GROWTH OF NUMERICAL RANGES OF POWERS OF HILBERT SPACE OPERATORS

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1. INTRODUCTION

Let $\mathscr{H}$ denote a complex Hilbert space with inner product $(\cdot, \cdot)$, and let $\mathscr{B}(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, let $\sigma(T)$ denote the spectrum of T, and let W(T) denote the numerical range of T, $W(T) = \{(Tx, x): x \in \mathscr{H}, \|x\| = 1\}$.

The power inequality $|W(T^n)| \leq |W(T)|^n$ for numerical radii was first proved by C. A. Berger ([1], [4, Problem 176]). This inequality gives an estimate of the maximum rate of growth of the numerical ranges of the powers of an operator. In this paper, we study the minimum rate of growth of the numerical ranges of the powers of non-Hermitian operators. Using von Neumann's theory of spectral sets, we prove that *if* $\sigma(T) \subset (\gamma, \infty)$ *with* $\gamma > 0$, *and if* T *is not Hermitian, then there is a positive integer* n_0 *such that* $\{z \in \mathbb{C}: |z| \leq \gamma^n\} \subset W(T^n)$ *whenever* $n \geq n_0$. In particular, if $\sigma(T) \subset (1, \infty)$, then either T is Hermitian, or for each bounded set Ω of complex numbers, there exists an integer $n_0(\Omega)$ such that $\Omega \subset W(T^n)$ whenever $n \geq n_0(\Omega)$. In the last part of this paper, we show that $\Re(T^n x, x) \geq 0$ for all $x \in \mathscr{H}$ and for $n = 1, 2, \cdots, k$, if and only if the closed sector

$$\{z \in \mathbb{C}: |\mathrm{Arg}\, z| \leq \pi/2k\} \cup \{0\}$$

is spectral for T; moreover, $\|\Im Tx\| \leq \tan(\pi/2k) \|\Re Tx\|$ for all $x \in \mathscr{H}$.

2. NOTATION AND PRELIMINARIES

Let $\mathbb{C}$ denote the set of complex numbers, and $\mathbb{R}$ the set of real numbers. For $\Omega \subset \mathbb{C}$, we denote by $\mathrm{Co}(\Omega)$ the convex hull, by $\mathrm{Cl}(\Omega)$ the closure, by $\mathrm{Int}(\Omega)$ the interior of Ω, and by $\Re(\Omega)$ the set $\{(z + \bar{z})/2: z \in \Omega\}$. For $\gamma \in \mathbb{R}$, we write $\Omega \geq \gamma$ (or $\Omega > \gamma$) if $\Omega \subset \mathbb{R}$ and each number λ in Ω satisfies the condition $\lambda \geq \gamma$ (or $\lambda > \gamma$). Let $\Delta(r) = \{z \in \mathbb{C}: |z| \leq r\}$. Let $\Sigma(\phi)$ denote the closed sector of the complex plane symmetric with respect to the real axis, with vertex at the origin, and with angular opening 2ϕ; and note that $\Sigma(\pi/2)$ denotes the right half-plane.

For $T \in \mathscr{B}(\mathscr{H})$, $\Re T = (T + T^*)/2$ and $\Im T = (T - T^*)/2i$. We say T is *positive* and write $T \geq 0$ if $W(T) \geq 0$; if T is also invertible, we write $T > 0$. By $T_1 \geq (>)\, T_2$, we mean $(T_1 - T_2) \geq (>)\, 0$. We say that T is *accretive* if $\Re T \geq 0$, *strictly accretive* if $\Re T > 0$.

Let $T \in \mathscr{B}(\mathscr{H})$, and let Ω be an open set of complex numbers containing $\sigma(T)$, whose boundary $\partial\Omega$ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense. If f is a function analytic on some neighborhood of $\mathrm{Cl}(\Omega)$, then we define f(T) by the Riesz-Dunford integral [3, Chapter VII]

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$$f(T) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\lambda)(\lambda - T)^{-1}\, d\lambda . \tag{1}$$

If $\Re(\sigma(T)) > 0$ and $\beta \in \mathbb{R}$, we define $T^\beta \in \mathscr{B}(\mathscr{H})$ by (1) with $f(\lambda) = \lambda^\beta$, $f(1) = 1$, and with $\Omega \subset \mathrm{Int}(\Sigma(\pi/2))$.

