A LOCAL FORM OF LAPPAN'S FIVE-POINT THEOREM FOR NORMAL FUNCTIONS

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A necessary and sufficient condition for a meromorphic function f to be normal in the unit disc D is that

$$\sup_{z \in D} f^{\#}(z)(1 - |z|^2) < \infty, \qquad \text{where } f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

(see [3]). Ch. Pommerenke [5] asked whether there exists a class of sets $\mathscr{E}$ such that if there is a set $E \in \mathscr{E}$ for which the quantity $\sup_{z \in f^{-1}(E)} f^{\#}(z)(1 - |z|^2)$ is sufficiently small then f is a normal function. P. Lappan's beautiful answer [2, Theorem 2] shows that f is a normal function in D if and only if for some five-point subset A of the extended plane Ω, $\sup_{z \in f^{-1}(A)} f^{\#}(z)(1 - |z|^2) < \infty$. This result has an elegant proof which combines a result of A. J. Lohwater and Ch. Pommerenke [4, Theorem 1] with the Nevanlinna theory on completely ramified values of a function meromorphic in the finite plane W. Our contribution is to make precise and to generalize slightly the result of Lohwater and Pommerenke, and this in turn leads to an extended form of Lappan's theorem.

Before proving the extended Lohwater-Pommerenke theorem we attend to a few details. Let ρ be the hyperbolic distance on D. For $a \in D$ and $r > 0$, let $D(a, r) = \{z \in D: |z - a| < r\}$ and $N(a, r) = \{z \in D: \rho(a, z) < r\}$. Note that if $\{z_n\}$ and $\{z_n'\}$ are two sequences in D such that $\rho(z_n, z_n') \to 0$ $(n \to \infty)$, then

$$(0) \qquad (1 - |z_n|)/(1 - |z_n'|) \to 1 \qquad (n \to \infty).$$

In the sequel we use the Hardy-Littlewood "o" notation.

THEOREM 1. *Let* f *be meromorphic in* D *and* $\alpha \geq 1$. *The following two statements are equivalent.*

(i) *There exists a sequence* $\{z_n\}$ *such that*

$$\lim_{n \to \infty} f^{\#}(z_n)(1 - |z_n|^2)^{\alpha} = \infty.$$

(ii) *There exist a sequence* $\{\zeta_n\}$ *and a sequence of positive numbers* $\{\varepsilon_n\}$ *satisfying*

$$(1) \qquad \varepsilon_n = o(1 - |\zeta_n|^2)^{\alpha} \qquad (n \to \infty),$$

Received June 13, 1975. Revisions received December 17, 1975 and April 12, 1976.

This research was conducted while the author was on sabbatical leave from The Pennsylvania State University, and was supported in part by a Canadian Research Council grant administered by Carleton University, Ottawa.

Michigan Math. J. 23 (1976).

such that the sequence of functions $\{g_n(t)\} = \{f(\zeta_n + \varepsilon_n t)\}$ *is a normal family in* W *with no constant limit functions.*

If either statement is true, the sequences $\{z_n\}$ *and* $\{\zeta_n\}$ *can be chosen such that*

$$\rho(z_n, \zeta_n) = o(1 - |z_n|^2)^{\alpha - 1}. \tag{2}$$

Proof. The proof is basically that used by Lohwater-Pommerenke with a minor modification. First we show (i) $\Rightarrow$ (ii), and so we suppose

$$f^{\#}(z_n)(1 - |z_n|^2)^{\alpha} = A_n \to \infty \quad \text{as } n \to \infty.$$

Letting $r_n = (1 - |z_n|^2)^{\alpha}/\sqrt{A_n}$ and noting that $d\rho = |dz|/(1 - |z|^2)$, we easily show that the discs $D(z_n, r_n)$ have hyperbolic diameters d_n satisfying

$$d_n = o(1 - |z_n|^2)^{\alpha - 1} \qquad (n \to \infty).$$

We are going to find the companion sequence $\{\zeta_n\}$ in the discs $D(z_n, r_n)$, and then (2) will hold.

