ON INNER FUNCTIONS WITH B^p DERIVATIVE

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As the title suggests, this paper contains results similar to those in [2], with the spaces H^p replaced by B^p. The basic problem we consider is that of determining the B^p classes $(p > 0)$ to which the derivative ϕ' of an inner function ϕ in the unit disk belongs. Recall that the space B^p is by definition the class of functions $f(z)$ analytic in the unit disk U and satisfying

$$\|f\|_p = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})| (1 - r)^{1/p-2} \, d\theta \, dr < \infty.$$

(Here and in what follows, $d\theta$ denotes *normalized* Lebesgue measure on the unit circle.)

M. R. Cullen [8] first considered the problem of determining the B^p classes of ϕ', for ϕ a singular inner function, and he conjectured that $\phi' \notin B^{1/2}$ for such a function. Cullen's idea was to use this to prove a conjecture of J. G. Caughran and A. L. Shields [6] to the effect that $\phi' \notin H^{1/2}$. H. A. Allen and C. L. Belna [3] disproved Cullen's conjecture by giving examples of singular inner functions ϕ with $\phi' \in B^p$ for all $p < 2/3$. The conjecture that $\phi' \notin B^{2/3}$ for inner functions with singular factors then seemed reasonable (see, for instance, Caughran and Shields [7]). Finally, D. Protas [11] gave a sufficient condition for $\phi' \in B^p$ $(p > 1/2)$ for ϕ a Blaschke product. (For $p < 1/2$, we have $\phi' \in B^p$ for any inner function [9, Theorem 5].)

In this paper we prove that if ϕ has a singular factor, then $\phi' \notin B^{2/3}$. To do this we develop (in Section 1) an integrated analogue of the angular derivative, the latter having been used in [2] to prove, among other things, the $H^{1/2}$ conjecture of Caughran and Shields. The methods of Section 1 are also applied to give a sufficient condition for the relation $\phi' \in B^p$ for ϕ a singular inner function (Section 3), to give a partial converse to Protas' condition for Blaschke products (Section 4), and to show that both Protas' condition and the partial converse are "best possible" (Sections 4 and 5).

The original $H^{1/2}$ conjecture of Caughran and Shields in [6] arose in connection with problems on exceptional sets, and the solution of the $B^{2/3}$ conjecture has applications to exceptional sets, as did our solution of the $H^{1/2}$ conjecture in [2]. These applications are discussed in Section 2.

Throughout this paper, the similarity of our results with those in [2] is apparent; however, it seems unlikely that the results of the present paper can be obtained directly from those in [2]. One reason for this is our example (Lemma 2) of a Blaschke product B with $B' \in B^{2/3}$ but $B' \notin H^{1/2}$.

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PART I. INNER FUNCTIONS

1. *Preliminaries.* In [2], it was seen that the classical angular derivative of an inner function ϕ was a useful tool in determining necessary conditions for the relation $\phi' \in H^p$. We introduce here an integrated analogue of the angular derivative which bears a similar relationship to the class B^p.

Let ϕ be an inner function and $1/2 < p < 1$. We define

$$I_p(\phi, \theta) = \int_0^1 \frac{1 - |\phi(re^{i\theta})|}{1 - r} (1 - r)^{1/p-2} dr.$$

The relationship between $I_p(\phi, \theta)$ and the main problem of the present paper is given in the following theorem.

THEOREM 1. *If* ϕ *is inner and if* $1/2 < p < 1$, *then*

$$\|\phi'\|_p \leq 2 \int_0^{2\pi} I_p(\phi, \theta)\, d\theta \leq 2p(2p - 1)^{-1} \|\phi'\|_p. \tag{1}$$

Proof. The inequality

$$|\phi'(z)| \leq (1 - |\phi(z)|^2)/(1 - |z|^2) \leq 2(1 - |\phi(z)|)/(1 - |z|)$$

holds, for any function ϕ holomorphic and bounded by 1 in the unit disk [4, p. 18], and it implies the first inequality in (1). To prove the second, we note first that, since ϕ is inner,

$$1 - |\phi(re^{i\theta})| \leq \int_r^1 |\phi'(te^{i\theta})|\, dt \qquad \text{a.e.}$$

Hence we have the inequalities

$$\begin{aligned} I_p(\phi, \theta) &\leq \int_0^1 (1 - r)^{1/p-3} \int_r^1 |\phi'(te^{i\theta})|\, dt\, dr \\ &= \int_0^1 |\phi'(te^{i\theta})| \int_0^t (1 - r)^{1/p-3}\, dr\, dt \\ &= p(2p - 1)^{-1} \int_0^1 |\phi'(te^{i\theta})| \, [(1 - t)^{1/p-2} - 1]\, dt \\ &\leq p(2p - 1)^{-1} \int_0^1 |\phi'(te^{i\theta})| (1 - t)^{1/p-2}\, dt, \end{aligned}$$

and integrating with respect to θ yields the second inequality in (1).

