A COVARIANT VERSION OF Ext

Vern Paulsen

1. Introduction. If H is a separable Hilbert space, and G is a topological group with a strongly continuous unitary representation on H, then G acts by conjugation on the bounded linear operators $L(H)$ and on the Calkin algebra $Q(H) = L(H)/K(H)$, where $K(H)$ denotes the compact operators. If G also acts on a C^*-algebra A, then $*$-monomorphisms of A into $Q(H)$ which are compatible with these two actions are called covariant extensions. In this paper we construct three groups out of equivalence classes of covariant extensions, when G is 2nd countable and compact, and A is separable and nuclear. Two of these groups are the analogues of the weak and strong Ext groups of [3].

A systematic study of covariant extensions and of several closely related topics has been undertaken in [4], [8], and [11]. In [4] an equivalence relation was defined on the set of covariant extensions and in the case when G is finite and $A = C(X)$, it was proved that the equivalence classes of covariant extensions together with a binary operation induced by direct sum forms an abelian group. We have been able to generalize this result somewhat, but at the expense of a slightly weaker notion of equivalence. However, we shall show that for G finite the two notions of equivalence coincide.

Section 2 contains some preliminary definitions together with a covariant version of Stinespring's Theorem [12]. In Section 3 we introduce three equivalence relations on the covariant extensions. Following the ideas of [2], we combine the covariant version of Stinespring's Theorem together with a result on the existence of covariant completely positive liftings to prove that, for each of the three equivalence relations, the collection of equivalence classes forms a group. We close Section 3 by describing some of the relationships between these three groups. Some discussion of the possibility of extending these results to noncompact G is included. In Section 4 we calculate each of these groups for the case of the circle group acting on itself by multiplication.

Our techniques and constructions are similar to those of Kasparov ([5], [6], and [7]), but the special case we are interested in, while considerably simpler, allows for some additional structure. In particular, our theory admits a natural action of the representation ring of G, while this does not appear to be possible in Kasparov's theory. It is this action that facilitates the calculations of Section 4. In addition, we feel that our construction is more amenable to generalizations to non-compact G. We discuss the relationships between our theory and Kasparov's in Section 5.

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2. Preliminaries. Let G be a topological group, A a C^*-algebra, and $\mathrm{Aut}(A)$ the group of automorphisms of A. We shall call a homomorphism $\alpha: G \to \mathrm{Aut}(A)$ an *action* of G on A and call A a *G-C^*-algebra.* If $g \to \alpha(g)(a)$ is continuous for all a in A, then following [9], we shall call (A, G, α) a *C^*-dynamical system.* If $\rho: G \to A$ is a homorphism of G into the unitary elements of A, then setting $\alpha(g)(a) = \rho(g)a\rho(g^{-1})$ defines an action of G on A, which we denote by $ad(\rho)$. If A and B are G-C^*-algebras with actions α and β, respectively, and $\varphi: A \to B$ is any map, then we say φ is *covariant* if $\varphi(\alpha(g)(a)) = \beta(g)(\varphi(a))$ for all $a \in A$ and $g \in G$.

The following is a spatial version of [6, Theorem 3]. The hypothesis that G be compact can be omitted in this version.

THEOREM 2.1. (Covariant Version of Stinespring) *Let (A, G, α) be a unital C^*-dynamical system, and let ρ be a strongly continuous unitary representation of G on a Hilbert space H. If $\psi: A \to L(H)$ is a unital covariant, completely positive map, then there exists:*

i) *a Hilbert space K,*

ii) *a representation π of A on K,*

iii) *a strongly continuous unitary representation $\tilde{\rho}$ of G on K,*

iv) *an isometry $V: H \to K$*

such that

1) $\psi(a) = V^* \pi(a) V$,

2) *$V(H)$ reduces $\tilde{\rho}$ and $\rho(g) = V^* \tilde{\rho}(g) V$,*

3) *π is covariant with respect to $ad(\tilde{\rho})$.*

Furthermore, if A and H are separable, then K may be taken to be separable.

Proof. Statements i), ii), and iv) are part of the standard version of Stinespring's Theorem. It will be sufficient to include in the construction of K a representation $\tilde{\rho}$. We only sketch the construction; the details are routine.

Recalling the proof of Stinespring's Theorem from [1], one first forms the algebraic tensor product, $A \otimes H$, and endows it with a pre-inner product by setting $\langle a \otimes v, b \otimes w \rangle_{A \otimes H} = \langle \psi(b^* a) v, w \rangle_H$ and extending linearly. To obtain K one divides by the kernel of $\langle\ ,\ \rangle_{A \otimes H}$ and completes. The representation π of A is defined by $\pi(a)(b \otimes w) = (ab) \otimes w$. The isometry $V: H \to K$ is defined by $V(w) = 1_A \otimes w$.

We define $\tilde{\rho}: G \to L(K)$, by setting $\tilde{\rho}(g)(a \otimes v) = (\alpha(g)a) \otimes (\rho(g)v)$ and extending linearly to $A \otimes H$. Since

$$\begin{aligned} \langle \tilde{\rho}(g)(a \otimes v), \tilde{\rho}(g)(b \otimes w) \rangle_{A \otimes H} &= \langle (\alpha(g)a) \otimes (\rho(g)v), (\alpha(g)b) \otimes (\rho(g)w) \rangle_{A \otimes H} \\ &= \langle \psi(\alpha(g)(b^* a)) \rho(g) v, \rho(g) w \rangle_H \\ &= \langle \psi(b^* a) v, w \rangle_H = \langle a \otimes v, b \otimes w \rangle_H, \end{aligned}$$

we have that $\tilde{\rho}(g)$ extends to an isometry on K. Furthermore, since $\tilde{\rho}$ is clearly a group homomorphism, we have that $\tilde{\rho}(g)$ is unitary.

