APPROXIMATION OF ANALYTIC FUNCTIONS SATISFYING A LIPSCHITZ CONDITION

A. Matheson

1. Let λ_α, $0 < \alpha < 1$, denote the class of functions f analytic in the open unit disk, continuous in the closed disk for which $t^{-\alpha}\omega(t) \to 0$ as $t \to 0$, where ω denotes the modulus of continuity of the boundary function of f.

Defining

$$\|f\|_\alpha = \|f\|_\infty + \sup t^{-\alpha}\omega(+) \tag{1.1}$$

yields a Banach algebra norm on λ_α. A theorem of Hardy and Littlewood [7, I, p.263] guarantees that $f \in \lambda_\alpha$ if and only if

$$|f'(z)| = o((1 - |z|)^{\alpha-1}) \qquad \text{as } |z| \to 1^-. \tag{1.2}$$

This theorem yields an equivalent Banach algebra norm on λ_α by setting

$$\|f\| = \|f\|_\infty + \sup \{(1 - |z|)^{1-\alpha}|f'(z)|: |z| < 1\}. \tag{1.3}$$

The norm given by (1.3) will be used exclusively in the sequel.

Every function $f \in \lambda_\alpha$ has a canonical factorization $f = FG$, where F is an outer function and G is an inner function. The purpose of this paper is to prove theorem A below, which states, in effect, that a function f in λ_α can be approximated by functions in λ_α with the same inner factor and with boundary zeros of arbitrarily high order.

THEOREM A. *Let $f \in \lambda_\alpha$ and let E be a closed set on the unit circle such that $f(z) = 0$ for all $z \in E$. Let $M > 0$ be given. Then for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in \lambda_\alpha$ such that*

(i) *the inner factors of f and f_ε coincide,*

(ii) $\|f - f_\varepsilon\| < \varepsilon$, *and*

(iii) $|f_\varepsilon(z)| = O(\text{dist}^M(z,E))$ *as* $\text{dist}(z,E) \to 0$.

Theorem A can be used to give a characterization of the closed ideals in λ_α analogous to the Rudin-Beurling characterization of the closed ideals in the disc algebra [5]. The argument is similar to that of Korenblum [2] and is presented in [4].

The principal difficulty in the proof of Theorem A is isolated in the next theorem.

THEOREM B. *Let $f \in \lambda_\alpha$ be of the form F^pG where F is outer, G is inner, $p > 1$ and $FG \in \lambda_\alpha$. Let Γ be an open subset of the unit circle such that f vanishes at the endpoints of each component interval of Γ. Define*

Received September 19, 1977.
Revision received February 28, 1978.

Michigan Math. J. 25 (1978).

$$(1.4)\qquad F_\Gamma(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|F(e^{i\theta})|\,d\theta\right\}.$$

Then there is a positive number N_o *independent of* Γ *such that if* $N > N_o$, *the function* fF_Γ^N *belongs to* λ_α *and satisfies*

$$(1.5)\qquad |(fF_\Gamma^N)'(z)| = o((1-|z|)^{\alpha-1})$$

uniformly with respect to Γ.

Section 2 will be devoted to presenting a series of lemmas needed in the proof of Theorem B. This theorem will be proved in Section 3 and Theorem A will be derived from Theorem B in Section 4.

2. The first lemma gives a useful characterization of convergence in λ_α.

LEMMA 1. *Let* $\{f_n\}$ *be a sequence in* λ_α *which converges uniformly on compact subsets of the open disk to some function* f. *If*

$$(2.1)\qquad \lim_{r\to 1^-}(1-r)^{1-\alpha}\sup_{|z|=r}|f_n'(z)| = 0$$

uniformly in n, *then* $f \in \lambda_\alpha$ *and* $\lim_{r\to\infty}\|f_n - f\| = 0$.

