COMPACTNESS OF λ-NUCLEAR OPERATORS

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1. INTRODUCTION

It is evident from the work of Persson and Pietsch [6] that the class of nuclear operators depends on ℓ_1, the space of absolutely convergent series. Replacing ℓ_1 by an arbitrary sequence space λ, we obtain a new class of operators called λ-nuclear, and we can pose questions motivated by known results in the case $\lambda = \ell_1$. The present work addresses the problem of under what restrictions the λ-nuclear operators are compact. Assuming that λ is a Banach space, Section 3 gives necessary and sufficient conditions on λ for λ-nuclear operators to be compact. Section 4 discusses a condition on the range of the operators that yields the same result.

2. PRELIMINARIES

We use λ to denote a *sequence space;* that is, a vector space whose elements are sequences of complex numbers, and we use $\lambda^\times$ for the *Köthe dual* of λ. ($\lambda^\times = \{b: \sum_{i=1}^\infty |a_i b_i| < \infty$ for all $a \in \lambda\}$.) A linear operator T between Banach spaces X and Y is *λ-nuclear* (respectively, *nuclear*) if

$$(1)\qquad Tx = \sum_{n=1}^{\infty} a_n \langle x, f_n \rangle y_n \quad \text{for all } x \in X,$$

where $\{a_n\}_{n=1}^\infty \in \lambda$ (respectively, ℓ_1), $f_n \in X'$ and $\sup_n \|f_n\| < \infty$, $y_n \in Y$ and $\{\langle y_n, g\rangle\}_{n=1}^\infty \in \lambda^\times$ for all $g \in Y'$. The series in (1) is required to converge in the norm topology on Y and (1) is referred to as a *λ-nuclear representation* for T. We will make use of some basic properties of λ-nuclear operators that have been discussed in sections (1.1) and (1.2) of [3].

All sequence spaces will be assumed to include ϕ, the set of finitely nonzero sequences, and to be *solid,* which means that $a \in \lambda$ if $b \in \lambda$ and $|a_i| \le |b_i|$ for all i. Recall that a sequence space is a BK-*space* if it is a Banach space and each of the coordinate maps $a \to a_i$ is continuous. A sequence space λ is an AK-*space* if it is a topological vector space and $x = \lim_n P_n x$ for each $x \in \lambda$, where $P_n x = (x_1, x_2, \cdots, x_n, 0, \cdots)$. We say that λ is *perfect* if $\lambda = \lambda^{\times\times}$. The abbreviation $\lambda\mu$ will be used for the set of products $\{a_i b_i\}_{i=1}^\infty$ formed by taking $a \in \lambda$ and $b \in \mu$. We say that λ is *μ-invariant* if $\lambda = \mu\lambda$. Finally, c_0 denotes the BK-AK-space of sequences convergent to zero; ℓ_∞ is the BK-space of bounded sequences. Both have sup norm.

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3. λ A BK-SPACE

It is known [7, p. 52] that every nuclear operator is the limit (in operator norm) of a sequence of finite rank operators and hence must be compact. A similar argument is employed in the following proposition.

PROPOSITION. *If* λ *is a* BK-AK-*space, then every* λ*-nuclear operator is compact.*

Proof. If T is an operator with λ-nuclear representation

$$Tx = \sum_{n=1}^{\infty} a_n \langle x, f_n \rangle y_n,$$

let T_n be the finite rank operator defined by

$$T_n x = \sum_{i=1}^{n} a_i \langle x, f_i \rangle y_i .$$

Then

$$\|T - T_n\| = \sup_{\|x\| \leq 1,\ \|g\| \leq 1} \left| \sum_{i=n+1}^{\infty} a_i \langle x, f_i \rangle \langle y_i, g \rangle \right|$$

$$\leq \sup_i \|f_i\| \, \| \, |a| - P_n |a| \, \| \sup_{\|g\| \leq 1} \| \{ | \langle y_i, g \rangle | \}_{i=1}^{\infty} \|_{\lambda^\times} .$$

The result now follows from the fact that $\lim_n \| \, |a| - P_n |a| \, \|_\lambda = 0$, while the collection $\{\{ | \langle y_i, g \rangle | \}_{i=1}^{\infty} : \|g\| \leq 1\}$ is pointwise bounded (Lemma 1.3 in [3]) and therefore norm bounded in $\lambda^\times$.

In [3] Dubinsky and Ramanujan remove the reference to a norm on λ and show that every λ-nuclear operator is compact if the Mackey topology $\tau(\lambda, \lambda^\times)$ is barrelled. Under the assumption that λ is a BK-space, the next theorem shows that this condition is also a necessary one. Further equivalent conditions are given. Compactness is verified using a criterion due to Terzioğlu [10] rather than the technique of finite rank operators.

