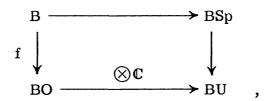
INDUCED COBORDISM THEORIES--AN EXAMPLE

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1. INTRODUCTION

The object of this note is to describe a way to construct new cobordism theories, basically by means of one example. First, recall that if M is a manifold, then the total space of the tangent bundle of M is an almost complex manifold, for $\tau(E(\tau_M)) \cong \pi^* \tau_M \oplus \pi^* \tau_M \cong \pi^* \tau_M \otimes_R \mathbb{C}$. If M is also a complex manifold, then τ_M is a complex bundle, so $\tau(E(\tau_M))$ is the complexification of a complex bundle and thus is a quaternionic (or symplectic) vector bundle.

One then introduces the notion of a weakly weakly almost complex manifold as a manifold together with a quaternionic structure on the complexification of the normal bundle. In bundle-theoretic terms, there is a fibering BSp \rightarrow BU obtained by considering a quaternionic bundle as just a complex bundle, and a map $\otimes \mathbb{C}$: BO \rightarrow BU obtained by classifying the complexification of the universal bundle. One may then form the induced fibering



and a weakly weakly almost complex manifold is a manifold M together with a chosen equivalence class of liftings of the normal map ν : M \rightarrow BO to B. (See Lashof [2] for the precise formalism of manifold with (B, f)-structure.)

Noting that the complexification of a complex bundle is quaternionic shows that

the composite BU $\xrightarrow{\pi}$ BO $\xrightarrow{\to}$ BU lifts to BSp, and hence every weakly almost complex manifold (for which ν lifts to BU) is weakly weakly almost complex.

Following Lashof, one may introduce the cobordism group $\Omega_*^{(B,f)}$ of weakly weakly almost complex manifolds. The main result of this paper is then:

THEOREM. The forgetful homomorphism $F\colon \Omega_*^{(B,f)} \to \mathfrak{R}_*$ into unoriented cobordism is monic, and one may choose generators x_i of $\mathfrak{R}_* = Z_2$ $[x_i \colon i \neq 2^s - 1]$ so that the image of F is the polynomial subalgebra on the x_i (i odd) and x_i^2 (i even).

Note. The image of the complex cobordism ring $\Omega^{\rm U}_*$ in \mathfrak{N}_* is the polynomial subalgebra consisting of the squares (Milnor [3]). The odd-dimensional generators needed may be taken to be U/O manifolds in the sense of Smith-Stong [4]; *i.e.*, manifolds for which the complexification of the normal bundle is trivial.

The results will include a general structure theorem (Remark following Lemma 3.1) showing that many theories are 2-torsion, and an analysis of Wall's cobordism theory W_{\star} (section 4) in a form similar to weakly weakly almost complex cobordism.

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2. COHOMOLOGY OF B

Following the procedure established by Thom, one first computes the cohomology of B.

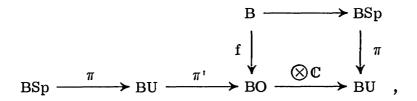
First, one has the fibering $\pi\colon \mathrm{BSp}\to\mathrm{BU}$ which has fiber U/Sp. The cohomology of BU with integer coefficients is $Z[c_i]$, where $c_i\in H^{2i}(\mathrm{BU};\, Z)$ is the universal Chern class (i>0), and that of BSp is $Z[\mathscr{P}_i^s]$, where \mathscr{P}_i^s is the symplectic Pontrjagin class in $H^{4i}(\mathrm{BSp};\, Z)$. Then $\pi^*(c_{2i+1})=0$ and $\pi^*(c_{2i})=\pm\mathscr{P}_i^s$ (the sign depending on your choice of definition). Since BU is simply connected, the Serre spectral sequence for π has a trivial coefficient system, and $H^*(U/\mathrm{Sp};\, Z)$ is the exterior algebra over Z on classes z_{4i+1} , $i\geq 0$, with z_{4i+1} transgressing to c_{2i+1} .

Note. One may also consider $Sp \xrightarrow{i} U \to U/Sp$, where i^* sends the exterior algebra on classes a_{2i+1} onto the exterior algebra on classes b_{4i+3} (i, $j \ge 0$).

Now considering the induced fibering U/Sp \rightarrow B $\stackrel{f}{\longrightarrow}$ BO, the coefficient system is again trivial. In mod 2 cohomology, H*(BO; Z₂) is the Z₂ polynomial ring on the Stiefel-Whitney classes $w_i \in H^i(BO; Z_2)$. The mod 2 reduction of the Chern class c_i is w_{2i} and $w(\gamma \bigotimes \mathbb{C}) = w(\gamma \bigoplus \gamma) = w(\gamma)^2$, so $(\bigotimes \mathbb{C})^*(w_{2i}) = w_i^2$. Thus, in the fibration f, z_{4i+1} transgresses to w_{2i+1}^2 and one has

LEMMA 2.1. The map $f: B \to BO$ induces an epimorphism in mod 2 cohomology, identifying $H^*(B; Z_2)$ with $Z_2[w_i]/(w_{2i+1}^2 = 0)$.