Proof. It suffices to show that $\{f_n\}$ is a Cauchy sequence in λ_α. Fix $\varepsilon > 0$ and choose R $(0 < R < 1)$ so that

$$(1-|z|)^{1-\alpha}|f_n'(z)| < \varepsilon \qquad \text{for all } |z| \geq R \text{ and for all } n.$$

Now choose $N > 0$ so that if $m,n > N$, then

$$|f_m(z) - f_n(z)| < \varepsilon,\quad |f_m'(z) - f_n'| < \varepsilon \qquad \text{for } |z| \leq R.$$

Let $z = re^{it}$ be a point such that $R < r < 1$. Then

$$f_m(z) - f_n(z) = \int_R^r [f_m'(se^{it}) - f_n'(se^{it})]\,e^{it}ds + f_m(Re^{it}) - f_n(Re^{it}),$$

so that

$$\begin{aligned}|f_m(z) - f_n(z)| &\leq \int_R^r \frac{2\varepsilon}{(1-s)^{1-\alpha}}\,ds + |f_m(Re^{it}) - f_n(Re^{it})|\\ &\leq \frac{2}{\alpha}\varepsilon + \varepsilon \qquad \text{for } m,n \geq N.\end{aligned}$$

Thus if $m,n > N$, $\|f_m - f_n\|_\infty < \left(1+\dfrac{2}{\alpha}\right)\varepsilon$. Also, if $|z| \geq R$,

$$(1-|z|)^{1-\alpha}|f_m'(z) - f_n'(z)| \leq 2\varepsilon \qquad \text{for } m,n \geq N.$$

Thus $\|f_m - f_n\| < \left(1 + \frac{2}{\alpha}\right)\varepsilon + 2\varepsilon$ which proves the lemma.

The second lemma, due to Havin [1] and Shamoyan [6] guarantees that if f is in λ_α then the outer factor of f is also.

LEMMA 2. *Let* $s \in H^\infty$. *For* $f \in H^\infty$ *define*

$$\text{(2.2)} \qquad Tf(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta} - z} f(e^{i\theta})\,\overline{s(e^{i\theta})}\,d\theta$$

for $|z| < 1$. *Then* T *is a bounded operator from* λ_α *to* λ_α. *In fact if* $f \in \lambda_\alpha$,

$$\text{(2.3)} \qquad \|Tf\| < C_\alpha \|s\|_\infty \|f\|$$

where C_α *is a constant depending only on* α. *In particular if* $f \in \lambda_\alpha$ *and* G *is an inner function dividing* f, *then* $G^{-1}f \in \lambda_\alpha$ *and* $\|G^{-1}f\|_\infty < C_\alpha\|f\|$.

The next two lemmas give restricted versions of Theorem A. Lemma 3 allows approximation by functions with boundary zeros of order greater than α, but the order depends in general on how close the approximating function is to the original function. In Lemma 4 E is restricted to being a finite set.

LEMMA 3. *Let* $f \in \lambda_\alpha$ *and let* $f = FG$ *where* G *is an inner function and* F *is an outer function. Then for all* $\varepsilon > 0$ *the functions* $F^{1+\varepsilon}G$ *are in* λ_α *and*

$$\text{(2.4)} \qquad \lim_{\varepsilon \to 0} \|F^{1+\varepsilon}G - f\| = 0.$$

Proof. It is clear that $F^\varepsilon f = F^{1+\varepsilon}G$ converges to f uniformly on compact subsets of the open disk. Since $F \in \lambda_\alpha$ by Lemma 2 and since

$$\begin{aligned}(F^{1+\varepsilon}G)' &= F^{1+\varepsilon}G' + (1+\varepsilon)F^\varepsilon F' G \\ &= F^\varepsilon f' + \varepsilon F^\varepsilon G F',\end{aligned}$$

it follows that

$$\lim_{r \to 1} (1-r)^{1-\alpha} \sup |(F^{1+\varepsilon}G)'(z)| = 0$$

uniformly in ε, and Lemma 3 follows from Lemma 1.