One example of the use of $I_p(\phi, \theta)$ is the following corollary of Theorem 1.

COROLLARY 1. *Suppose* ϕ *is inner and* $1/2 < p < 1$. *Then the relation* $\phi' \in B^p$ *implies* $\psi' \in B^p$ *for any divisor* ψ *of* ϕ.

Proof. If ψ is a divisor of ϕ, we have that $|\psi(z)| \geq |\phi(z)|$ for all $|z| < 1$, and hence that

$$I_p(\psi, \theta) \leq I_p(\phi, \theta).$$

The result now follows from Theorem 1.

Next we use Theorem 1 to give a "geometric" criterion which is necessary in order that the derivative of an inner function belong to B^p. The criterion involves the $R(\delta, \gamma, \zeta)$ regions of G. T. Cargo [5]. If $\delta > 0$, $\gamma \geq 1$ and $|\zeta| = 1$, we have, by definition,

$$R(\delta, \gamma, \zeta) = \{z\colon 1 - |z| \geq \delta\,|\arg(\zeta - z)|^{\gamma}\}.$$

THEOREM 2. *Suppose that* ϕ *is inner and that* $|\phi(z)|$ *is bounded away from* 1 *in* $R(\delta, \gamma, \zeta)$, *for some* δ, γ, *and* ζ, *with* $\delta > 0$, $\gamma \geq 1$ *and* $|\zeta| = 1$. *Then*

$$\int_0^{2\pi} (1 - |\phi(re^{i\theta})|)\, d\theta \geq \varepsilon(1 - r)^{1/\gamma}$$

for some $\varepsilon > 0$. *In particular* $\phi' \notin B^{\gamma/(2\gamma-1)}$.

H. Somadasa [12] has studied conditions sufficient for $|\phi|$ to tend to 0 uniformly in a region $R(\delta, \gamma, \zeta)$; we shall refer to some of his results later in the construction of examples.

Proof. Let

$$\alpha_r = \{\theta\colon |\theta - \theta_0| < [(1 - r)/\delta]^{1/\gamma}\}$$

and suppose that $|\phi(z)| \leq \rho < 1$ in $R(\delta, \gamma, \zeta)$, where $\zeta = e^{i\theta_0}$. Then

$$\begin{aligned}\int_0^{2\pi} |\phi(re^{i\theta})|\, d\theta &\leq \int_{\alpha_r} |\phi(re^{i\theta})|\, d\theta + \int_{\alpha_r^c} |\phi(re^{i\theta})|\, d\theta \\ &\leq \rho|\alpha_r| + 1 - |\alpha_r| = 1 - (1 - \rho)|\alpha_r| \\ &= 1 - (2/\delta^{1/\gamma})(1 - \rho)(1 - r)^{1/\gamma},\end{aligned}$$

and the theorem follows.

COROLLARY 2. *If* $\phi_\lambda(z) = \exp[-\lambda(\zeta + z)/(\zeta - z)]$ *for* $\lambda > 0$ *and* $|\zeta| = 1$, *then there is an* $\varepsilon > 0$ *such that*

$$\int_0^{2\pi} (1 - |\phi_\lambda(re^{i\theta})|)\, d\theta \geq \varepsilon(1 - r)^{1/2}.$$

In particular, if ϕ *is any inner function having* ϕ_λ *as a divisor, then* $\phi' \notin B^{2/3}$.

Proof. It is enough to show that $|\phi_\lambda|$ is bounded from 1 in $R(\delta, 2, \zeta)$, for some $\delta > 0$. Now $|\phi_\lambda(z)| = \exp[-\lambda(1 - |z|^2)/|\zeta - z|^2]$, and

$$\{z: (1 - |z|^2)/|\zeta - z|^2 > 1/2\}$$

is a disk inside U and tangent to ∂U at ζ. It is not hard to see that, for suitable δ, $R(\delta, 2, \zeta)$ lies inside this disk.

