathrm{BMO}$, we have

$$\begin{aligned}\int_0^\rho|\mu|(\Delta_r)\,dr&\le\frac{1}{\pi}\int_0^\rho\int_0^{2\pi}|\hat{\mu}(re^{i\theta})-c(\Delta_\rho)|r\,d\theta\,dr\\&=\frac{1}{\pi}\iint_{\Delta_\rho}|\hat{\mu}(\zeta)-c(\Delta_\rho)|\,dx\,dy\\&\le A_{\hat{\mu}}\rho^2.\end{aligned}$$

Hence,

$$\int_{\rho/2}^\rho|\mu|(\Delta_r)\,dr\le A_{\hat{\mu}}\rho^2.$$

In particular, $\exists r_\rho\colon\rho/2<r_\rho<\rho$ such that

$$|\mu|(\Delta_{r_\rho})\le 4A_{\hat{\mu}}r_\rho, \tag{**}$$

and this is true for an arbitrary $\rho > 0$. Letting ρ tend to zero we obtain

$$(***)\qquad \frac{|\mu|(\Delta_\rho)}{\rho} \le \frac{|\mu|(\Delta_{r_{2\rho}})}{r_{2\rho}} \cdot \frac{r_{2\rho}}{\rho} \le 8A_{\hat{\mu}},$$

and the statement follows.

The proof of (iv) is very similar to (iii) and we shall omit it. □

COROLLARY 3. *Let μ, ζ_0, Δ_r be the same as in Theorem 2. If $\hat{\mu} \in \mathrm{Lip}(\alpha, C)$, $0 < \alpha \le 1$, then*

$$\varlimsup_{r \to 0+} \frac{|\mu|(\Delta_r)}{r^{1+\alpha}} \le \text{const} < +\infty,$$

where the constant does not depend on ζ_0.

REMARK. The reader has undoubtedly noticed that by being a little bit more careful with the estimates (*), (**), and (***) we could improve the constants in (i)–(iv) and Corollary 3, obtaining the sharper constants $\|\hat{\mu}\|_{L^\infty}$, 1, $A_{\hat{\mu}}$, 1, and $\|\hat{\mu}\|_{\mathrm{Lip}(\alpha, C)}$, respectively. Since we are here interested only in qualitative estimates of the measures, for the sake of brevity we do not pursue the sharp constants.

We recall that if $h(r) > 0$ is an increasing continuous function and $h(0) = 0$, then m_h denotes the Hausdorff measure in $\mathbf{C}$ associated with the function $h(r)$. In particular, $m_2 = m_{r^2}$ and $m_1 = m_{2r}$ denote (respectively) area measure and 1-dimensional Hausdorff measure in $\mathbf{C}$. Details concerning the definitions and properties of Hausdorff measures can be found, for instance, in [4] and [8].

COROLLARY 4. *Let μ be a compactly supported Borel measure in $\mathbf{C}$.*

(i) *If $\hat{\mu} \in \mathrm{BMO}$ (in particular, if $\hat{\mu} \in L^\infty(dm_2)$), then the measure $|\mu|$ is absolutely continuous with respect to m_1, that is, $m_1(E) = 0 \Rightarrow |\mu|(E) = 0$ for any Borel set $E \subset \mathbf{C}$. Moreover, the Radon–Nikodym derivative of $|\mu|$ with respect to m_1 is bounded.*

(ii) *If $\hat{\mu} \in \mathrm{VMO}$, then the measure $|\mu|$ is absolutely continuous with respect to m_h, where $h(r) = r\epsilon(r)$, $\epsilon(r) = \epsilon_{\hat{\mu}}(r)$. Moreover, the Radon–Nikodym derivative of $|\mu|$ with respect to m_h is bounded.*

(iii) *If $\hat{\mu} \in C(\mathbf{C})$ and $\omega(r) = \omega(r, \hat{\mu})$ is its modulus of continuity, then $|\mu|$ is absolutely continuous with respect to m_h, where $h(r) = r\omega(r)$; the Radon–Nikodym derivative of $|\mu|$ with respect to m_h is bounded.*

The proof follows from Theorem 2 by means of the standard measure-theoretic argument (cf. [4, Ch. II, §2.10.17]). For the sake of completeness we will sketch the argument for $\hat{\mu} \in L^\infty$.

Since $\mu(A) = \mu(A \cap E_\mu)$ for all Borel sets A, we need only to consider the subsets of E_μ.

Let C be the constant defined in (i) of Theorem 2. We set

$$B(\delta) = \{z \in A \colon |\mu|(A \cap \Delta_r) \le 2Cr \text{ for all } r \le \delta \text{ wherever } z \in \Delta_r\}.$$

Then, clearly $|\mu|[B(\delta)] \le 2C \sum_j r_j$ for any countable system of disks Δ_{r_j} ($r_j \le \delta$) covering $B(\delta)$. Therefore,

$$|\mu|[B(\delta)] \le 2C \inf\left\{\sum_j r_j \text{ over all } \{\Delta_{r_j}\}, \text{ such that } r_j \le \delta \text{ and } \bigcup_{j=1}^{\infty} \Delta_{r_j} \supset B(\delta)\right\}$$
$$\le Cm_1(B(\delta)) \le Cm_1(A).$$

According to Theorem 2, for every $\zeta_0 \in A \cap E_\mu$ we have ($\Delta_r = \Delta_r(\zeta_0)$):

$$\overline{\lim_{r\to 0+}} \frac{|\mu|(\Delta_r)}{r} \le C.$$

Hence, $A = \bigcup_{n=1}^{\infty} B(1/n)$. Since $B(1) \subset B(1/2) \subset \cdots$, we obtain

$$|\mu|(A) = \lim_{n\to\infty} \mu[B(1/n)] \le Cm_1(A).$$

From this the corollary follows. □

It seems appropriate to single out the following statement, which follows immediately from the above corollary.

COROLLARY 5. *Let $\mu \ne 0$ and E_μ be as above. Then the following statements hold.*

(i) *If $\hat{\mu} \in$ BMO (in particular, if $\hat{\mu} \in L^\infty(dm_2)$), then $m_1(E_\mu) > 0$.*
(ii) *If $\hat{\mu} \in$ VMO (in particular, if $\hat{\mu} \in C(\mathbf{C})$), then $m_1(E_\mu) = \infty$.*
(iii) *If $\hat{\mu} \in \mathrm{Lip}(\alpha, C)$, $0 < \alpha \le 1$, then $m_{1+\alpha} \overset{\text{def}}{=} m_{r1+\alpha}(E_\mu) > 0$. Moreover, $\mu \ll m_{1+\alpha}$ and $d\mu/dm_{1+\alpha} \in L^\infty(m_{1+\alpha}, \mathbf{C})$. In particular, if $\hat{\mu} \in \mathrm{Lip}(1, C)$, then $\mu = g\,dm_2$, $g \in L^\infty$.*

REMARK. The results stated in part (iii) of Corollary 5 are closely related to a well-known theorem (due to E. Dolženko) concerning the removable sets for analytic functions in the Lipschitz classes (see [8, Ch. III, §4] for detailed discussion). The same note applies to part (i) and the results of Kaufman and Wu concerning removable sets for analytic functions f of BMO (see [9], [10]).

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