LEMMA 4. *Let* $f \in \lambda_\alpha$ *and let* E *be a finite subset of the unit circle on which* f *vanishes. Let* $N > 0$ *be given. Then for each* $\varepsilon > 0$ *there is an outer function* F *in* λ_α *such that*

(i) $\|Ff - f\| < \varepsilon$,

(ii) $\{z \mid F(z) = 0\} = E$, and

(iii) $|F(z)| = O(\text{dist}^N(z,E))$ *as* $\text{dist}(z,E) \to 0$.

Proof. It is permissible to assume that $N = 1$ and that E consists of a single point, say $E = \{1\}$, since the general result will follow by induction. Set

$$F_\delta(z) = \frac{z-1}{z-1-\delta}, \qquad \delta > 0.$$

Clearly F_δ is an outer function satisfying (ii) and (iii). To finish the proof it suffices to show that $\lim_{\delta\to 0} \|fF_\delta - f\| = 0$. It is clear that $F_\delta \to 1$ uniformly on compact subsets of the open disk; so in view of Lemma 1, it will be enough to prove that

$$\lim_{r\to 1} (1-r)^{1-\alpha} \sup_{|z|=r} |(F_\delta f)'(z)| = 0$$

uniformly in δ. Since $\|F_\delta\|_\infty \le 1$, this reduces to showing that

$$\lim_{r\to 1} (1-r)^{1-\alpha} \sup_{|z|=r} |(f\,F'_\delta)(z)| = 0$$

uniformly with respect to δ.

Given $\varepsilon > 0$, choose $\eta > 0$ so that $|f(z)| < \varepsilon\,|1-z|^\alpha$ whenever $|1-z| < \eta$, and choose R, $(0 < R < 1)$, so that $(1-R)^{1-\alpha} < \eta\varepsilon$. Since $F'_\delta(z) = -\delta(z-1-\delta)^{-2}$, and $|z-1-\delta| \ge \delta$ for $|z| < 1$, it follows that $|F'_\delta(z)| \le |z-1-\delta|^{-1}$. It is also clear that $|z-1-\delta| \ge |z-1| \ge 1-|z|$ for $|z| < 1$. Hence

$$|F'_\delta(z)| \le (1-|z|)^{\alpha-1}|1-z|^{-\alpha} \qquad \text{for } |z| < 1.$$

If $|z-1| < \eta$, then

$$(1-|z|)^{1-\alpha}|f(z)|\,|F'_\delta(z)| < (1-|z|)^{1-\alpha}\varepsilon|1-z|^\alpha(1-|z|)^{-\alpha+1}|1-z|^{-\alpha} = \varepsilon,$$

while if $|z-1| > \eta$, and $|z| \ge R$, then

$$\begin{aligned}(1-|z|)^{1-\alpha}|f(z)|\,|F'_\delta(z)| &\le (1-R)^{1-\alpha}\|f\|_\infty|1-z-\delta|^{-1}\\ &\le \eta\varepsilon\|f\|_\infty\,|1-z|^{-1} < \varepsilon\|f\|_\infty\end{aligned}$$

which proves the lemma.

The next lemma, due to Korenblum [3] is an essential ingredient in the proof of Theorem B.

LEMMA 5. *Let g be a bounded analytic function in the disk with* $\|g\|_\infty < 1$. *Let* $\gamma = \{e^{i\theta} | \alpha < \theta < \beta\}$, *with* $\beta - \alpha < 1$. *Suppose that*

$$|g(e^{i\theta})| \le a\,d^n(e^{i\theta}) \qquad \textit{for } e^{i\theta} \in \gamma, \tag{2.5}$$

where $a \le 1$ *and* $d(z) = \min\{|z-e^{i\alpha}|, |z-e^{i\beta}|\}$. *Let*

$$\gamma^k = \{z\colon |z| < 1,\ z|z|^{-1} \in \gamma,\ kd(z|z|^{-1}) = 1-|z|\}$$

where k < 1. *Then*

$$|g(z)| \leq C^n a^{1-qk}[d(z)]^{n(1-qk)} \tag{2.6}$$

for all $z \in \gamma^k$, *where* C *and* q *are positive constants independent of* k, $\beta - \alpha$ *and* n.