2. *Inner functions.* In this section, we prove a slightly strengthened version of the $B^{2/3}$ conjecture. We begin with a simple inequality for the Poisson integral of a measure.

LEMMA 1. *If* σ *is a positive measure and* $1/2 \leq r < 1$, *and if*

$$|e^{i\theta} - e^{i\lambda}| \geq (1 - r)^{1/2}$$

for all $\lambda \in \operatorname{supp} \sigma$, *then*

$$\int_0^{2\pi} (1 - r^2)\, |e^{i\lambda} - re^{i\theta}|^{-2}\, d\sigma(\lambda) \leq 4 \int_0^{2\pi} d\sigma\ .$$

Proof. We have the inequalities

$$\begin{aligned}\int_0^{2\pi} (1 - r^2)\, |e^{i\lambda} - re^{i\theta}|^{-2}\, d\sigma(\lambda) &= \int_0^{2\pi} (1 - r^2)/[(1 - r^2) + r\, |e^{i\theta} - e^{i\lambda}|^2]\, d\sigma(\lambda) \\ &\leq \int_0^{2\pi} (1 - r^2)/[(1 - r)^2 + r(1 - r)]\, d\sigma(\lambda) \\ &\leq 2r^{-1} \int_0^{2\pi} d\sigma(\lambda)\ .\end{aligned}$$

THEOREM 3. *If* ϕ *is an inner function that is not a Blaschke product, then, for some* $\varepsilon > 0$,

$$\int_0^{2\pi} (1 - |\phi(re^{i\theta})|)\, d\theta \geq \varepsilon(1 - r)^{1/2}.$$

In particular $\phi' \notin B^{2/3}$.

Proof. By Corollary 2, we may assume ϕ has no divisor of the form $\phi_\lambda(z) = \exp[-\lambda(\zeta + z)/(\zeta - z)]$. Suppose, therefore, that ϕ has a divisor of the form

$$\psi(z) = \exp\left[- \int_0^{2\pi} (e^{i\lambda} + z)/(e^{i\lambda} - z)\, d\sigma(\lambda) \right],$$

where σ is a singular measure with no atoms. By the proof of Theorem 1.1 of [1] (specifically, by the second and ninth lines following (1.4) on p. 194 of [1]), we may write

$$\frac{1 - |\psi(re^{i\theta})|^2}{1 - r^2} = \int_0^{2\pi} \frac{|\psi_\lambda(re^{i\theta})|^2}{|e^{i\lambda} - re^{i\theta}|^2}\, d\sigma(\lambda),$$

where $\psi_\lambda(z) = \exp\left[- \int_0^\lambda (e^{it} + z)/(e^{it} - z)\, d\sigma(t) \right]$.

Without loss of generality, we may suppose the measure σ to carry some mass on the interval $(0, \pi)$. We have

$$\int_0^{2\pi} (1 - |\phi(re^{i\theta})|)\, d\theta \geq \int_0^{2\pi} (1 - |\psi(re^{i\theta})|)\, d\theta \geq \frac{1}{2} \int_0^{2\pi} (1 - |\psi(re^{i\theta})|^2)\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} |\psi_\lambda(re^{i\theta})|^2 (1 - r^2)\, |e^{i\lambda} - re^{i\theta}|^{-2}\, d\theta\, d\sigma(\lambda)$$

$$\geq \frac{1}{2} \int_0^{\pi} \int_{\alpha_r(\lambda)} |\psi_\lambda(re^{i\theta})|^2 (1 - r^2)\, |e^{i\lambda} - re^{i\theta}|^{-2}\, d\theta\, d\sigma(\lambda),$$

where $\alpha_r(\lambda) = \{\theta\colon (1 - r)^{1/2} \leq |e^{i\theta} - e^{i\lambda}| \leq 2(1 - r)^{1/2}\}$. By Lemma 1, there exists $\varepsilon > 0$ such that $|\psi_\lambda(re^{i\theta})|^2 \geq \varepsilon$, if $\theta \in \alpha_r(\lambda)$, and we arrive at the relations

$$\int_0^{2\pi} (1 - |\phi(re^{i\theta})|)\, d\theta \geq (\varepsilon/2) \int_0^{\pi} \int_{\alpha_r(\lambda)} (1 - r^2)\, |e^{i\lambda} - re^{i\theta}|^{-2}\, d\theta\, d\sigma(\lambda)$$

$$= (\varepsilon/2) \int_0^{\pi} \int_{\alpha_r(\lambda)} (1 - r)/[(1 - r)^2 + r\, |e^{i\theta} - e^{i\lambda}|^2]\, d\theta\, d\sigma(\lambda)$$

$$\geq (\varepsilon/8r) \int_0^{\pi} |\alpha_r(\lambda)|\, d\sigma(\lambda) \geq \varepsilon_0 (1 - r)^{1/2}$$

if $r \geq 1/2$, since $|\alpha_r(\lambda)| \geq c(1 - r)^{1/2}$ for some constant c, and $\int_0^{\pi} d\sigma(\lambda) > 0$. This completes the proof.