Now, letting p be an odd prime, one considers the diagram



in which $(\bigotimes \mathbb{C}) \circ \pi'$ and $(\bigotimes \mathbb{C}) \circ \pi' \circ \pi$ both lift to BSp. Since π is a principal fibration (U/Sp may be realized as a group), one has an induced fibering

$$\begin{array}{c|c} \operatorname{BSp} \times \operatorname{U/Sp} & \longrightarrow & \operatorname{B} \\ \pi_1 & \downarrow & \downarrow & \operatorname{f} \\ \operatorname{BSp} & \xrightarrow{\pi^{\scriptscriptstyle \mathsf{I}} \circ \pi} & \operatorname{BO} \end{array}$$

Again the coefficient systems are trivial and $\pi' \circ \pi$ induces an isomorphism in Z_p cohomology, so $H^*(B; Z_p) \cong H^*(BO; Z_p) \bigotimes_{Z_p} H^*(U/Sp; Z_p)$.

3. CALCULATION OF THE COBORDISM

Beginning with the fibration f: B \to BO, one forms the induced fibering f_n : $B_n \to BO_n$, and if γ_n is the universal n-plane bundle over BO_n , one lets TB_n

be the Thom space of $f_n^*(\gamma_n)$. From the Pontrjagin-Thom theorem (Lashof [2]), the cobordism group $\Omega_*^{(B,f)}$ may be identified with the homotopy of the spectrum $\{TB_n\}$. Since $\pi\colon BSp\to BU$ is an H-map, B is an H-space and f is an H-map (i.e., if $\xi\otimes\mathbb{C}$ and $\eta\otimes\mathbb{C}$ have quaternionic structures, so does $(\xi\oplus\eta)\otimes\mathbb{C}$), so that $\{TB_n\}$ is a ring spectrum, giving $\Omega_*^{(B,f)}$ the structure of a graded ring. The map on Thom spaces $T_f\colon TB_n\to T_n$ is a map of ring spectra inducing the forgetful homomorphism $F\colon \Omega_*^{(B,f)}\to \mathfrak{N}_*$, a ring homomorphism.

Now considering the inclusion i: $S(\gamma_n) \to D(\gamma_n)$ of the sphere bundle in the disc bundle, the projection identifies $D(\gamma_n)$ with BO_n , and $S(\gamma_n)$ may be identified with BO_{n-1} , so that the inclusion i may be identified with the usual map $BO_{n-1} \to BO_n$ classifying the Whitney sum with a trivial bundle (see [5], page 72). In particular, taking induced bundles, one has a cofibration

$$B_{n-1} \xrightarrow{j_{n-1}} B_n$$

$$\exists i : \vdots$$

$$S(f_n^* \gamma_n) \xrightarrow{} D(f_n^* \gamma_n) \xrightarrow{} TB_n .$$

Letting p be an odd prime and n an odd integer, one has the fiberings

$$\begin{array}{ccc}
B_{n-1} & \xrightarrow{j_{n-1}} & B_n \\
f_{n-1} & \downarrow & f_n \\
BO_{n-1} & \xrightarrow{i_{n-1}} & BO_n
\end{array}$$

with fiber U/Sp, with i_{n-1} being an isomorphism on Z_p cohomology, and with trivial coefficient system. Thus j_{n-1} induces an isomorphism on Z_p cohomology and $\widetilde{H}^*(TB_n; Z_p) = 0$ for n odd. Thus the Z_p cohomology of the Thom spectrum $\{TB_n\}$ is zero, giving

LEMMA 3.1. The groups $\Omega_{\star}^{(B,f)}$ are entirely 2-torsion.

Note. Any fibration $f: B \to BO$ with trivial coefficient system gives a 2-primary cobordism theory using this argument. In particular, this holds whenever f is induced from a fibering over a simply connected space.

To study the 2-primary structure, one considers Z_2 cohomology for which there is a Thom isomorphism. Thus $\widetilde{H}^*(TB; Z_2) \cong Z_2[w_i]/(w_{2i+1}^2 = 0) \cdot U$ is a free $H^*(B; Z_2)$ module of rank 1 on the Thom class U. Since $TB = \{TB_n\}$ is a ring spectrum, $\widetilde{H}^*(TB; Z_2)$ is a connected coalgebra over the Hopf algebra \mathscr{A} (the mod 2 Steenrod algebra). Let $\nu \colon \mathscr{A} \to \widetilde{H}^*(TB; Z_2) : \alpha \to \alpha(U)$ denote the action on the Thom class.