A similar calculation, using the norm continuity of $\alpha(g)(a)$ and the strong continuity of ρ shows that $\tilde{\rho}$ is strongly continuous on finite sums of elementary tensors and then the fact that $\|\tilde{\rho}(g)\| \leq 1$ allows one to pass to limits. Finally, we note that $\rho(g) = V^*\tilde{\rho}(g)V$, from which it follows that $V(H)$ reduces $\tilde{\rho}$, since $\rho(g)$ and $\tilde{\rho}(g)$ are unitary, and that

$$\pi(\alpha(g)a)(b\otimes w) = ((\alpha(g)a)b)\otimes w = \tilde{\rho}(g)\pi(a)\tilde{\rho}(g^{-1})(b\otimes w).$$

This completes the proof of the theorem. □

3. The main results. Throughout this section we shall assume that (A, G, α) is a covariance system, with G a 2nd countable, compact group and A a separable, nuclear, unital C^*-algebra. For T in $L(H)$, we shall let $\dot{T}$ denote its image in the Calkin algebra, $Q(H) = L(H)/K(H)$.

By a *covariant extension* we mean a pair (τ, ρ) where τ is a unital *-monomorphism of A into Q and ρ is a strongly continuous unitary representation of G on H, such that τ and $ad(\dot{\rho})$ are covariant. We define a covariant extension to be *split*, if there is a unital *-monomorphism θ of A into $L(H)$ with $\dot{\theta} = \tau$ such that θ and $ad(\rho)$ are covariant. Two covariant extensions (τ, ρ) and (τ', ρ') are *rigidly equivalent* (denoted: $(\tau, \rho) \sim_r (\tau', \rho')$), if there exists a unitary U in $L(H)$ with $\dot{U}^*\tau(a)\dot{U} = \tau'(a)$ and $U^*\rho(g)U = \rho'(g)$ for every $a \in A$ and $g \in G$. We shall call two covariant extensions *strongly equivalent* $((\tau, \rho) \sim_s (\tau', \rho'))$, if there exists a unitary U in $L(H)$ with $\dot{U}^*\tau(a)\dot{U} = \tau'(a)$ and $\dot{U}^*\dot{\rho}(g)\dot{U} = \dot{\rho}'(g)$ for every $a \in A$ and $g \in G$. If U is only required to be unitary in $Q(H)$ in the above relationships then we shall call (τ, ρ) and (τ', ρ') *weakly equivalent* $((\tau, \rho) \sim_w (\tau', \rho'))$. Finally, two covariant extensions (τ, ρ) and (τ', ρ') are (respectively, *rigidly, strongly, weakly*) *stably equivalent* if there exist split extensions (φ, σ) and (φ', σ') such that $(\tau\oplus\varphi, \rho\oplus\sigma)$ and $(\tau'\oplus\varphi', \rho'\oplus\sigma')$ are respectively, rigidly, strongly, or weakly equivalent. We shall use $(\tau, \rho) \approx_r (\tau', \rho')$, $(\tau, \rho) \approx_s (\tau', \rho')$, and $(\tau, \rho) \approx_w (\tau', \rho')$ to denote, respectively, rigid, strong, and weak, stable equivalence.

We note that all six of the above relationships are, indeed, equivalence relations and that the rigid, strong, and weak (respectively: stably rigid, stably strong, stably weak) are linearly ordered from finest to coarsest. In addition, all split extensions are equivalent in all three of the stable relations. We denote the stable equivalence class of (τ, ρ) by $[\tau, \rho]_r$, $[\tau, \rho]_s$, or $[\tau, \rho]_w$, respectively, and the collection of stable equivalence classes by $\mathrm{Ext}_G^r(A)$, $\mathrm{Ext}_G^s(A)$, or $\mathrm{Ext}_G^w(A)$. We shall frequently omit the subscripts, or superscripts, and refer only to equivalence or stable equivalence when the particular equivalence relation is unimportant.

REMARK 3.1. The set $\mathrm{Ext}_G(A)$ is non-empty; that is, there always exists a split covariant extension. To see this let θ be a faithful representation of A on a separable Hilbert space, H, with $\theta(A) \cap K(H) = (0)$, and define a faithful representation Θ of A on $L^2(G, H)$ by $(\Theta(a)\Phi)(g) = \theta(\alpha(g^{-1})a)\Phi(g)$. Let ρ be the strongly continuous unitary representation of G on $L^2(G, H)$ given by left translation, i.e., $(\rho(g)\Phi)(h) = \Phi(g^{-1}h)$. Then since Θ and ρ are covariant and since $\Theta(A)$ misses the compact operators on $L^2(G, H)$ we have $(\dot{\Theta}, \rho)$ is a covariant extension. Note that we used the 2nd countability of G to insure that $L^2(G, H)$ was separable.

REMARK 3.2. We note that in the above construction of the covariant extension $(\dot{\Theta}, \rho)$, every irreducible representation of G occurs as a subrepresentation of ρ with infinite multiplicity. If (τ, σ) is any covariant extension, then $(\tau, \sigma) \approx (\tau \oplus \dot{\theta}, \sigma \oplus \rho)$ as can be seen by direct summing the left- and right-hand sides with the split extensions $(\dot{\theta} \oplus \dot{\theta}, \rho \oplus \rho)$ and $(\dot{\theta}, \rho)$, respectively. Thus, every covariant extension is stably equivalent to a covariant extension (τ, ρ_u) where every irreducible representation of G occurs as a subrepresentation of ρ_u of infinite multiplicity.