The following lemma gives a sufficient condition for two nth roots of an operator to be equal [5, Section 4].

LEMMA 1. *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *be such that* $\sigma(A) \cap \omega^j \sigma(B) = \emptyset$ $(1 \le j \le n-1$, $\omega = \exp(2\pi i/n))$. *If* $A^n = B^n$, *then* $A = B$.

Proof. There exists an open neighborhood of $\sigma(A) = \sigma(B)$, not containing 0, on which the function z^n is one-to-one. Apply [3, Chapter VII, Theorem 3.12].

3. MAPPINGS OF SPECTRAL SETS

Let $T \in \mathscr{B}(\mathscr{H})$, and let Λ be a closed subset of $\mathbb{C}$ containing $\sigma(T)$. We say that Λ is spectral (in the sense of von Neumann) for T, if whenever q is a rational complex-valued function with poles outside Λ, $\|q(T)\| \le \sup_{\lambda\in\Lambda} |q(\lambda)|$. See [11, Chapter XI] for details. We list some properties of spectral sets: (i) If Λ is spectral for T, then $\mathrm{Co}(\Lambda) \supset W(T)$ and each closed set containing Λ is spectral for T; (ii) $\Sigma(\pi/2)$ is spectral for T if and only if T is accretive; (iii) $\mathbb{R}$ is spectral for T if and only if T is Hermitian.

If Λ and Λ_n $(n = 1, 2, 3, \cdots)$ are closed convex subsets of $\mathbb{C}$ such that $\Lambda_n \supset \Lambda$, we say Λ_n tends to Λ $(\Lambda_n \to \Lambda)$ whenever for each $\varepsilon > 0$ and each compact set K, there exists an integer $n_0(\varepsilon, K)$ such that $n \ge n_0(\varepsilon, K)$ implies

$$\Lambda_n \cap K \subset (\Lambda \cap K) + \Delta(\varepsilon) .$$

The following two results about spectral sets are proved in [2]. They are the principal tools in this paper.

LEMMA 2. *Let* f *be an analytic function in* $\mathrm{Int}(\Sigma(\pi/2))$, *and suppose* T *is accretive and* $\Re(\sigma(T)) > 0$. *Then* $\mathrm{Cl}(\mathrm{Co}(f(\mathrm{Int}(\Sigma(\pi/2)))))$ *is spectral for* f(T).

LEMMA 3. *Let* Λ *and* Λ_n $(n = 1, 2, 3, \cdots)$ *be closed convex subsets of* $\mathbb{C}$ *such that* $\Lambda_n \supset \Lambda$ *and* $\Lambda_n \to \Lambda$. *Let* $T_n \in \mathscr{B}(\mathscr{H})$, *with* Λ_n *spectral for* T_n. *If* $T_n \to T$ *(in the uniform operator topology), then* Λ *is spectral for* T.

4. GROWTH OF NUMERICAL RANGES OF POWERS

The following theorem shows that the numerical ranges of large powers of a non-Hermitian operator with positive spectrum must have a certain minimum rate of growth.

THEOREM 1. *Let* $T \in \mathscr{B}(\mathscr{H})$ *and* $\sigma(T) > \gamma > 0$. *If* $\Delta(\gamma^n) \not\subset W(T^n)$ *for infinitely many positive integers* n, *then* $T > \gamma I$.

Proof. Since there exists some positive number ε such that $\sigma(T) > \gamma + \varepsilon$, we may assume $\Delta(\gamma^n) \not\subset \mathrm{Cl}(W(T^n))$ for infinitely many integers n. Let M be an infinite sequence of positive integers such that for each $m \in M$, there is a complex number k_m satisfying the conditions $|k_m| \le \gamma^m$ and $k_m \notin \mathrm{Cl}(W(T^m))$. We shall show that T is Hermitian.

For each $m \in M$, $0 \notin \mathrm{Cl}(W(T^m - k_m))$; by the convexity of the numerical range [4, Problem 166], there is a number $\theta_m \in (-\pi, \pi)$ such that $A_m = e^{i\theta_m}(T^m - k_m)$ is strictly accretive. By Lemma 2, the closed sector $\Sigma(\pi/2m)$ is spectral for $A_m^{1/m}$.