LEMMA 3.2. $\nu: \mathcal{A} \to \widetilde{H}^*(TB; Z_2)$ is monic.

Proof. Let $U/O \otimes \mathbb{C}$ denote the fiber of the map $\otimes \mathbb{C}$: BO \to BU. The inclusion i: $U/O \otimes \mathbb{C} \to BO$ then lifts to B, giving maps $U/O \otimes \mathbb{C} \xrightarrow{j} B \xrightarrow{f} BO$, and induced maps of spectra. In Smith-Stong [4], it was verified that the \mathscr{A} action on the Thom class in $T(U/O \otimes \mathbb{C})$ is monic, and hence ν is monic.

Following the ideas of Browder, Liulevicius, and Peterson [1], $\widetilde{H}^*(TB; \mathbb{Z}_2)$ is a free \mathscr{A} module and TB has the homotopy type of a product of Eilenberg-MacLane spectra $K(\mathbb{Z}_2)$. Since f is epic in mod 2 cohomology, one then has:

PROPOSITION 3.3. The forgetful homomorphism $F: \Omega_*^{(B,f)} \to \mathfrak{R}_*$ is a monomorphism.

To determine the image of F, one recalls (from Smith-Stong [4]) that generators for $\mathfrak{R}_* = \mathbb{Z}_2[x_i \colon i \neq 2^s - 1]$ may be chosen so that x_{2i+1} is a $U/O \otimes \mathbb{C}$ manifold, and following Milnor [3], the image of complex cobordism in \mathfrak{R}_* consists of the squares. Since $U/O \otimes \mathbb{C}$ manifolds and weakly almost complex manifolds are weakly weakly almost complex, this choice of generators shows that the image of F contains $X = \mathbb{Z}_2[x_{2i+1}, x_{2j}^2 \colon 2i + 1 \neq 2^s - 1]$.

Now the Steenrod algebra has the same dimension (degree by degree) as a \mathbb{Z}_2 polynomial algebra on classes $\mathbb{X}_{2^{S}-1}$, so $\mathbb{X} \otimes \mathscr{A}$ has the same dimension as $\mathbb{Z}_2[\mathbb{X}_{2i+1}, \mathbb{X}_{2j}^2]$, which is identical with the dimension of

$$\tilde{H}^*(TB; Z_2) \cong Z_2[w_i]/(w_{2i+1}^2 = 0).$$

Thus, the homotopy of TB has the same dimension as X in each degree, and image F = X, completing the proof of

THEOREM 3.4. The forgetful homomorphism $F: \Omega_*^{(B,f)} \to \mathfrak{R}_*$ into unoriented cobordism is monic, and one may choose generators x_i of $\mathfrak{R}_* = Z_2[x_i \colon i \neq 2^s - 1]$, so that the image of F is the polynomial subalgebra on the x_i (i odd) and x_i^2 (i even).

4. REMARK ON WALL MANIFOLDS

Surprisingly, the notion of weakly weakly almost complex manifolds is very similar to the manifolds introduced by Wall [6] in computing oriented cobordism. Specifically, Wall considered manifolds together with a reduction of the first Stiefel-Whitney class to an integral class.

To see the analogy, consider a manifold M for which the complexification of the normal bundle has a special unitary structure. In bundle-theoretic terms, one has the induced fibering

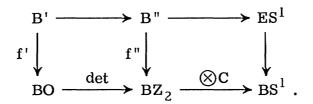
$$\begin{array}{ccc}
B' & \longrightarrow & BSU \\
f' \downarrow & & \downarrow \pi \\
BO & \longrightarrow & BU
\end{array}$$

The fibering $f': B' \to BO$ is exactly the fibration which gives rise to Wall manifolds.

First one notes that π is induced from the universal bundle $\pi' \colon ES^1 \to BS^1$ by the map det: $BU \to BS^1$ which classifies the determinant bundle, but

$$\det \circ (\bigotimes \mathbb{C}) = (\bigotimes \mathbb{C}) \circ \det,$$

and so f' is induced by



If λ denotes the universal line bundle over $BZ_2 = RP(\infty)$, B" is the sphere bundle of $\lambda \otimes C = \lambda \oplus \lambda$ and f" may be identified with the projection

$$S^{\infty} \times S^{1}/-1 \times -1 \rightarrow S^{\infty}/-1$$
.

The projection $S^{\infty} \times S^1/-1 \times -1 \to S^1/-1 = S^1$ is a homotopy equivalence, and so f'' is just the usual map $K(Z, 1) \to K(Z_2, 1)$. A lifting to B' is then precisely a reduction of the first Stiefel-Whitney class to an integral class.

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