We define a binary operation on $\mathrm{Ext}_G(A)$ by $[\tau, \rho] + [\tau', \rho'] = [\tau \oplus \tau', \rho \oplus \rho']$, where the usual identification of $H \oplus H$ with H has been made. The following is immediate:

PROPOSITION 3.3. $\mathrm{Ext}_G^r(A)$, $\mathrm{Ext}_G^s(A)$, *and* $\mathrm{Ext}_G^w(A)$ *are abelian semigroups with identity. The identity is, respectively, the rigid, strong, or weak stable equivalence class of the split extensions. Furthermore the maps* $i_{r,s} : [\tau, \rho]_r \to [\tau, \rho]_s$; $i_{r,w} : [\tau, \rho]_r \to [\tau, \rho]_w$, *and* $i_{s,w} : [\tau, \rho]_s \to [\tau, \rho]_w$ *define semigroup homomorphisms which are onto, with* $i_{s,w} \circ i_{r,s} = i_{r,w}$.

We note that if (τ, ρ) is a covariant extension with a faithful representation θ of A and a strongly continuous unitary representation σ of G such that θ and σ are covariant with $\dot{\theta} = \tau$, $\dot{\sigma} = \dot{\rho}$, then (τ, ρ) is strongly equivalent to a split extension.

We shall occasionally need to integrate strongly continuous operator valued functions. We record the following observation for future reference.

LEMMA 3.4. *Let X be a compact topological space, μ a finite measure on X, and let $T(x)$ be a strongly continuous operator valued function. If $T(x)$ is a compact operator for all x, then $\int T(x)\, d\mu$ is a compact operator.*

Proof. Let P_n be a sequence of finite rank projections converging strongly to 1 and let $K_n = \int P_n T(x) P_n \, d\mu$, then K_n is finite rank. Since $\sup_{x \in X} \| T(x) \| < +\infty$ by the uniform boundedness principle and the fact that X is compact, we have

$$\|K - K_n\| \leqslant \int \| T(x) - P_n T(x) P_n \| \, d\mu \to 0,$$

as $n \to \infty$ by the Dominated Convergence Theorem and since

$$\| T(x) - P_n T(x) P_n \| \to 0$$

pointwise. Hence K is compact. □

We are now in a position to proceed with the proof that $\mathrm{Ext}_G(A)$ is a group. We begin with a lemma.

LEMMA 3.5. (Existence of Covariant Completely Positive Liftings) *Let (A, G, α) be a C^*-dynamical system with G compact and A separable and nuclear with unit. If (τ, ρ) is a covariant extension of A, then there exists a unital completely positive map $\psi : A \to L(H)$ with $\dot{\psi} = \tau$ such that ψ and $ad(\rho)$ are covariant.*

Proof. Since A is nuclear, there exists a unital completely positive lifting φ of τ. We set

$$\psi(a) = \int \rho(g)^* \varphi(\alpha(g) a) \rho(g) \, dg,$$

where dg is Haar measure on G.

It is easily checked that ψ is unital and completely positive. Furthermore, $\rho(g)^*\varphi(\alpha(g)a)\rho(g)-\varphi(a)$ is strongly continuous and since (τ,ρ) is a covariant extension it assumes compact values. Hence, by Lemma 3.4, $\dot{\psi}=\dot{\varphi}=\tau$.

Finally, ψ is covariant with respect to $ad(\rho)$, since for $h\in G$,

$$\begin{aligned}\rho(h)\psi(a)\rho(h^{-1})^* &= \int\rho(hg^{-1})\varphi(\alpha(g)a)\rho(gh^{-1})\,dg\\ &= \int\rho(g')\varphi(\alpha(g'h)a)\rho(g')\,dg' = \psi(\alpha(h)a),\end{aligned}$$

where $g'=gh^{-1}$. □

THEOREM 3.6. *Let (A,G,α) be a C^*-dynamical system with G compact and 2nd countable and with A separable and nuclear with unit, then* $\mathrm{Ext}_G^r(A)$, $\mathrm{Ext}_G^s(A)$, *and* $\mathrm{Ext}_G^w(A)$ *are groups.*

Proof. By Proposition 3.1 it will be sufficient to show that $\mathrm{Ext}_G^r(A)$ is a group. Let (τ,ρ) be a covariant extension, again by Proposition 3.1 it will be sufficient to construct a covariant extension (τ',ρ') such that $(\tau\oplus\tau',\ \rho\oplus\rho')$ is rigidly stably equivalent to a split extension. To this end let $\psi:A\to L(H)$ be a completely positive lifting of τ which is covariant with respect to $ad(\rho)$. Applying the covariant version of Stinespring, we have a representation $\theta:A\to L(K)$, with K separable and a strongly continuous unitary representation $\tilde{\rho}:G\to L(K)$. If we let P denote the projection onto $V(H)^{\perp}$, where $V:H\to K$ is the isometry guaranteed by Proposition 2.1, then as noted in [2], P essentially reduces θ. Furthermore, by Proposition 2.1, P reduces $\tilde{\rho}$. Thus, we may define (τ',ρ') by $\tau'(a)=\dot{P}\theta(\dot{a})\dot{P}\oplus\tau_0(a)$ and $\rho'(g)=\tilde{\rho}(g)|_{PK}\oplus\rho_0(g)$ where (τ_0,ρ_0) is a split extension. It is straightforward to check that $(\tau\oplus\tau',\ \rho\oplus\rho')$ is split. □

REMARK 3.7. We note that the proof of the theorem shows that even if G is non-separable but compact, then the existence of any covariant extension (τ,ρ) insures the existence of a split extension (on a separable Hilbert space) and hence an identity for $\mathrm{Ext}_G^r(A)$. Thus, even for non-separable, compact G, if there exist any covariant extensions at all, then the sets $\mathrm{Ext}_G^r(A)$, $\mathrm{Ext}_G^s(A)$, and $\mathrm{Ext}_G^w(A)$ are groups.