Since $\limsup_M \|k_m T^{-m}\|^{1/m} \leq \gamma/|\sigma(T)| < 1$, where $|\sigma(T)|$ denotes the spectral radius of T, we see that $\lim_M \|k_m T^{-m}\| = 0$. Let M_1 be the subsequence $\{m \in M: \|k_m T^{-m}\| < 1/2\}$. For each $m \in M_1$, put $B_m = (I - k_m T^{-m})$; then $B_m^{1/m}$ exists and

$$B_m^{1/m} - I = B_m^{1/m} - I^{1/m} = \frac{1}{2\pi i}\int_\Gamma \lambda^{1/m}((\lambda - B_m)^{-1} - (\lambda - I)^{-1})\, d\lambda,$$

where Γ is the positively oriented circle centered at 1 with radius 1/2. There exists a constant C, independent of m, such that

$$\|B_m^{1/m} - I\| \leq C \sup_{\lambda \in \Gamma} \|(\lambda - B_m)^{-1} - (\lambda - I)^{-1}\|.$$

Applying [3, Chapter VII, Lemma 6.3], we obtain the relation $\lim_{M_1} \|B_m^{1/m} - I\| = 0$. Thus

$$\lim_{M_1} \|e^{i\theta_m/m} B_m^{1/m} - I\| = 0.$$

Consequently, if we show that $e^{i\theta_m/m} B_m^{1/m} T = A_m^{1/m}$, then the positivity of T follows immediately, by Lemma 3. Since $\sigma(T) > 0$, it follows from Lemma 1 that $A_m^{1/m} = (e^{i\theta_m} B_m)^{1/m} T$. Using the facts that $\theta_m \in (-\pi, \pi)$, $\Re(\sigma(B_m)) > 0$, and $\Re(\sigma(e^{i\theta_m} B_m)) > 0$, we get the identity $(e^{i\theta_m} B_m)^{1/m} = e^{i\theta_m/m} B_m^{1/m}$.

The following is an immediate consequence of Theorem 1.

COROLLARY 1. *Let* $T \in \mathscr{B}(\mathscr{H})$ *with* $\sigma(T) > 0$. *If* $0 \notin \mathrm{Int}(W(T^n))$ *for infinitely many integers* n, *then* $T > 0$.

We note that the methods used above are not applicable to a singular operator with real nonnegative spectrum. However, the following is proved in [12].

THEOREM 2. *Let* $T \in \mathscr{B}(\mathscr{H})$. *Suppose* T *is singular*, $\sigma(T) \geq 0$, *and* $0 \notin \mathrm{Int}(W(T^n))$ $(n = 1, 2, 3, \cdots)$. *If* 0 *is an isolated point of* $\sigma(T)$, *then* $T \geq 0$.

We do not know whether we can omit the condition "0 is an isolated point of $\sigma(T)$" in Theorem 2.

Theorem 1 has the following generalization.

THEOREM 1'. *Let* $T, D \in \mathscr{B}(\mathscr{H})$, *with* $\sigma(T) > \gamma > 0$ *and* $TD = DT$. *Suppose there are infinitely many integers* n *such that* $\Delta(\gamma^n) \not\subset W(T^n D)$. *Then* (i) *there exists some real number* θ *such that* $\sigma(D) \subset e^{i\theta}\Sigma(\pi/2)$, *and* (ii) $T > \gamma I$ *if* D *is invertible.*

We remark that because of (i), D is invertible whenever D is surjective. The proof of Theorem 1' is given in [12], and it is more technical than that of Theorem 1.

It is clear that the proof of Theorem 1 does not extend to the case of an arbitrary Banach algebra. In fact, Theorem 1 does not hold in the general setting; for a counterexample, see [2, Section 3.(1)]. However, the above results remain valid in C^*-algebras because of the Gelfand-Naimark theorem that every C^*-algebra has a faithful $*$-representation as a closed self-adjoint subalgebra of $\mathscr{B}(\mathscr{H})$ for some suitable Hilbert space $\mathscr{H}$. The following theorem is obtained by working in the Calkin algebra (see [12], [13, Section 4]).