Proof. Define

$$\phi(\theta) = \begin{cases} 1 & \text{if } e^{i\theta} \notin \gamma \\ ad^n(e^{i\theta}) & \text{if } e^{i\theta} \in \gamma \end{cases},$$

and let

$$F(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \phi(\theta)\, d\theta\right\}.$$

Then F is an outer function satisfying the hypotheses and $|g(z)| \leq |F(z)|$ for all $|z| < 1$, so it suffices to prove the lemma for F. Let $P_r(\theta)$ denote Poisson's kernel. Then, if $z = re^{it}$,

$$\log|F(z)| = \frac{\log a}{2\pi}\int_\alpha^\beta P_r(\theta - t)\, d\theta + \frac{n}{2\pi}\int_\alpha^\beta \log d(e^{i\theta})\, P_r(\theta - t)\, d\theta$$

so it suffices to find constants q and C independent of k, $\beta - \alpha$ and n such that $1 - qk \leq \frac{1}{2\pi}\int_\alpha^\beta P_r(\theta - t) d\theta$ and

$$\frac{1}{2\pi}\int_\alpha^\beta P_r(\theta - t) \log d(e^{i\theta}) d\theta \leq \log C + (1 - qk) \log d(re^{it})$$

whenever $z = re^{it} \in \gamma^k$. A simple calculation yields the first inequality with $q = 4/\pi$. Assume, *e.g.*, that $d(e^{it}) = |e^{it} - e^{i\alpha}|$. Then $[\alpha, 2t - \alpha] \subseteq [\alpha\ \beta]$, so

$$\begin{aligned} 2\frac{1}{\pi}\int_\alpha^\beta P_r(\theta - t) \log d(e^{i\theta})\, d\theta &\leq \frac{1}{2\pi}\int_\alpha^{2t-\alpha} P_r(\theta - t) \log d(e^{i\theta})\, d\theta \\ &\leq \frac{\log 2d(z)}{2\pi}\int_\alpha^{2t-\alpha} P_r(\theta - t)\, d\theta \\ &\leq \left[1 - \frac{4}{\pi}k\right][\log d(z) + \log 2]. \end{aligned}$$

The lemma follows with $q = 4/\pi$ and $c = 2$.

3. This section is devoted to proving Theorem B. Let Γ be an open subset of the unit circle and let E denote the boundary (on the circle) of Γ. Let $f \in \lambda_\alpha$

be of the form F^pG with $p > 1$, and $F\,G \in \lambda_\alpha$, such that f vanishes on E. Since F also must vanish on E and $F \in \lambda_\alpha$, it follows that

$$|f(z)| \le C \operatorname{dist}^{\beta}(z,E), \qquad |z| < 1 \tag{3.1}$$

for some $C > 0$ and some $\beta > \alpha$. Without loss of generality, assume that $\|f\|_\alpha < 1$ and $\|F\|_\infty < 1$.

Define

$$F_\Gamma(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|F(e^{i\theta})|\,d\theta\right\} \tag{3.2}$$

$$A(z) = \frac{1}{\pi}\int_\Gamma \frac{e^{i\theta}}{(e^{i\theta}-z)^2}\log|F(e^{i\theta})|\,d\theta \tag{3.3}$$

$$B(z) = \frac{1}{\pi}\int_{C\Gamma} \frac{e^{i\theta}}{(e^{i\theta}-z)^2}\log|F(e^{i\theta})|\,d\theta \tag{3.4}$$

where $C\Gamma$ denotes the complement of Γ. Since $F' = F(A+B)$ and $(F_\Gamma^N)' = N\,F_\Gamma^N\,A$, it follows that