REMARK 3.8. Let (A,G,α) be a C^*-dynamical system with G locally compact and 2nd countable, and A separable and nuclear with unit. Remark 3.1 combined with the proof of Theorem 3.6 actually shows that the set of stable equivalence classes of covariant extensions, that have covariant, completely positive liftings, forms a group. We do not know whether or not all covariant extensions enjoy this lifting property, or how to characterize those that do.

We shall now partially describe the relationship between the groups $\mathrm{Ext}_G^r(A)$, $\mathrm{Ext}_G^s(A)$ and $\mathrm{Ext}_G^w(A)$. We obtain a generalization of the fact that Ext^w is a quotient of Ext^s by an action of Z. Recall the definition of ind_G from [8]. If ρ is a strongly continuous unitary representation of G on H, then an operator T is called G-Fredholm if T is Fredholm and $\dot{T}$ is fixed by $ad(\dot{\rho})$. In this case $T=S+K$ where S is fixed by $ad(\rho)$ and K is compact [8, Proposition 3.1]. Since S is fixed, $\ker(S)$ and

$\ker(S^*)$ are G-invariant subspaces, and one defines

$$\mathrm{ind}_G(S) = [\ker S] - [\ker S^*] \in R(G)$$

where $R(G)$ is the representation ring of G. One sets $\mathrm{ind}_G(T) = \mathrm{ind}_G(S)$ and this definition is independent of the particular fixed operator S. If every irreducible representation of G appears infinitely often, then ind_G maps onto $R(G)$, [8, Theorem 5.1].

Let (τ, ρ) be a split extension of A, with every irreducible representation of G appearing infinitely often. Let $m \in R(G)$ and let U be G-Fredholm, and essentially unitary with $\mathrm{ind}_G(U) = m$. Note that $(\dot{U}^*\tau\dot{U}, \rho)$ defines another covariant extension. We claim that the rigid equivalence class of this extension is independent of the choice of U. That is, if V is G-Fredholm and essentially unitary with $\mathrm{ind}_G(V) = m$, then $(\dot{V}^*\tau\dot{V}, \rho) \sim_r (\dot{U}^*\tau\dot{U}, \rho)$. To see this, note that $\mathrm{ind}_G(V^*U) = 0$ and so by [8, Theorem 3.2] there exists a fixed unitary W with $W - V^*U$ compact. Hence, $\dot{W}(\dot{U}^*\tau\dot{U})\dot{W}^* = \dot{V}^*\tau\dot{V}$ and $W\rho W^* = \rho$. A similar check shows that the rigid stable equivalence class of $(\dot{U}^*\tau\dot{U}, \rho)$ is independent of the split extension (τ, ρ), and we denote it by $[\tau, \rho]_m$.

In what follows we let $R_0(G)$ denote the subgroup of $R(G)$ consisting of formal differences of G-modules, $m_1 - m_2$ for which $\dim(m_1) = \dim(m_2)$.

THEOREM 3.9. *The following sequences are exact:*

$$R(G) \to \mathrm{Ext}_G^r(A) \xrightarrow{i_{r,w}} \mathrm{Ext}_G^w(A) \to 0,$$

and

$$R_0(G) \to \mathrm{Ext}_G^r(A) \xrightarrow{i_{r,s}} \mathrm{Ext}_G^s(A) \to 0.$$

Proof. First we show that $m \to [\tau, \rho]_m$ is a homomorphism. Let $m_1, m_2 \in R(G)$, and let U_1, U_2 be G-Fredholm, essential unitaries with $\mathrm{ind}_G(U_i) = m_i$, $i = 1, 2$. We have $(\dot{U}_1^*\dot{U}_2^*\tau\dot{U}_2\dot{U}_1, \rho) \approx_r (\dot{U}_1^*\dot{U}_2^*\tau\dot{U}_2\dot{U}_1 \oplus \tau, \rho \oplus \rho)$. Since $\mathrm{ind}_G(U_1^* \oplus U_1) = 0$, by [8, Theorem 3.2] there is a unitary V which commutes with $\rho + \rho$, $\mathrm{ind}_G(V) = 0$, and $U_1^* \oplus U_1 - V$ is compact. Thus, $(\dot{U}_1^*\dot{U}_2^*\tau\dot{U}_2\dot{U}_1 \oplus \tau, \rho \oplus \rho) \sim_r (\dot{V}^*(\dot{U}_1^*\dot{U}_2^*\tau\dot{U}_2\dot{U}_1 \oplus \tau)\dot{V}, \dot{V}^*(\rho \oplus \rho)V) = (\dot{U}_2^*\tau\dot{U}_2 \oplus \dot{U}_1^*\tau\dot{U}_1, \rho \oplus \rho)$, and so $[\tau, \rho]_{m_1+m_2} = [\tau, \rho]_{m_1} + [\tau, \rho]_{m_2}$. Hence, $m \to [\tau, \rho]_m$ is a homomorphism.

Next, let $[\tau', \rho']_r \in \mathrm{Ext}_G^r(A)$ be an element in the kernel of $i_{r,w}$. That is, $(\tau', \rho') \approx_w (\tau, \rho)$, where (τ, ρ) is split, and ρ contains every irreducible representation with infinite multiplicity. Let (γ, σ) and (γ', σ') be split extensions such that $(\tau' \oplus \gamma', \rho' \oplus \sigma') \sim_w (\tau \oplus \gamma, \rho \oplus \sigma)$. Without loss of generality, we may assume that $\rho' \oplus \sigma' = \rho \oplus \sigma$. Thus, since we have an essential unitary U with $\dot{U}^*(\tau \oplus \gamma)\dot{U} = \tau' \oplus \gamma'$ and $\dot{U}^*(\dot{\rho} \oplus \dot{\sigma})\dot{U} = \dot{\rho} \oplus \dot{\sigma}$, we see that U is also G-Fredholm and if $m = \mathrm{ind}_G(U)$, then $[\tau', \rho'] = i_{r,w}([\tau \oplus \gamma, \rho \oplus \sigma]_m)$. Hence, the top sequence is exact.