THEOREM 1''. *Let* $T \in \mathscr{B}(\mathscr{H})$ *and* $\bigcap\{\sigma(T+K): K$ *compact*$\} > \gamma > 0$. *If* $\Im T$ *is not a compact operator, then there is a positive integer* n_0 *such that* $\Delta(\gamma^n) \subset W(T^n + K)$ *for every integer* $n \geq n_0$ *and for every compact operator* K.

5. OPERATORS WITH ACCRETIVE POWERS

THEOREM 3. *Let* $T \in \mathscr{B}(\mathscr{H})$. *Then* T^n *is accretive* $(n = 1, 2, \cdots, k)$ *if and only if* $\Sigma(\pi/2k)$ *is spectral for* T.

Proof. The sufficiency follows from the definition of a spectral set. If T^n is accretive $(n = 1, 2, \cdots, k)$, then $\Sigma(\pi/2k) \supset \sigma(T)$; by Lemma 1, $((T+\gamma)^k)^{1/k} = T + \gamma$, for each $\gamma > 0$. Thus $\Sigma(\pi/2k)$ is spectral for $T + \gamma$ by Lemma 2. We let γ tend to 0 and apply Lemma 3.

Theorem 3 has a "dual" version. Given an accretive operator $A \in \mathscr{B}(\mathscr{H})$ and a number $\alpha \in (0, 1)$, we define the fractional power A^α by

$$A^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}(A+\lambda)^{-1} Ax \, d\lambda \tag{2}$$

for each $x \in \mathscr{H}$. The integral in (2) is convergent in the Bochner or absolute sense; and if $\Re(\sigma(A)) > 0$, then the fractional power defined by (2) is the same as the one defined by (1). Furthermore, $\lim_{\gamma \to 0^+} \|(A+\gamma)^\alpha - A^\alpha\| = 0$. See [8, Chapter V, Section 3.11].

THEOREM 3'. *For an accretive operator* $A \in \mathscr{B}(\mathscr{H})$ *and a positive integer* k, *there exists a unique operator* B *such that* $A = B^k$ *and* $\Sigma(\pi/2k)$ *is spectral for* B.

Theorem 3' generalizes a theorem of V. I. Macaev and Ju. A. Palant [10]. See also [9] and [14, Proposition 5.5].

THEOREM 4. *Let* $T \in \mathscr{B}(\mathscr{H})$. *If* T^n *is accretive* $(n = 1, 2, \cdots, k)$, *then* $\|\Im Tx\| \leq \tan(\pi/2k)\, \|\Re Tx\|$ *for each* $x \in \mathscr{H}$.

Proof. Apply [7, Theorem 1.1].

The following is an immediate consequence of Theorem 3 or Theorem 4.

THEOREM 5 ([6], [2]). *Let* $T \in \mathscr{B}(\mathscr{H})$. *Then* $T \geq 0$ *if and only if* T^n *is accretive* $(n = 1, 2, 3, \cdots)$.

It is interesting to note that there is an elementary proof for Theorem 5. We conclude this paper with this proof.

LEMMA 4. *Let* $A, B \in \mathscr{B}(\mathscr{H})$. *If* $A = A^*$, $B \geq 0$, *and* $B^2 \geq A^2$, *then* $B \geq A$.

Proof. Pick $\lambda < 0$ and $x \in \mathscr{H}$, with $\|x\| = 1$. Then

$$\|(B-\lambda)x\|^2 - \|Ax\|^2 = ((B^2 - A^2)x, x) - 2\lambda(Bx, x) + \lambda^2 \geq \lambda^2 .$$

Hence

$$\|(B - A - \lambda)x\| \geq \|(B - \lambda)x\| - \|Ax\| \geq \frac{\lambda^2}{\|B - \lambda\| + \|A\|} > 0.$$

Since $B - A$ is Hermitian, we see that $\sigma(B - A) \geq 0$. Consequently, $B \geq A$.

LEMMA 5. *Let* $T \in \mathcal{B}(\mathcal{H})$. *If* T *and* T^2 *are accretive, then* $W(T) \subset \Sigma(\pi/4)$.