As stated in the introduction, one point of interest in the $B^{2/3}$ conjecture is its relationship to exceptional sets. We now give an application of Theorem 3 in this direction. Recall that, for an inner function ϕ, we define the exceptional set $E(\phi)$ as

$$E(\phi) = \{\mu\colon |\mu| < 1 \text{ and } (\phi - \mu)/(1 - \bar{\mu}\phi) \text{ is not a Blaschke product}\}.$$

See [2] for a brief discussion of $E(\phi)$.

COROLLARY 3. *If* ϕ *is inner and satisfies* $\phi' \in B^{2/3}$, *then* $E(\phi) = \emptyset$.

Proof. Let $\phi_\mu = (\phi - \mu)/(1 - \bar{\mu}\phi)$. Then

$$|\phi_\mu'| = (1 - |\mu|^2)\, |\phi'|\, |1 - \bar{\mu}\phi|^{-2} \leq c|\phi'|,$$

so that $\phi' \in B^{2/3}$ implies $\phi_\mu' \in B^{2/3}$. Theorem 3 now tells us that ϕ_μ must be a Blaschke product.

COROLLARY 4. *If* B *is a Blaschke product with zeros* $\{a_n\}$ *and if* $\Sigma(1 - |a_n|)^{1/2} < \infty$, *then* $E(B) = \emptyset$.

Proof. The condition $\Sigma(1 - |a_n|)^{1/2} < \infty$ implies $B' \in B^{2/3}$, by a theorem of Protas [11] (to which we shall return in Section 4).

In [2, Theorem 6], we proved that if ϕ is an inner function satisfying $\phi' \in H^{1/2}$, then $E(\phi) = \emptyset$. Combining this with another theorem of Protas ([11, Theorem 2]) we see that [2, Theorem 6] implies a result very close to Corollary 4: *if* B *is a Blaschke product with* $\Sigma(1 - |a_n|)^p < \infty$ *for some* $p < 1/2$, *then* $E(B) = \emptyset$. We assert that Corollary 4 above is actually stronger than [2, Theorem 6]. Indeed, there exist Blaschke products B with $\Sigma(1 - |a_n|)^{1/2} < \infty$ (hence with $B' \in B^{2/3}$) but with $B' \notin H^{1/2}$. The existence of such B is obtained from the following lemma.

LEMMA 2. *Suppose a sequence* $\{d_n\}$ *is given, with* $1 > d_n > 0$, $\Sigma d_n^{1/2} < \infty$ *and* $\Sigma d_n^{1/2} \log(1/d_n) = \infty$. *Then there is a Blaschke product* B *with zeros* $\{a_n\}$ *satisfying* $1 - |a_n| = d_n$ (*in particular* $B' \in B^{2/3}$) *and with* $B' \notin H^{1/2}$. *Moreover, the* a_n *may be chosen so that* $a_n \to 1$ *as* $n \to \infty$.

Proof. Assume for convenience that $\Sigma d_n^{1/2} < \pi/2$, and define

$$\theta_n = \sum_{k=n}^{\infty} d_k^{1/2} \quad \text{and} \quad a_n = (1 - d_n)e^{i\theta_n}.$$

By Lemma 1 of [2], we need only show that

$$f(\theta) \equiv \sum_{n=1}^{\infty} d_n/[d_n^2 + (\theta - \theta_n)^2] \notin L^{1/2}.$$

Now, in the interval (θ_{n+1}, θ_n), we have

$$f(\theta) \geq d_n/[d_n^2 + (\theta - \theta_n)^2],$$

so that

$$\int_0^{2\pi} f(\theta)^{1/2}\, d\theta \geq \sum_n d_n^{1/2} \int_{\theta_{n+1}}^{\theta_n} [d_n^2 + (\theta - \theta_n)^2]^{-1/2}\, d\theta$$

$$= \sum_n d_n^{1/2} \int_0^{d_n^{1/2}} (d_n^2 + t^2)^{-1/2}\, dt$$

$$= \sum_n d_n^{1/2} \left[\frac{1}{2}\log(1/d_n) + \log(1 + (1 + d_n^2)^{1/2}) \right] = \infty.$$