The proof that the second sequence is exact is identical, once one observes that $m \in R_0(G)$ and $\mathrm{ind}_G(U) = m$ implies $\mathrm{ind}(U) = 0$. □

We close this section with a result relating the covariant Ext groups to the ordinary Ext groups in the case where G is finite. We let $G \times_\alpha A$ denote the crossed product (or covariance algebra) [6] of (A, G, α). We recall that $G\times_\alpha A$ is the C^*-completion of the continuous functions from G to A with respect to a certain involution and product. For finite G, the map $g \to \delta_g$, where δ_g is the characteristic function of $\{g\}$ times the identity of A, defines a homomorphism from G into the unitaries in $G\times_\alpha A$. Also, for a in A, if we set $\tilde{a} = \delta_e \cdot a$, then $a \to \tilde{a}$ defines a *-monomorphism of A into $G\times_\alpha A$, and $G\times_\alpha A$ is the C^*-algebra generated by the set of δ_g's and $\tilde{a}$'s. Further, if ρ is a homomorphism from G into the unitary group of some C^*-algebra B, and θ is a *-homomorphism of A into B such that θ and $ad(\rho)$ are covariant, then there is a *-homomorphism of $G\times_\alpha A$ into B given by $\delta_g \to \rho(g)$, $\tilde{a} \to \theta(a)$. Finally, if $C^*(G)$ denotes the subalgebra of $G\times_\alpha A$ generated by the image of G, then since G is finite $C^*(G)$ will be finite dimensional. The following result is essentially contained in [4].

PROPOSITION 3.10. *Let G be a finite group, then* $\mathrm{Ext}^w(G\times_\alpha A)$ *and* $\mathrm{Ext}^w_G(A)$ *are isomorphic. In addition, there is an exact sequence*

$$0 \to \mathrm{Ext}^s_G(A) \to \mathrm{Ext}^s(G\times_\alpha A) \xrightarrow{i^*} \mathrm{Ext}^s(C^*(G)) \to 0,$$

where $i: C^*(G) \to G\times_\alpha A$ denotes inclusion.

Proof. We begin by proving the second statement. Given a covariant extension (τ, ρ), by the above observations there is a *-homomorphism of $G\times_\alpha A$ into $Q(H)$ given by $\delta_g \to \dot{\rho}(g)$, $\tilde{a} \to \tau(a)$. By direct summing (τ, ρ) with a split extension, we can make the above map into a *-monomorphism without changing the stable equivalence class of (τ, ρ). A moment's reflection shows that given two covariant extensions (τ, ρ) and (τ', ρ') the *-monomorphisms of $G\times_\alpha A$ obtained in this fashion will be strongly equivalent if and only if (τ, ρ) and (τ', ρ') are strongly stably equivalent. This shows exactness at $\mathrm{Ext}^s_G(A)$.

If $[\sigma] \in \mathrm{Ext}^s(G\times_\alpha A)$ with $i^*[\sigma] = 0$, then we may define a covariant extension (τ, ρ) by setting $\tau(a) = \sigma(\tilde{a})$, and letting $\rho(g) = \varphi(\delta_g)$ where φ is a lifting of σ restricted to $C^*(G)$. Clearly, σ will be the *-monomorphism associated with (τ, ρ). This shows exactness at $\mathrm{Ext}^s(G\times_\alpha A)$.

Finally, to see that i^* is onto, we note that since $C^*(G)$ is finite dimensional, $\mathrm{Ext}^w(C^*(G)) = 0$. Thus, every element of $\mathrm{Ext}^s(C^*(G))$ is weakly equivalent to the trivial element. Hence, by conjugating the trivial element of $\mathrm{Ext}^s(G\times_\alpha A)$ by essential unitaries of non-zero index, one obtains elements of $\mathrm{Ext}^s(G\times_\alpha A)$ which restrict to any element of $\mathrm{Ext}^s(C^*(G))$.

To obtain the isomorphism between $\mathrm{Ext}^w_G(A)$ and $\mathrm{Ext}^w(G\times_\alpha A)$, one repeats the above argument with strong equivalence replaced by weak equivalence and observes that since $\mathrm{Ext}^w(C^*(G)) = 0$ one actually obtains an isomorphism. □

In the next section we give an example of a non-discrete group for which $\mathrm{Ext}^w_G(A)$ and $\mathrm{Ext}^w(G\times_\alpha A)$ are not isomorphic.

4. The circle group. Let T denote the circle group. We define an action of T on $C(T)$ by $(\alpha(\lambda)f)(z) = f(\lambda z)$, so that $(C(T), T, \alpha)$ is a C^*-dynamical system. In this section we shall prove that

$$\mathrm{Ext}_T^r(C(T)) \simeq \sum_{n=-\infty}^{+\infty} \oplus Z,$$

$$\mathrm{Ext}_T^s(C(T)) \simeq Z \oplus Z, \quad \text{and} \quad \mathrm{Ext}_T^w(C(T)) \simeq Z,$$

where Z denotes the integers and the direct sum denotes the collection of infinite-tuples of integers which are 0 in all but finitely many entries. For convenience we write an element of the direct sum as $\sum n_j e_j$ where e_j denotes the vector which is 1 in the jth coordinate and 0 elsewhere. In addition, one has

$$\mathrm{Ext}^s(T \times_\alpha C(T)) \simeq \mathrm{Ext}^w(T \times_\alpha C(T)) \simeq (0).$$