Proof. $(((\Re T)^2 - (\Im T)^2)x, x) = 1/2(T^2x + T^{*2}x, x) = \Re(T^2x, x)$. Hence T^2 is accretive if and only if $(\Re T)^2 \geq (\Im T)^2$. By Lemma 4 and the fact that T is accretive, we get the inequalities $\Re T \geq \Im T$ and $\Re T \geq -\Im T$. Therefore, $W(T) \subset \Sigma(\pi/4)$.

COROLLARY 2. *Let* $T \in \mathcal{B}(\mathcal{H})$. *If* T *is accretive*, $0 \leq \alpha < 1$, *and* $W(T^2) \subset \Sigma(\alpha\pi/2)$, *then* $W(T) \subset \Sigma(\alpha\pi/4)$.

Proof. By Lemma 5, the operators $\exp(\pm i\pi/4)T$ are accretive. Hence the operators $\exp(\pm i(1 - \alpha)\pi/4)T$ are accretive. The hypothesis $W(T^2) \subset \Sigma(\alpha\pi/2)$ implies that the operators $\exp(\pm i(1 - \alpha)\pi/2)T^2$ are accretive. Applying Lemma 5 again, we deduce that both $\exp(i\pi/4)(\exp(i(1 - \alpha)\pi/4)T)$ and

$$\exp(-i\pi/4)(\exp(- i(1 - \alpha)\pi/4)T)$$

are accretive. Therefore, the operators $\exp(\pm i(1 - \alpha/2)\pi/2)T$ are accretive and $W(T) \subset \Sigma(\alpha\pi/4)$.

THEOREM 5'. *Let* $T \in \mathcal{B}(\mathcal{H})$. *If* T^{2^n} *is accretive for* $n = 0, 1, 2, \cdots$, *then* $T \geq 0$.

Proof. Corollary 2 shows that if T^{2^n} is accretive $(n = 0, 1, \cdots, k)$, then $W(T) \subset \Sigma(\pi/2^{k+1})$.

REFERENCES

1. C. A. Berger and J. G. Stampfli, *Mapping theorems for the numerical range.* Amer. J. Math. 89 (1967), 1047-1055. MR 36 #5744.
2. C. R. DePrima and B. K. Richard, *A characterization of the positive cone of* B(𝔥). Indiana Univ. Math. J. 23 (1973), 163-172. MR 47 #4052.
3. N. Dunford and J. T. Schwartz, *Linear Operators, Part I. General Theory.* Interscience, New York, 1958. MR 22 #8302.
4. P. R. Halmos, *A Hilbert space problem book.* Van Nostrand, Princeton, N.J., 1967. MR 34 #8178.
5. E. Hille, *On roots and logarithms of elements of a complex Banach algebra.* Math. Ann. 136 (1958), 46-57. MR 20 #2632.
6. C. R. Johnson, *Powers of matrices with positive definite real part.* Proc. Amer. Math. Soc. 50 (1975), 85-91. MR 51 #5628.
7. T. Kato, *Fractional powers of dissipative operators.* J. Math. Soc. Japan 13 (1961), 246-274. MR 25 #1453.
8. ———, *Perturbation theory for linear operators.* Grundlehren, Band 132. Springer-Verlag, New York, 1966. MR 34 #3324.

9. H. Langer, *Über die Wurzeln eines maximalen dissipativen Operators.* Acta Math. Acad. Sci. Hungar. 13 (1962), 415-424. MR 26 #1757.

10. V. I. Macaev and Ju. A. Palant, *On the powers of a bounded dissipative operator.* (Russian) Ukrain. Mat. Ž. 14 (1962), 329-337. MR 26 #4184.

11. F. Riesz and B. Sz.-Nagy, *Functional analysis.* Ungar, New York, 1955. MR 17, p. 175.

12. E. S. W. Shiu, *Numerical ranges of powers of operators.* Thesis, California Institute of Technology, 1975.

13. ———, *Commutators and numerical ranges of powers of operators.* Pacific J. Math., to appear.

14. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space.* Akadémiai Kiadó, Budapest; American Elsevier Publ. Co., New York, 1970. MR 43 #947.

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