3. *Singular inner functions.* In this section, we show how Theorem 1 may be used to obtain a sufficient condition for the relation $\phi' \in B^p$, when ϕ is a singular inner function. Our theorem contains as a special case the result of Allen and Belna [3] that if the singular measure associated with ϕ consists of a finite number of point masses, then $\phi' \in B^p$ for all $p < 2/3$. It also enables us to give examples of singular inner functions ϕ having purely nonatomic singular measures and satisfying $\phi' \in B^p$, for all $p < 2/3$.

We say a compact subset E of $[0, 2\pi]$ is *of type* β $(0 < \beta \leq 1)$ if there is a constant c such that $|E_\varepsilon| \leq c\varepsilon^\beta$, where

$$E_\varepsilon = \{\theta: \text{dist}(\theta, E) < \varepsilon\}.$$

Roughly speaking, if β is close to 1, then a set E of type β has small ε-neighborhoods. For example, if $E = \{0, 1/2, 1/3, \cdots\}$, then E is of type 1/2, but of no larger type; if $E = \{0, 1/2, 1/4, \cdots\}$, then E is of type β for all $\beta < 1$; and E is finite if and only if E is of type 1.

THEOREM 4. *Suppose* σ *is a singular measure carried on a set* E *of type* β $(\beta > 0)$. *Let* ϕ *be the corresponding singular inner function. Then there is a constant* c *such that*

$$\int_0^{2\pi} (1 - |\phi(re^{i\theta})|)\,d\theta \leq c(1 - r)^q$$

for all $q > \beta/2$. *In particular,* $\phi' \in B^p$ *for all* $p < 2/(4 - \beta)$.

Proof. Let $\hat{\sigma}$ denote the Poisson integral of σ:

$$\hat{\sigma}(re^{i\theta}) = \int_0^{2\pi} (1 - r^2)\,|e^{i\lambda} - re^{i\theta}|^{-2}\,d\sigma(\lambda).$$

There is a constant c such that

$$\hat{\sigma}(re^{i\theta}) \leq c(1 - r)\,d(\theta)^{-2},$$

where $d(\theta)$ is the distance from θ to E. Now

$$|\phi(re^{i\theta})| = \exp[-\hat{\sigma}(re^{i\theta})],$$

and since $1 - e^{-x} \leq x$ for $x \geq 0$, we have the inequality

$$1 - |\phi(re^{i\theta})| < \min\{1, \hat{\sigma}(re^{i\theta})\}.$$

Now fix $q < \beta/2$, and define a sequence $\gamma_0, \gamma_1, \cdots$, by: $\gamma_0 = q/\beta$ and, if $\gamma_0, \cdots, \gamma_{k-1}$ are defined, $\gamma_k = \gamma_0 + \beta^{-1}(2\gamma_{k-1} - 1)$. Notice that $\gamma_0 < 1/2$ and hence that $\varepsilon_0 = -\beta^{-1}(2\gamma_0 - 1) > 0$. We assert that

(2) $$\gamma_k \leq \gamma_0 - k\varepsilon_0 \qquad (k = 0, 1, \cdots).$$

For $k = 0$, this is clear. Proceeding by induction, suppose that (2) holds for a given value of k. Then

$$\begin{aligned}\gamma_{k+1} &= \gamma_0 + \beta^{-1}(2\gamma_{k-1} - 1) \leq \gamma_0 + \beta^{-1}(2(\gamma_0 - k\varepsilon_0) - 1)\\ &= \gamma_0 - \varepsilon_0 - 2k\varepsilon_0\beta^{-1} \leq \gamma_0 - (k+1)\varepsilon_0,\end{aligned}$$

since $2/\beta > 1$. This proves (2), which we use to infer that eventually we have $\gamma_k < 0$. Corresponding to the γ_k, define a sequence of sets

$$\alpha_0 = \{\theta\colon d(\theta) \le (1-r)^{\gamma_0}\}$$

$$\alpha_k = \{\theta\colon (1-r)^{\gamma_{k-1}} < d(\theta) \le (1-r)^{\gamma_k}\} \qquad (k = 1, 2, \cdots).$$

Since eventually $\gamma_k < 0$, a finite number of the α_j will cover $[0, 2\pi]$, the number required being independent of r.