We begin by making a few observations. First, given a covariant extension (τ, ρ), by Remark 3.2 we may assume that $\rho = \rho_\mu$ without changing the stable equivalence class of (τ, ρ). For the circle group, ρ_μ is the strongly continuous unitary representation on $\sum_{n=-\infty}^{+\infty} \oplus H_n$ by $\rho_\mu(\lambda) = \sum_{n=-\infty}^{+\infty} \oplus \lambda^n 1_{H_n}$ where each $H_n = H$ and $\dim(H) = +\infty$. Let ψ be a completely positive covariant lifting of τ, and let Z denote the coordinate function on T. A simple check shows that the doubly infinite operator matrix, $\psi(Z) = (V_{i,j})_{i,j=-\infty}^{+\infty}$ must satisfy $V_{i,j} = 0$ when $i - j \neq 1$. For convenience we set $V_i = V_{i,i-1}$. Since $\psi(Z)$ is essentially unitary, we have that $V_i^* V_i - 1$, $V_i V_i^* - 1$ are compact for all i, and that as $|i| \to +\infty$, their norms tend to 0. Using the polar decomposition, we write $V_i = U_i P_i$, then each U_i is also essentially unitary, $V_i - U_i$ is compact, and as $|i| \to +\infty$, it tends in norm to 0. Furthermore, if $\| V_i^* V_i - 1 \| < 1$ and $\| V_i V_i^* - 1 \| < 1$, then U_i is unitary. We set $U = (U_{i,j})_{i,j=-\infty}^{+\infty}$ with $U_{i,j} = 0$ when $i - j \neq 1$, and $U_{i,i-1} = U_i$. By the above we have that $\psi(Z) - U$ is compact, U is essentially unitary, and there is an N such that for $|n| > N$, U_n is unitary.

Now given two covariant extensions (τ, ρ_μ) and (τ', ρ_μ), let U and U', respectively, be the essential unitaries associated with them by the above construction. We claim $(\tau, \rho_\mu) \approx_r (\tau', \rho_\mu)$ if and only if $\mathrm{ind}(U_i) = \mathrm{ind}(U_i')$ for all i. Note that if (γ, ρ_μ) is split and W is the associated essential unitary, then necessarily $\mathrm{ind}(W_i) = 0$ for all i. Thus to prove the above claim, it will suffice to prove that $(\tau, \rho_\mu) \sim_r (\tau', \rho_\mu)$ if and only if $\mathrm{ind}(U_i) = \mathrm{ind}(U_i')$ for all i.

So assume $(\tau, \rho_\mu) \sim_r (\tau', \rho_\mu)$. Thus, there is a unitary W, which commutes with ρ_μ such that $W^* U W - U'$ is compact. Since W commutes with ρ_μ, we have that $W = (W_{i,j})_{i,j=-\infty}^{+\infty}$ with $W_{i,j} = 0$ when $i \neq j$. Setting $W_i = W_{i,i}$, we need $W_i^* U_i W_{i-1} - U_i'$ to be compact and tend in norm to 0 as $|i| \to +\infty$. Since each W_i is unitary we have $\mathrm{ind}(U_i) = \mathrm{ind}(U_i')$ for all i.

Conversely, assume $\mathrm{ind}(U_i) = \mathrm{ind}(U_i')$ for all i. We know that there is an N such that for $|n| > N$, U_n and U_n' are unitary. We define W as follows.

Let $W_0 = 1$. Assume W_{i-1} is defined and unitary for $0 < i \leqslant N$. Since $\mathrm{ind}(U_i) = \mathrm{ind}(U_i')$, $\mathrm{ind}(U_i W_{i-1} U_i'^*) = 0$, and thus there is a compact operator K_i such that $U_i W_{i-1} U_i'^* + K_i = W_i$ is a unitary. For $i > N$, let $W_i = U_i W_{i-1} U_i'^*$. Similarly, for $N \leqslant i \leqslant 0$, we set $W_{i-1} = U_i^* W_i U_i' + K_{i-1}$, where K_{i-1} is compact and chosen to make W_{i-1} unitary. For $i \leqslant -N-1$, we simply set $W_{i-1} = U_i^* W_i U_i'$. Let $W = \sum_{i=-\infty}^{+\infty} \oplus W_i$.

Since $W_i^* U_i W_{i-1} - U_i'$ is compact for $-N-1 < i < N$ and is 0 for all other values of i, we have $W^* U W - U'$ is compact, where W is a unitary that commutes with ρ_μ. Thus, $(\tau, \rho_\mu) \sim_r (\tau', \rho_\mu)$, which completes the proof of the claim.

The above claim shows that we have a well defined, one-to-one homomorphism,

$$r: \mathrm{Ext}_T^r(C(T)) \to \sum_{-\infty}^{+\infty} \oplus Z,$$

given by $r([\tau, \rho_\mu]_r) = \Sigma_{-\infty}^{+\infty} \operatorname{ind}(U_j)e_j$, where the essential unitaries U_j are as above. Furthermore, this map is onto since given any sequence $\Sigma_{-\infty}^{+\infty} n_j e_j$, we may define $U = (U_{i,j})_{i,j=-\infty}^{+\infty}$ by setting $U_{i,j} = 0$ if $i-j \neq 1$ and letting $U_{i,i-1}$ be an essential unitary of index n_i. We then have a covariant extension (τ, ρ_μ) given by $\tau(Z) = \dot{U}$ and $r([\tau, \rho_\mu]_r) = \Sigma n_j e_j$.