THEOREM 1. *If* λ *is a* BK-*space contained in* c_0, *then the following conditions are equivalent:*

(i) λ *is* c_0-*invariant;*

(ii) λ *is an* AK-*space;*

(iii) $\tau(\lambda, \lambda^\times)$ *is barrelled;*

(iv) *every* λ*-nuclear operator is compact;*

(v) λ *is separable.*

If, in addition, λ *is perfect, then the following conditions are equivalent to those above:*

(vi) λ *contains no isomorphic copy of* ℓ_∞;

(vii) λ *contains no isomorphic copy of* c_0 ;

(viii) $\overline{\phi}_\lambda$ *contains no isomorphic copy of* c_0, *where* $\overline{\phi}_\lambda$ *denotes the closure of* ϕ *in* λ.

Proof. (i) $\Leftrightarrow$ (ii). This is a result of Garling [4].

(ii) $\Rightarrow$ (iii). Assuming (ii), the dual λ' and the Köthe dual $\lambda^\times$ are identified under the correspondence $f \leftrightarrow \{f(e^i)\}_{i=1}^\infty$. Now (iii) is a well known fact concerning Banach spaces [8, p. 67].

(iii) $\Rightarrow$ (ii). If $\tau(\lambda, \lambda^\times)$ is barrelled, then the closed graph theorem [8, p. 116] may be applied to the identity map $(\lambda, \tau(\lambda, \lambda^\times)) \to (\lambda, \|\cdot\|)$, showing that $\tau(\lambda, \lambda^\times)$ refines the norm topology on λ. But $(\lambda, \tau(\lambda, \lambda^\times))$ is an AK-space (Proposition 2 in [1]), so $(\lambda, \|\cdot\|)$ must also be an AK-space.

(ii) $\Rightarrow$ (v). Garling [4] shows that (ii) implies $\lambda = \overline{\phi}_\lambda$. It follows that λ is separable.

(v) $\Rightarrow$ (ii). We proceed exactly as for (iii) $\Rightarrow$ (ii), using the closed graph theorem as it appears in [5] (Theorems 2.4 and 2.6) and the fact that $\lambda^\times$ is $\tau(\lambda^\times, \lambda)$ sequentially complete (Proposition 3 in [1]).

(i) $\Rightarrow$ (iv). Assuming that λ is c_0-invariant, we consider any λ-nuclear operator $T: X \to Y$ with λ-nuclear representation

$$Tx = \sum_{n=1}^\infty a_n \langle x, f_n \rangle y_n .$$

Using the main result in [10], we can conclude that T is compact if we exhibit a sequence $\{h_n\}_{n=1}^\infty$ in X' satisfying $\lim_n \|h_n\| = 0$ and

$$\|Tx\| \leq \sup_n |\langle x, h_n \rangle| \qquad \text{for all } x \in X.$$

To that end, choose $b \in c_0$ and $c \in \lambda$ so that $a = bc$. By Lemma (1.3) of [3], we have

$$\sup_{\|g\| \leq 1} \left| \sum_{n=1}^\infty c_n \langle y_n, g \rangle \right| < \infty . \tag{2}$$

Using s for the quantity in (2), let $h_n = b_n s f_n$. Then $\lim_n \|h_n\| = 0$. Furthermore,

$$\|Tx\| = \sup_{\|g\| \leq 1} \left| \sum_{n=1}^\infty c_n \langle x, b_n f_n \rangle \langle y_n, g \rangle \right|$$

$$\leq \sup_n |\langle x, b_n f_n \rangle| \sup_{\|g\| \leq 1} \sum_{n=1}^\infty |c_n \langle y_n, g \rangle| = \sup_n |\langle x, h_n \rangle| ,$$

as desired.

(iv) $\Rightarrow$ (i). This will be established by exhibiting a noncompact, λ-nuclear operator for any BK-space λ which fails to be c_0-invariant.

If λ is not c_0-invariant, choose $a \in (\lambda - c_0\lambda)$ and consider the operator $a: c_0 \to \lambda$ given by $a(x) = \{a_i x_i\}_{i=1}^{\infty}$. Since $c_0\lambda = \{x \in \lambda: \lim_n P_n x = x\}$ (Theorem 4 in [4]), we have

$$a(x) = \sum_{n=1}^{\infty} a_n \langle x, e^n \rangle e^n, \tag{3}$$

where e^n denotes the sequence having one in the nth position and zero elsewhere. Moreover, (3) is a λ-nuclear representation for the operator a, since $a \in \lambda$, $\sup_n \|e^n\|_{c_0'} = 1$, and for all $f \in \lambda'$ we have $\{f(e^n)\}_{n=1}^{\infty} \in \lambda^\times$ as an easy consequence of results in [4].

Now since $a \notin c_0\lambda$, there exist an $\varepsilon > 0$ and integer sequences $\{n_i\}_{i=1}^{\infty}$, $\{m_i\}_{i=1}^{\infty}$, with $1 < m_1 < n_1 < m_2 < \cdots$ and

$$\inf_i \|P_{n_i} a - P_{m_i} a\| \geq \varepsilon .$$

Letting

$$x^i = \sum_{j=m_i+1}^{n_i} e^j ,$$

we have a sequence $\{x^i\}_{i=1}^{\infty}$ from c_0 with $\sup_i \|x^i\| = 1$. But the sequence $\{ax^i\}_{i=1}^{\infty}$ in λ is coordinatewise convergent to zero and bounded below, since $a(x^i) = P_{n_i} a - P_{m_i} a$. It follows that $\{ax^i\}_{i=1}^{\infty}$ has no (norm) convergent subsequence, and therefore the mapping a is not compact.