Now we have

$$\int_{\alpha_0} (1 - |\phi(re^{i\theta})|)\,d\theta \le |\alpha_0| \le (1-r)^{\gamma_0\beta} = (1-r)^q,$$

$$\int_{\alpha_k} (1 - |\phi(re^{i\theta})|)\,d\theta \le c(1-r)(1-r)^{-2\gamma_{k-1}}|\alpha_k|$$

$$\le c(1-r)^{1-2\gamma_{k-1}+\gamma_k\beta} \le c(1-r)^q,$$

and hence

$$\int_0^{2\pi} (1 - |\phi(re^{i\theta})|)\,d\theta \le c(1-r)^q.$$

The proof is complete.

COROLLARY 5. *If* supp σ *is of type* β *for every* $\beta < 1$, *then* $\phi' \in B^p$, *for all* $p < 2/3$.

If supp σ is a finite set, then Corollary 5 yields the above mentioned result of Allen and Belna. We now show how to construct a set E which is of type β for every $\beta < 1$ and which supports a continuous measure. Take a sequence $\{\delta_n\}$ with $\delta_0 = 2\pi$ and $\delta_n \downarrow 0$ and construct a Cantor set in the usual way: E_n consists of 2^n intervals each of length $2^{-n}\delta_n$ and E_{n+1} is obtained from E_n by deleting an open interval from the center of each of the intervals in E_n, so that E_{n+1} consists of 2^{n+1} intervals, each of length $2^{-(n+1)}\delta_{n+1}$. Let $E = \bigcap_{n=0}^{\infty} E_n$. If $\varepsilon > 0$ is given, choose n so that

$$2^{-(n+1)}\delta_{n+1} \le \varepsilon < 2^{-n}\delta_n.$$

Thus

$$|E_\varepsilon| \le 3\delta_n = 3(\delta_n/\delta_{n+1})2^{n+1}(2^{-(n+1)}\delta_{n+1}) \le 3\cdot 2^{n+1}(\delta_n/\delta_{n+1})\varepsilon.$$

Now pick $\rho \in (0, 1)$ and $\delta_n = 2^n\rho^{n^2}$. We have

$$|E_\varepsilon| \le 3\cdot 2^{n+1}[2^n\rho^{n^2}/(2^{n+1}\rho^{(n+1)^2})]\varepsilon = 3(2^n/\rho^{2n+1})\varepsilon = 3\rho^{-1}(2/\rho^2)^n\varepsilon.$$

Since $\varepsilon < \rho^{n^2}$, we have $\log\varepsilon < n^2\log\rho$ and $n^2 \le \log\varepsilon/\log\rho$ so that

$$(2/\rho^2)^n \le \exp[(\log\varepsilon/\log\rho)^{1/2}\log(2/\rho^2)] = \exp[\alpha(\log\varepsilon^{-1})^{1/2}]$$

for some $\alpha > 0$. It follows easily that for every $\beta < 1$, there is a constant C such that

$$|E_\varepsilon| \le C\varepsilon^\beta.$$

Thus E is of type β and the Cantor function on E induces the required type of measure σ.

PART II. BLASCHKE PRODUCTS

4. *Arbitrary Blaschke products.* In this section, we consider a Blaschke product B(z) with zeros $\{a_n\}$; $1 - |a_n|$ will be denoted by d_n. As in [2, Section 3], we deal with two theorems giving a sufficient condition for $B' \in B^p$ and a partial converse. Both theorems are shown to be the best possible of their type. The first theorem is due to Protas [11], and we include only the statement, as the use of Theorem 1 does not appear to simplify Protas' proof.

THEOREM 5. *Suppose* $\Sigma d_n^\alpha < \infty$, *for some* $\alpha < 1$. *Then* $B' \in B^{1/(1+\alpha)}$.

THEOREM 6. *Suppose* $B' \in B^p$, *for some* $p > 2/3$. *Then* $\Sigma d_n^\alpha < \infty$ *for all* $\alpha > (1 - p)/(2p - 1)$.

Before beginning the proof, we prove a lemma.

LEMMA 3. *If* $\rho > 1$, *if* $0 < x < 1$, *and if* $\rho \log(1/x) \le \log 2$, *then* $1 - x^\rho \ge (\rho/2)(1 - x)$.