In $D(z_n, r_n)$ consider the continuous function

$$f^{\#}(z)(r_n - |z - z_n|), \tag{3}$$

which is zero on the boundary of $D(z_n, r_n)$ and equals $\sqrt{A_n}$ at $z = z_n$. Let ζ_n be a point in $D(z_n, r_n)$ such that

$$f^{\#}(\zeta_n)(r_n - |\zeta_n - z_n|) = \max_{z \in D(z_n, r_n)} f^{\#}(z)(r_n - |z - z_n|).$$

Then we have

$$\sqrt{A_n} \leq f^{\#}(\zeta_n)(r_n - |\zeta_n - z_n|) = M_n, \tag{4}$$

and $M_n \to \infty$ as $n \to \infty$. Also $\varepsilon_n = (f^{\#}(\zeta_n))^{-1} \to 0$ as $n \to \infty$. In addition, if $z \in D(z_n, r_n)$,

$$f^{\#}(z) \leq f^{\#}(\zeta_n)\left(\frac{r_n - |\zeta_n - z_n|}{r_n - |z - z_n|}\right). \tag{5}$$

We need $\zeta_n + \varepsilon_n t \in D(z_n, r_n)$, and this occurs if

$$|t| < \frac{r_n - |\zeta_n - z_n|}{\varepsilon_n} = M_n.$$

Fix $R > 0$ and suppose $|t| < R$. For all n such that $M_n > R$,

$$z = \zeta_n + \varepsilon_n t \in D(z_n, r_n).$$

Thus (5) and the definition of M_n combine to give

$$g_n^{\#}(t) = \varepsilon_n f^{\#}(\zeta_n + \varepsilon_n t) \leq \frac{1}{1 - \frac{R}{M_n}}. \tag{6}$$

Notice also

$$g_n^{\#}(0) = \varepsilon_n f^{\#}(\zeta_n) = 1, \quad \text{for all } n. \tag{7}$$

By Marty's criterion, $\{g_n(t)\}$ is a normal family in W. For any subsequence $\{g_{n_i}(t)\}$ which converges locally uniformly to $g(t)$ on W, $\{g_{n_i}^{\#}(t)\}$ converges locally uniformly to $g^{\#}(t)$ also, so that by (7), $g(t)$ is meromorphic and nonconstant, and by (6), is of order ≤ 2.

It follows from (4) that $\sqrt{A_n} < f^{\#}(\zeta_n) r_n$, and since $r_n = (1 - |z_n|^2)^{\alpha}/\sqrt{A_n}$,

$$\varepsilon_n = (f^{\#}(\zeta_n))^{-1} < (1 - |z_n|^2)^{\alpha}/A_n .$$

This last inequality, together with (2) and (0), yields (1). This completes the first part of the proof.

The reverse implication is essentially a backwards glance at the above, but for future considerations we go into some detail. Suppose $\{f(\zeta_n + \varepsilon_n t)\} = \{g_n(t)\}$ has a convergent subsequence which we relabel as $\{g_n(t)\}$. Let $g(t)$ be the nonconstant limit function. That the following are true is an easy exercise:

(a) $g(t)$ *is meromorphic in* D;

(8) (b) $\{g_n^{\#}(t)\}$ *converges locally uniformly to* $g^{\#}(t)$;

(c) *If* $\lambda \in g(W)$ *and* $g(t_\lambda) = \lambda$, *then there exists a sequence of points* $\{t_n\}$, *where* $t_n \to t_\lambda$ *as* $n \to \infty$, *such that*

$$f^{\#}(z_n^{(\lambda)})\,\varepsilon_n = g_n^{\#}(t_n) \to g^{\#}(t_\lambda) \text{ as } n \to \infty \quad \text{and} \quad f(z_n^{(\lambda)}) = g_n(t_n) = \lambda,$$

where $z_n^{(\lambda)} = \zeta_n + \varepsilon_n t_n$.