Assuming that λ is perfect, we proceed.

(v) $\Rightarrow$ (vi). This follows from the nonseparability of ℓ_∞.

(vi) $\Rightarrow$ (vii). Bessaga and Pełczyński prove in [2] that any dual space of a Banach space contains an isomorphic copy of ℓ_∞ whenever it contains an isomorphic copy of c_0. To apply this result, we note first that $\lambda^\times$ is a BK-space with respect to the norm given by

$$\|y\| = \sup_{\|x\|_\lambda \leq 1} \sum_{n=1}^{\infty} |x_n y_n| .$$

Then we see that λ is the dual of the BK-AK-space $\overline{\phi}_{\lambda^\times}$ because

$$\lambda = \lambda^{\times\times} = (\overline{\phi}_{\lambda^\times})^\times = (\overline{\phi}_{\lambda^\times})' .$$

(vii) $\Rightarrow$ (viii). This is immediate.

(viii) $\Rightarrow$ (ii). Bessaga and Pełczyński prove in [2] that in any Banach space not containing an isomorphic copy of c_0, every weakly unconditionally Cauchy series must be unconditionally convergent. We apply this result to the BK-AK-space $\overline{\phi}_\lambda$. If a is an arbitrary element of λ, the series $\sum_{n=1}^{\infty} a_n e^n$ is a weakly unconditionally Cauchy series of $\overline{\phi}_\lambda$ elements. Thus the series converges (to a) in $\overline{\phi}_\lambda$, showing that λ is an AK-space. This completes the proof.

4. THE RANGE SPACE

In [3] Dubinsky and Ramanujan prove that reflexivity of Y is a sufficient condition for compactness of every λ-nuclear operator into Y. This result carries no restriction on λ. The following theorem modifies their technique to show that a weaker condition suffices and is, in a sense, the best possible.

THEOREM 2. *If* λ *is a sequence space and* Y *is a Banach space not containing an isomorphic copy of* c_0, *then every* λ*-nuclear mapping into* Y *is compact. Conversely, if* Y *contains an isomorphic copy of* c_0, *then there is a Banach space* X, *a sequence space* λ, *and a noncompact,* λ*-nuclear mapping* T: X → Y.

Proof. Assume that T: X → Y has λ-nuclear representation

$$Tx = \sum_{n=1}^{\infty} a_n \langle x, f_n \rangle y_n,$$

and that Y contains no isomorphic copy of c_0. Following Theorem 1.3 of [3], we define adjoint maps $\mu: Y' \to \lambda^\times$ and $\nu: \lambda \to Y''$ by the equations:

$$\mu(g) = \{ \langle y_n, g \rangle \}_{n=1}^{\infty}, \tag{4}$$

$$\nu(a)(g) = \sum_{n=1}^{\infty} a_n \langle y_n, g \rangle . \tag{5}$$

Now the series $\sum_{n=1}^{\infty} a_n y_n$ in Y is weakly unconditionally Cauchy for each $a \in \lambda$, and therefore must be unconditionally convergent (Theorem 5 in [2]). Thus the range of ν is contained in Y, and μ must be continuous with respect to the weak topologies $\sigma(Y', Y)$ and $\sigma(\lambda^\times, \lambda)$ [9, p. 128]. It now follows that μ(V) is a $\sigma(\lambda^\times, \lambda)$ compact set, where V denotes the unit ball in Y'. This implies that

$$\lim_n \sup_{b \in \mu(V)} \sum_{k=n+1}^{\infty} |a_k b_k| = 0 \tag{6}$$

(Lemma 1.2 in [3]). Using (4) and (5), we rewrite (6) as

$$\lim_n \sup_{\|g\| \leq 1} \sum_{k=n+1}^{\infty} |a_k \langle y_k, g \rangle| = 0. \tag{7}$$

Letting T_n denote the finite rank operator

$$T_n x = \sum_{i=1}^{n} a_i \langle x, f_i \rangle y_i,$$

we have

$$\|T - T_n\| = \sup_{\|x\| \leq 1,\ \|g\| \leq 1} \left| \sum_{k=n+1}^{\infty} a_k \langle x, f_k \rangle \langle y_k, g \rangle \right| \tag{8}$$

$$\leq \sup_i \|f_i\| \sup_{\|g\| \leq 1} \sum_{k=n+1}^{\infty} |a_k \langle y_k, g \rangle| \ .$$

Using (7) we see from (8) that $\lim_n \|T - T_n\| = 0$, and therefore T is compact.

For the converse statement, we first observe that the inclusion map I: $\ell_1 \to c_0$ has an ℓ_∞-nuclear representation

$$Ix = \sum_{n=1}^{\infty} \langle x, e^n \rangle e^n ,$$

but is not a compact operator. Thus if Y is a Banach space and S: $c_0 \to Y$ is an isomorphism, then $S \circ I$: $\ell_1 \to Y$ is a noncompact, ℓ_∞-nuclear mapping into Y. This completes the proof.

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