Proof. We have the relations

$$1 - x^\rho = \rho \int_x^1 t^{\rho-1}\,dt \ge \rho(1 - x)x^{\rho-1} \ge \rho(1 - x)x^\rho \ge (\rho/2)(1 - x)$$

if $x^\rho \ge 1/2$, that is, if $\rho \log(1/x) \ge \log 2$.

Proof of Theorem 6. Let $\alpha_0 = \inf\{\alpha\colon \Sigma d_n^\alpha < \infty\}$, and suppose that $\alpha_0 > (1 - p)/(2p - 1)$. Choose $\alpha > \alpha_0$ (so $\Sigma d_n^\alpha < \infty$) and let $\rho_n = d_n^{\alpha-1}$. Then $\prod |a_n|^{\rho_n}$ converges, since

$$\log |a_n|^{\rho_n} = \rho_n \log |a_n| = d_n^\alpha\,[(\log |a_n|)/d_n].$$

In order to estimate the partial products of B from below, we want to find the least number r_j such that

$$\frac{|z| - |a_j|}{1 - |a_j|\,|z|} \ge |a_j|^{\rho_j},$$

for all $|z| \ge r_j$. A straightforward calculation shows that

$$r_j = (|a_j| + |a_j|^{\rho_j})/(1 + |a_j|^{1+\rho_j}).$$

Now if we assume, as we may, that the $|a_j|$ form an increasing sequence it follows easily that the numbers r_j form an increasing sequence also. So, if $B_n(z) = \prod_{j=1}^{n-1} (z - a_j)/(1 - \bar{a}_j z)$, then,

$$|B_n(z)| \geq \prod_{j=1}^{n-1} \left|\frac{z - a_j}{1 - \bar{a}_j z}\right| \geq \prod_{j=1}^{n-1} \frac{|z| - |a_j|}{1 - |a_j|\,|z|} \geq \prod_{j=1}^{n-1} |a_j|^{\rho_j} \geq \prod_{j=1}^{\infty} |a_j|^{\rho_j} = \varepsilon_0 > 0$$

if $|z| \geq r_{n-1}$.

Next we observe that, for every n,

$$\frac{1 - |B(re^{i\theta})|^2}{1 - r^2} \geq \frac{1 - |B_n(re^{i\theta})|^2}{1 - r^2} = \sum_{j=1}^{n} |B_j(re^{i\theta})|^2 \frac{1 - |a_j|^2}{|1 - \bar{a}_j re^{i\theta}|^2},$$

the last equality being easily proved by induction. If we let n go to infinity we obtain the inequality

$$\frac{1 - |B(re^{i\theta})|^2}{1 - r^2} \geq \sum_{j=1}^{\infty} |B_j(re^{i\theta})|^2 \frac{1 - |a_j|^2}{|1 - \bar{a}_j re^{i\theta}|^2}.$$

(Equality actually holds, as can be proved by the methods of [1]; we do not use this fact however.) Now multiply both sides by $(1 - r)^{1/p-2}$ and integrate to get

$$\int_0^{2\pi} I_p(B, \theta)\, d\theta \geq \frac{1}{2} \int_0^{2\pi} \sum_n \int_{r_n}^1 \frac{|B_n(re^{i\theta})|^2 (1 - |a_n|^2)(1 - r)^{1/p-2}}{|1 - \bar{a}_n re^{i\theta}|^2}\, dr\, d\theta$$

$$\geq \frac{1}{2}\varepsilon_0^2 \sum_n \int_{r_n}^1 (1 - r)^{1/p-2} \int_0^{2\pi} (1 - |a_n|^2)/|1 - \bar{a}_n re^{i\theta}|^2\, d\theta\, dr$$

$$= \frac{1}{2}\varepsilon_0^2 \sum_n (1 - |a_n|^2) \int_{r_n}^1 (1 - r)^{1/p-2}/(1 - |a_n|^2 r^2)\, dr$$

$$\geq \varepsilon_1 \sum_n d_n \int_{r_n}^1 (1 - r)^{1/p-2}/[d_n + (1 - r)]\, dr$$

$$= \varepsilon_1 \sum_n d_n^{1/p-1} \int_0^{(1-r_n)/d_n} (t^{1/p-2})/(1 + t)\, dt$$

$$\geq \varepsilon_2 \sum_n d_n^{1/p-1} \int_0^{(1-r_n)/d_n} t^{1/p-2}\, dt = \varepsilon_2 \sum_n (1 - r_n)^{1/p-1}.$$