Now suppose a λ is selected for which $g^{\#}(t_\lambda) \neq 0$. The proof is complete if we look at (8) (c) and combine (1) and (0) to show

$$\lim_{n \to \infty} f^{\#}(z_n^{(\lambda)})(1 - |z_n^{(\lambda)}|^2)^{\alpha} = \lim_{n \to \infty} \frac{g_n^{\#}(t_n)}{\varepsilon_n}(1 - |\zeta_n|^2)^{\alpha}$$

$$= g^{\#}(t_\lambda) \lim_{n \to \infty} \frac{(1 - |\zeta_n|^2)^{\alpha}}{\varepsilon_n} = \infty .$$

Actually, we have proved more than (ii) $\Rightarrow$ (i). The key point of Lappan's proof is that for any $\lambda \in g(W)$ for which there is a $t_\lambda \in g^{-1}(\lambda)$ with $g^{\#}(t_\lambda) \neq 0$, there exists a sequence $\{z_n^{(\lambda)}\}$ for which $f(z_n^{(\lambda)}) = \lambda$, for all n, *and* $f^{\#}(z_n^{(\lambda)})(1 - |z_n^{(\lambda)}|^2)^{\alpha} \to \infty$. By a result from Nevanlinna theory, the nonconstant meromorphic function g can have at most four distinct values λ such that $g^{\#}(t_\lambda) = 0$ for all $t_\lambda \in g^{-1}(\lambda)$; that is, g can have at most four completely ramified values. By combining these two facts, Lappan obtained his five-point theorem. We observe here that the Nevanlinna theory actually says more: for each value g omits (no more than two), the possible number of completely ramified values actually assumed by g decreases by two.

Lohwater and Pommerenke considered the function (3) (for $\alpha = 1$) over the discs $D(0, s_n)$, where the sequence $\{s_n\}$ was chosen so that $\overline{\lim}_{n \to \infty} \rho(|z_n|, s_n) < \infty$, although this condition was not stated in terms of the hyperbolic distance. Our proof differs mostly in the choice of discs.

We define one more cluster set. If $G \subseteq D$ such that $\overline{G} \cap \partial D \neq \emptyset$, if f is a function from D into Ω, and if $\alpha \geq 1$, let $R_\alpha(f, G)$ be the set of $w \in W$ for which there exists a sequence $\{z_n\}$ with $\rho(z_n, G) = o(1 - |z_n|^2)^{\alpha-1}$ and $f(z_n) = w$ for each n. In case $G = D$, α has no significance and we write R(f). We come to an extended version of Lappan's theorem which is also a generalization to $\alpha > 1$ from Lappan's $\alpha = 1$. This theorem merely summarizes the idea contained in the proof of Theorem 1 and the remarks thereafter.

THEOREM 2. *Let* f *be meromorphic in* D *and* $G \subseteq D$ *such that* $\overline{G} \cap \partial D \neq \emptyset$. *For each fixed* $\alpha \geq 1$, *the following two statements are equivalent.*

(i) *For each sequence* $\{z_n\}$ *such that* $\rho(z_n, G) = o(1 - |z_n|^2)^{\alpha-1} < \infty$, *we have* $\overline{\lim}_{n\to\infty} f^\#(z_n)(1 - |z_n|^2)^\alpha < \infty$.

(ii) *If* $R_\alpha(f, G)$ *omits* i *values*, $0 \leq i \leq 2$, *then there exists a set* $E \subseteq R_\alpha(f, G)$ *containing* 5 - 2i *distinct elements with the property that for each sequence* $\{\zeta_n\}$ *with* $f(\zeta_n) \in E$ *and* $\rho(\zeta_n, G) = o(1 - |\zeta_n|^2)^{\alpha-1}$, *we have*

$$\overline{\lim_{n\to\infty}}\ f^\#(\zeta_n)(1 - |\zeta_n|^2)^\alpha < \infty .$$