$$f(F_\Gamma^N)' = N\,f\,F_\Gamma^N\,A \tag{3.5}$$

$$= F^{p-1}\,N\,G\,F'\,F_\Gamma^N - N\,G\,F^p F_\Gamma^N B. \tag{3.6}$$

Also $|F^p(z)| \le |F(z)| \operatorname{dist}^{\varepsilon}(z,E)$ where $\varepsilon = \varepsilon(p) > 0$. Let q be as in Lemma 5 and choose k so that $1 - qk > 0$. Choose $N_0 > 0$ so that $N_0\alpha(1-qk) \ge 2-\alpha$. Let $d(z) = \operatorname{dist}(z,E)$ and divide the open disk into three disjoint sets:

$$\begin{aligned} G_1 &= \{z : kd(z) < 1-|z|\} \\ G_2 &= \{z : kd(z) \ge 1-|z|,\ z|z|^{-1} \in \Gamma\}, \\ G_3 &= \{z : kd(z) \le 1-|z|,\ z|z|^{-1} \notin \Gamma\}. \end{aligned}$$

Fix $N \ge N_0$. To prove Theorem B it suffices to show that

$$\lim_{r\to 1}(1-r)^{1-\alpha}\sup_{|z|=r}|f(z)(F_\Gamma^N(z))'| = 0 \tag{3.7}$$

uniformly with respect to Γ, separately in each of the three sets G_1, G_2, and G_3.

First suppose $z \in G_1$. Since $\|F_\Gamma^N\|_\infty < 1$, it follows from Cauchy's estimates that $|(F_\Gamma^N)'(z)| < (1-|z|)^{-1}$, hence

$$(1-|z|)^{1-\alpha}\,|f(z)(F_\Gamma^N(z))'| \le (1-|z|)^{-\alpha}d^\beta(z)$$
$$\le k^{-\beta}(1-|z|)^{\beta-\alpha}$$

so (3.7) holds in G_1.

If $z \in G_2$, it follows from Lemma 5 that $|F_\Gamma^N(z)| \le C^{N\beta}[d(z)]^{N\beta(1-qk)}$. It is clear that $|B(z)| \le 2Q[d(z)]^{-2}$ where $Q = -\log|F(0)|$, so that

$$|NG(z)F(z)F_\Gamma^N(z)B(z)| \le 2NC_1C^{N\beta}Q[d(z)]^{\alpha+N\beta(1-qk)-2},$$

which is bounded, since $N\beta(1-qk) > 2-\alpha$. Thus (3.7) follows from (3.6) in G_2.

For $z \in G_3$, following Korenblum, let $a(z) = -\frac{1}{2\pi}\int_\Gamma \frac{\log|F(e^{i\theta})|}{|e^{i\theta}-z|^2}\,d\theta$. The proof divides into two cases. If $a(z)d(z) \le 18|\log d(z)|$, $z \in G_3$, then

$$|A(z)| \le 2\,a(z) \le 36\,|\log d(z)|\,[d(z)]^{-1},$$

so that

$$(1-|z|)^{1-\alpha}N\,|f(z)F_\Gamma^N(z)A(z)| \le 36N(1-|z|)^{1-\alpha}|\log d(z)|\,[d(z)]^{\beta-1}$$
$$\le 36N(1-|z|)^{\delta}k^{1-\alpha-\delta}[d(z)]^{\delta}|\log d(z)|,$$

where $\delta = \frac{1}{2}(\beta-\alpha)$. Since $\beta-\alpha-\delta- > 0$, $|\log d(z)|\,[d(z)]^{\delta}$ is bounded, hence (3.7) follows from (3.5).