REMARK 4.1. We remark that even though the U defined above is not in the usual domain of the T-index map, since $\rho_\mu(\lambda) U \rho_\mu(\lambda^{-1}) = \lambda U$, we have that $\ker(U)$ and $\operatorname{coker}(U)$ are T-modules. Thus, we obtain an element of $R(T)$ by setting $\operatorname{ind}_G(U) = [\ker U] - [\operatorname{coker} U]$. If we let $\hat{j}(\lambda) = \lambda^j$, then we may identify $R(T)$ with $\Sigma \oplus Z$ via $\Sigma n_j \hat{j} \to \Sigma n_j e_j$. We see that with these identifications, if U is the essential unitary associated with $[\tau, \rho_\mu]$, then $r([\tau, \rho_\mu]) = \operatorname{ind}_T(U)$.

To prove that $\mathrm{Ext}_T^s(C(T)) \simeq Z \oplus Z$ we shall make use of Theorem 3.9. Let (τ, ρ_μ) be the split extension defined by $\tau(Z) = \dot{B}$, where B denotes the block bilateral shift, i.e., $B = (B_{i,j})$ with $B_{i,j} = 0$ if $i - j \neq 1$ and $B_{i,i-1} = 1_H$. Let $\Sigma n_j \hat{j}$ be an arbitrary element of $R_0(T)$, so that $\Sigma n_j = 0$, and let W be an essentially T-Fredholm unitary with $\operatorname{ind}_T(W) = \Sigma n_j \hat{j}$. Thus, we may set $W = \Sigma \oplus W_j$ where each W_j is essentially unitary with $\operatorname{ind}(W_j) = n_j$, and when $n_j = 0$, W_j is unitary. We have $W^* B W = (U_{i,j})$, where $U_{i,j} = 0$, if $i - j \neq 1$ and $U_i = U_{i,i-1}$ is an essential unitary of index $n_{i-1} - n_i$. Thus, the exact sequence of Theorem 3.9 becomes

$$R_0(T) \xrightarrow{\varphi} \Sigma \oplus Z \to \mathrm{Ext}_T^s(C(T)) \to 0$$

with $\varphi(\Sigma n_j \hat{j}) = \Sigma (n_{j-1} - n_j) e_j$. To identify $\mathrm{Ext}_T^s(C(T))$ we need only compute the cokernel of this map. To this end, we define $h: \Sigma \oplus Z \to Z \oplus Z$ by $h(\Sigma m_j e_j) = (\Sigma n_j, \Sigma j n_j)$. We claim that h is onto and that $\ker(h) = \operatorname{im}(\varphi)$. First, $h(e_0) = (1,0)$ and $h(e_1 - e_0) = (0,1)$, so that h is onto. Next, note that

$$h\varphi\left(\Sigma n_j j\right) = h\left(\Sigma (n_{j-1} - n_j) e_j\right) = \left(\sum_j (n_{j-1} - n_j),\ \Sigma j (n_{j-1} - n_j)\right) = (0,0).$$

Thus, $\operatorname{im}(\varphi) \subseteq \ker(h)$. Let $\Sigma m_j e_j \in \ker(h)$, and say $m_j = 0$ for $|j| > N$. Set $n_j = 0$ for $j \geqslant N$, and let $n_{j-1} = m_j + n_j$; then $n_{j-1} = 0$ for $j < -N$. Also,

$$\Sigma n_j = \Sigma j (n_{j-1} - n_j) = \Sigma j m_j = 0.$$

Hence, $\Sigma n_j \hat{j} \in R_0(T)$ and $\varphi(\Sigma n_j \hat{j}) = \Sigma m_j e_j$, so that $\ker(h) \leqslant \operatorname{im}(\varphi)$. Thus, the cokernel of φ and hence $\mathrm{Ext}_T^s(C(T))$ are isomorphic to $Z \oplus Z$. □

A similar application of Theorem 3.9 yields the result for $\mathrm{Ext}_T^w(C(T))$. We summarize the above results in the following:

THEOREM 4.2. *Let* $\alpha: T \to C(T)$ *be given by* $(\alpha(\lambda) f)(Z) = f(\lambda Z)$; *then*

$$\mathrm{Ext}^r_T(C(T)) \simeq \sum_{-\infty}^{+\infty} \oplus Z,$$

$$\mathrm{Ext}^s_T(C(T)) \simeq Z \oplus Z, \quad \textit{and} \quad \mathrm{Ext}^w_T(C(T)) \simeq Z.$$

Furthermore, if (τ, ρ_μ) *is a covariant extension with associated essential unitary* $U = (U_{i,j})$, *then the above isomorphisms are given by the maps*:

$$[\tau, \rho_\mu]_r \to \sum \mathrm{ind}(U_{i,i-1}) e_i,$$

$$[\tau, \rho_\mu]_s \to \left(\sum \mathrm{ind}(U_{i,i-1}), \sum i \cdot \mathrm{ind}(U_{i,i-1})\right),$$

$$[\tau, \rho_\mu]_w \to \sum \mathrm{ind}(U_{i,i-1}).$$

REMARK 4.3. A calculation similar to the one carried out in Theorem 4.2 shows that for $(C(T), T, \alpha^n)$ one has $\mathrm{Ext}^w_T(C(T)) \simeq Z^n$. As before, one associates to a covariant extension (τ, ρ_μ) an essential unitary $U = (U_{i,j})$ and finds that $U_{i,j} = 0$, unless $i - j = n$. Setting $U_i = U_{i,i-n}$, one obtains an n-tuple $\left(\sum_{k=-\infty}^{+\infty} \mathrm{ind}(U_{kn+j})\right)_{j=1}^n$ which only depends on the weak equivalence class of (τ, ρ_μ). This correspondence defines the isomorphism between $\mathrm{Ext}^w_G(C(T))$ and Z^n. However, $T \times_{\alpha^n} C(T)$ is isomorphic to the direct sum of n copies of the compacts [10], and hence $\mathrm{Ext}^w(T \times_{\alpha^n} C(T)) = (0)$. The above examples show that Ext^w_G can distinguish between group actions which Ext^w of the corresponding covariance algebra identifies.