In Theorems 1 and 2, if we restrict f to be a holomorphic function, a bit more can be said. We restrict ourselves to $\alpha = 1$ although what follows can be adapted to the more general case. Suppose there is a sequence $\{z_n\}$ with $\rho(z_n, G) \to 0$ $(n \to \infty)$, and

$$f^\#(z_n)(1 - |z_n|^2) \to \infty \qquad (n \to \infty). \tag{9}$$

Then the limit functions of the sequence $\{f(\zeta_n + \varepsilon_n t)\} = \{g_n(t)\}$ must be holomorphic. The constant function ∞ is not allowed. Consequently,

$$\{g_n'(t)\} = \{\varepsilon_n f'(\zeta_n + \varepsilon_n t)\}$$

is also a normal sequence and its limit functions are derivatives of the corresponding limit functions of $\{g_n(t)\}$. The convergence is locally uniform in both situations (relative to the Euclidean metric), and Hurwitz's theorem now applies also to the derivative sequence. If a limit function g(t) has a completely ramified value λ, then there is a sequence of points $\{\zeta_n\}$ with $\rho(\zeta_n, G) \to 0$, on which $f'(\zeta_n) = 0$ for each n. Suppose we assume f is holomorphic and locally univalent in D (that is, $f'(z) \neq 0$, $z \in D$) and satisfies (9). Then any limit function g(t) has no completely ramified values, and we can apply the remarks following the proof of Theorem 1 to conclude that for *each* $\lambda \in g(W) \subseteq R_1(f, G)$, there exists a sequence $\{z_n^{(\lambda)}\}$, satisfying

(i) $f(z_n^{(\lambda)}) = \lambda$, for all n;

(ii) $\rho(z_n^{(\lambda)}, G) \to 0$ $(n \to \infty)$;

(iii) $\lim_{n\to\infty} f^\#(z_n^{(\lambda)})(1 - |z_n^{(\lambda)}|^2) = \infty$.

We summarize this in

THEOREM 3. *Let* f *be holomorphic and locally univalent in* D *and* $G \subseteq D$ *such that* $\overline{G} \cap \partial D \neq \emptyset$. *Then the following statements are equivalent.*

(i) *There exists a value* $\lambda \in R_1(f, G)$ *with the property that for all sequences* $\{z_n^{(\lambda)}\}$ *with* $f(z_n^{(\lambda)}) = \lambda$ *and* $\rho(z_n^{(\lambda)}, G) \to 0$ $(n \to \infty)$, *we have*

$$\overline{\lim_{n\to\infty}}\ f^{\#}(z_n^{(\lambda)})(1 - |z_n^{(\lambda)}|^2) < \infty. \tag{10}$$

(ii) *For each sequence* $\{z_n\}$ *such that* $\rho(z_n, G) \to 0$ $(n \to \infty)$, *we have*

$$\overline{\lim_{n\to\infty}}\ f^{\#}(z_n)(1 - |z_n|) < \infty.$$

We give an area condition which implies statement (i) and state it for $G = D$.

COROLLARY 1. *Let* f *be holomorphic and locally univalent in* D. *Then* f *is normal in* D *if for some value* $\lambda \in R(f)$ *there is a value* $\rho_0 > 0$ *such that*

$$\sup_{z\in f^{-1}(\lambda)} \iint_{N(z,\, \rho_0)} |f'(\zeta)|^2\, dA < \infty. \tag{11}$$

Proof. It is easy to see that we can find a $t > 0$ so that for all $z \in f^{-1}(\lambda)$, $D(z, (1 - |z|^2)t) \subseteq N(z, \rho_0)$. Then for $z \in f^{-1}(\lambda)$,

$$\pi(|f'(z)|(1 - |z|^2)t)^2 \le \iint_{D(z,\, (1 - |z|^2)t)} |f'(\zeta)|^2\, dA,$$

whence (10) follows from (11).

Some condition is necessary on locally univalent functions in order that they be normal. Lappan [1] has constructed a holomorphic function f which is not normal but which is univalent in any $N(z, \rho_0)$, $z \in D$, with ρ_0 sufficiently small but independent of z. It is curious to note that by a recent result of Yamashita [6, Theorem 1], this univalent property implies that f' is normal.

The author acknowledges with thanks the many helpful suggestions of the referee.

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