Finally assume that $a(z)d(z) > 18|\log d(z)|$, $z \in G_3$, and let

$$\rho = 1 - 18\frac{|\log d(z)|}{a(z)}.$$

The first step is to estimate $|F(\rho z)|$. Since $1-\rho \le d(z)$, it follows from a calculation that $\frac{|e^{i\theta}-z|}{|e^{i\theta}-\rho z|} > \frac{1}{3}$ for $e^{i\theta} \in \Gamma$, and hence

$$\operatorname{Re}\left(\frac{e^{i\theta}+\rho z}{e^{i\theta}-\rho z}\right) = \frac{1-\rho^2|z|^2}{|e^{i\theta}-\rho z|^2}$$
$$\ge \frac{1}{9}\,\frac{1-\rho^2}{|e^{i\theta}-z|^2}$$
$$= 2(1+\rho)\frac{|\log d(z)|}{a(z)}\,\frac{1}{|e^{i\theta}-z|^2}$$
$$> 2\frac{|\log d(z)|}{a(z)}\,\frac{1}{|e^{i\theta}-z|^2}, \qquad \text{if } e^{i\theta} \in \Gamma.$$

Therefore, since $\log|F(e^{i\theta})| < 0$,

$$\begin{aligned}\log|F(\rho z)| &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta}+\rho z}{e^{i\theta}-\rho z}\right)\log|F(e^{i\theta})|d\theta \\ &\le \frac{1}{2\pi}\int_\Gamma \operatorname{Re}\left(\frac{e^{i\theta}+\rho z}{e^{i\theta}-\rho z}\right)\log|F(e^{i\theta})|d\theta \\ &< 2\frac{|\log d(z)|}{a(z)}\cdot\frac{1}{2\pi}\int_\Gamma \frac{\log|F(e^{i\theta})|}{|e^{i\theta}-z)|^2}d\theta \\ &= -2|\log d(z)|,\end{aligned}$$

so $|F(\rho z)| \le d^2(z)$. Now

$$\begin{aligned}|F(z)| &= |F(\rho z) + \int_\rho^1 z F'(sz)ds| \\ &\le |F(\rho z)| + (1-\rho)|z| \sup_{p<s<1}|F'(sz)| \\ &\le |F(\rho z)| + (1-\rho) \sup_{|\zeta|=|z|}|F'(\zeta)|\end{aligned}$$

by the maximum modulus theorem. Hence,

$$\begin{aligned}|Nf(z)F_\Gamma^N(z)A(z)| &\le N|F(z)F_\Gamma^N(z)A(z)|d^\varepsilon(z) \\ &\le N|F(\rho z)|2Qd^{\varepsilon-2}(z) \\ &+ N(1-\rho)\sup_{|\zeta|=|z|}|F'(\zeta)|\,|A(z)|d^\varepsilon(z) \\ &< Nd^\varepsilon(z)\,[2Q + (1-\rho)|A(z)| \sup_{|\zeta|=|z|}|F'(\zeta)|].\end{aligned}$$

But $|A(z)| \le 2a(z)$ and $1-\rho = 18|\log d(z)|[a(z)]^{-1}$, so

$$(1-\rho)|A(z)|\,d^\varepsilon(z) \le 36|\log d(z)|\,d^\varepsilon(z),$$

which is bounded. Thus there are positive constants C_1 and C_2 such that

$$\begin{aligned}|Nf(z)F_\Gamma^n(z)A(z)| &\le C_1 + C_2 \sup_{|\rho|=|z|}|F'(\rho)| \\ &= o((1-|z|)^{\alpha-1}),\end{aligned}$$

since $F \in \lambda_\alpha$. This completes the proof of the Theorem B.

4. Let f, E and M be as in Theorem A, and fix $\varepsilon > 0$. By lemma 3, there exists $p > 1$ such that $\|F^pG - f\| < \varepsilon/3$, where F is the outer part and G is the inner part of f. Let $\{I_k\}_{k=1}^\infty$ be the sequence of complimentary intervals to E and set $B_n = \bigcup_{k=n}^\infty I_k$. Define

$$(4.1) \qquad F_n(z) = \exp\left\{\frac{1}{2\pi}\int_{B_n} \frac{e^{i\theta}+z}{e^{i\theta-z}} \log|F(e^{i\theta})|\,d\theta\right\}.$$