5. Relationship with Kasparov's Ext(A, B). Kasparov has constructed a group $\mathrm{Ext}(A, B)$ from equivalence classes of extensions of C^*-algebras of the form

$$0 \to B \otimes K(H) \to E \to A \to 0 \tag{1}$$

where A is nuclear and separable, B has countable approximate unit, a 2nd countable, compact group G acts on both A and B, and all homomorphisms respect the actions of G. The Busby invariant identifies extensions of the above form with *-monomorphisms of the type we consider and leads to an identification of one of our groups with one of Kasparov's. It is the purpose of this section to describe this identification.

Unfortunately, Kasparov is somewhat vague in describing the action of G on $B \otimes K(H)$ and this will be necessary to our identification. A careful reading of [6, Lemma 4] shows that for the isomorphism between K_B and $B \otimes K(H)$ to respect the G-actions, the action of G on $1 \otimes K(H)$ cannot be the trivial action as is implied in [7, Section 7(2)]. Instead, the action of G on $B \otimes K(H)$ must be taken to be $\beta \otimes ad(\rho_\mu)$, where β is the action on B and $ad(\rho_\mu)$ is the action on $K(H)$ induced by a strongly continuous unitary representation ρ_μ of G on H in which every irreducible representation of G occurs with infinite multiplicity.

When $B = \mathbf{C}$, $\mathbf{C} \otimes K(H) = K(H)$ and an exact sequence of the above form induces a possibly non-unital *-homomorphism τ_E of A into $Q(H)$. The requirement that the homomorphisms in (1) respect the actions of G, is equivalent to requiring that

(τ_E, ρ_μ) be covariant. Conversely, given a covariant extension it is stably equivalent to one with a representation unitarily equivalent to ρ_μ and determines an extension of the form (1).

The group $\operatorname{Ext}(A, \mathbf{C})$ consists of equivalence classes of extensions of the form (1), where two extensions τ_E and $\tau_{E'}$ are equivalent if there exist (not necessarily unital) split extensions (φ, ρ_μ) and (φ', ρ_μ), and a unitary U on $L(H \oplus H)$ fixed by $\rho_\mu \oplus \rho_\mu$ such that $\dot{U}^*(\tau_E \oplus \varphi)\dot{U} = \tau_{E'} \oplus \varphi'$. The existence of nonunital split extensions makes Kasparov's seemingly rigid equivalence relation identical with our weak stable equivalence relation. An analogous identification occurs between the weak and strong Ext [3], without group actions, when the unital hypothesis is dropped and our argument is the same. For completeness we sketch the argument.

If (τ_0, ρ_0) and (τ_1, ρ_1) are two covariant extensions which are stably weakly equivalent then we shall show they are equivalent in Kasparov's sense. To see this it will be sufficient to assume that $\rho_0 = \rho_1 = \rho_\mu$ and that there exist unital split extensions (φ_0, ρ_μ), (φ_1, ρ_μ), and an essential unitary U in $L(H \oplus H)$ which essentially commutes with $\rho_\mu \oplus \rho_\mu$ such that $\dot{U}^*(\tau_0 \oplus \varphi_0)\dot{U} = \tau_1 \oplus \varphi_1$. Since $\operatorname{ind}_G(U \oplus U^*) = 0$, there will exist a unitary W in $L(H \oplus H \oplus H \oplus H)$ which commutes with

$$\rho_\mu \oplus \rho_\mu \oplus \rho_\mu \oplus \rho_\mu,$$

is a compact perturbation of $U \oplus U^*$, and hence

$$\dot{W}^*(\tau_0 \oplus \varphi_0 \oplus 0 \oplus 0)\dot{W} = \tau_1 \oplus \varphi_1 \oplus 0 \oplus 0.$$

Thus, (τ_0, ρ_0) and (τ_1, ρ_1) are equivalent in Kasparov's sense.

Conversely, suppose there are (not necessarily unital) split extensions (φ_0, ρ_μ), (φ_1, ρ_μ) and a unitary U which commutes with $\rho_\mu \oplus \rho_\mu$ such that

$$\dot{U}^*(\tau_0 \oplus \varphi_0)\dot{U} = \tau_1 \oplus \varphi_1.$$

Let γ_0 and γ_1 be covariant *-homomorphisms which are lifting of φ_0 and φ_1, respectively, and set $P_i = \gamma_i(1)$ for $i = 1, 2$. Setting $V = (1 \oplus P_0)U(1 \oplus P_1)$ defines an essential unitary from the range of $1 \oplus P_1$ to the range of $1 \oplus P_0$, which commutes with $\rho_\mu \oplus \rho_\mu$. Thus, the compression of $(\tau_i \oplus \varphi_i,\ \rho_\mu \oplus \rho_\mu)$ to the range of $1 \oplus P_i$ for $i = 1, 2$ are unital covariant extensions which are weakly equivalent. Finally, since (φ_i, ρ_μ) compressed to the range of P_i is still split for $i = 1, 2$, we have (τ_0, ρ_μ) and (τ_1, ρ_μ) are stably weakly equivalent.

The above identification of Kasparov's equivalence classes of extensions with our stable weak equivalence classes of covariant extensions is a bijection which preserves the group operations.

THEOREM 5.1. *Let (A, G, α) be a C^*-dynamical system with, G 2nd countable and compact and A separable and nuclear with unit, then* $\operatorname{Ext}_G^w(A)$ *and* $\operatorname{Ext}(A, \mathbf{C})$ *are isomorphic.*

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Department of Mathematics
University of Houston
Houston, Texas 77004