Choose $N > \max\{N_o, M\alpha^{-1}\}$ where M is the constant of Theorem A and N_o the constant of Theorem B. Define $g_n = F_n^N F^p G$. It is clear that g_n converges to F^pG uniformly on compact subsets of the open disk, as $n \to \infty$ and, since $N \geq N_o$, Theorem B implies that $\lim_{r\to 1}(1-r)^{1-\alpha} \sup_{|z|=r} |g_n'(z)| = 0$ uniformly with respect to n. It follows by Lemma 1 that $\|g_n - F^pG\| \to 0$ as $n \to \infty$. Choose n so that $\|g_n - F_pG\| < \varepsilon/3$. The function g_n satisfies conditions (i) and (ii) of Theorem A, but does not necessarily satisfy (iii). To rectify this, notice that the set $E_1 = (E - \bar{B}_n) \cup (E \cap \partial\bar{B}_n)$ is a finite set upon which g_n vanishes. By Lemma 4, there is an outer function $H \in \lambda_\alpha$ such that $\|g_n - Hg_n\| < \varepsilon/3$, H vanishes exactly on E_1, and

$$H(z) < C\,\text{dist}^N(z,E_1) \qquad \text{for } |z| < 1.$$

By construction, it is evident that $|H(z)g_n(z)| < C_1\,\text{dist}^N(z,E)$ for $|z| = 1$. In particular, if $\zeta \in E$,

$$(4.2) \qquad |H(z)g_n(z)| \leq C_1|z-\zeta|^N \qquad \text{for } |z| = 1.$$

But $(z-\zeta)^N$ is an outer function, so (4.2) holds for all $|z| < 1$. Hence

$$|H(z)g_n(z)| \leq C_1 \inf_{s\in E}|z-\zeta|^N = C_1\,\text{dist}^N(z,E), \qquad |z| < 1.$$

Since $\|Hg_n - f\| < \|Hg_n - g_n\| + \|g_n - F^pG\| + \|F^pG - f\| < \varepsilon$, and $HF_n^NF^p$ is an outer function, Theorem A is proved.

Theorem A is analogous to the results of Korenblum in [2] and [3]. The method of proof in this paper is similar to, and inspired by the proof in [3]. The principal difference between [3] and the present paper is that in [3] the norm estimates are carried out on the boundary of the disk, whereas here it seems to be expedient to work in the interior via the theorem of Hardy and Littlewood. The referee has pointed out that Shamoyan has proved a result similar to Theorem A for a restricted class of closed sets [7].

This work constitutes a portion of the author's doctoral dissertation, written under the direction of Robert Kaufman at the University of Illinois. The author would like to thank Professor Kaufman for suggesting this problem and for his patience in seeing it through to solution.

REFERENCES

1. V. P. Havin, *On the factorization of analytic functions smooth on the boundary.* Zapiski Nauknikh Seminarov LOMI, 22 (1971).

2. B. I. Korenblum, *Closed ideals in the ring A^n.* Functional analysis and its applications, 6 No. 3 (1972), 203-214.

3. ———, *Invariant subspaces of a shift operator in a weighted Hilbert space.* Matem. Sb., 89 (1972), 110–137.

4. A. Matheson, *Closed ideals in rings of analytic functions satisfying a Lipschitz condition.* Banach spaces of analytic functions (Proc. Conf., Kent State University, Kent, Ohio, July 1976). Lecture Notes in Mathematics, Vol. 604. Springer-Verlag, Berlin, 1977.

5. W. Rudin, *The closed ideals in an algebra of analytic functions.* Canad. J. Math., 9 (1957), 426–434.

6. F. A. Shamoyan, *Division by an inner function in some spaces of functions analytic in the disk.* Zapiski Nauknikh Seminarov LOMI, 22 (1971), 206–208.

7. ———, *The structure of closed ideals in several algebras of analytic functions.* Dokl. Akad. Nauk Armyan. SSR, LV, 1972 (4) 207–210.

8. A. Zygmund, *Trigonometric Series.* 2nd ed. Cambridge University Press, New York, 1959.

Department of Mathematics
Oklahoma State University
Stillwater, Oklahoma 74074