Computing $1 - r_n$ gives

$$1 - r_n = d_n(1 - |a_n|^{\rho_n})/(1 + |a_n|^{1+\rho_n}) \geq (1/4)\, d_n \rho_n d_n$$

(by Lemma 3) so that $1 - r_n \geq (1/4)\, d_n^{1+\alpha}$. It follows that

$$\sum d_n^{(1+\alpha)(1/p-1)} < \infty,$$

for any $\alpha > \alpha_0$. But if $\alpha_0 > (1 - p)/(2p - 1)$, then $(1 + \alpha_0)(1/p - 1) < \alpha_0$ and hence $(1 + \alpha)(1/p - 1) < \alpha_0$ for all α such that $\alpha < \alpha_0 + \varepsilon$. But this contradicts the definition of α_0, and completes the proof.

We shall now show (in the rest of this section and in the next) that Theorems 5 and 6 represent the best possible results obtainable for general Blaschke products, involving a relationship between $B' \in B^p$ and $\Sigma d_n^\alpha < \infty$.

To show Theorem 5 is best possible, we will show that if $0 < \alpha < \alpha_0 < 1$, there is a Blaschke product B such that $\Sigma d_n^{\alpha_0} < \infty$, but $B' \notin B^{1/(1+\alpha)}$. In fact, choose γ and β with $\gamma > 1$, $\beta > 1$ and $(\beta - 1)/\beta < 1/\gamma$. Somadasa has shown [12] that there exists a Blaschke product B such that $d_n = n^{-\beta}$ and $B(z)$ tends uniformly to 0, as $z \to 1$, $z \in R(\delta, \gamma, 1)$, for any $\delta > 0$. By Theorem 2, this implies $B' \notin B^{\gamma/(2\gamma-1)}$. If we let $(\alpha + 1)^{-1} = \gamma/(2\gamma - 1)$, we see that there is a Blaschke product B with $d_n = n^{-\beta}$ and $B' \notin B^{1/(1+\alpha)}$, as long as $\alpha\beta < 1$. So our goal is achieved if we pick β so that $\alpha\beta < 1$ but $\alpha_0 \beta > 1$.

Theorem 6 will be shown to be best possible in the next section.

5. *Blaschke products with* $\arg a_n$ *in a set of type* β. In this section, we obtain a sufficient condition for B' to belong to B^p, in case the arguments of the zeros lie in a set of type β (as defined in Section 3). In the case of a set of type β for all $\beta < 1$, this condition yields the same degree of convergence of Σd_n^α as our general necessary condition (Theorem 6), and so in the case of real zeros, parallelism with the problem of $B' \in H^p$ [2, Section 4] is again to be noted.

THEOREM 7. *Suppose* B *is a Blaschke product with zeros* $\{a_n\}$ *having* $\arg a_n \in E$ *for some set* E *of type* β $(0 < \beta \leq 1)$. *Suppose also that* $\Sigma d_n^\alpha < \infty$, *for some* α $(0 < \alpha \leq 1)$. *Then*

$$\int_0^{2\pi} (1 - |B(re^{i\theta})|)\, d\theta \leq c(1 - r)^q$$

for all $q < \beta/(1 + \alpha)$. *In particular,* $B' \in B^p$ *for*

$$p < (1 + \alpha)/(2 + 2\alpha - \beta).$$

Proof. The proof is similar to that of Theorem 4 and will only be sketched. First by [10, p. 170],

$$1 - |B(re^{i\theta})| \leq 4 \sum d_n(1 - r)\, |1 - \bar{a}_n re^{i\theta}|^{-2} \equiv \tau(re^{i\theta}),$$

and so we get the inequality

$$1 - |B(re^{i\theta})| \leq \min\{1, \tau(re^{i\theta})\}.$$

Now by the proof of [2, Theorem 11], there is a constant c such that

$$\tau(re^{i\theta}) \leq c(1 - r)/d(\theta)^{1+\alpha},$$

since $\Sigma d_n^\alpha < \infty$ (recall that $d(\theta) = \mathrm{dist}(\theta, E)$). Choose

$$\gamma_0 = q/\beta \quad \text{and} \quad \gamma_k = \gamma_0 + \beta^{-1}[\gamma_{k-1}(1 + \alpha) - 1],$$

$$\alpha_0 = \{\theta\colon d(\theta) < (1-r)^{\gamma_0}\} \quad \text{and} \quad \alpha_k = \{\theta\colon (1-r)^{\gamma_{k-1}} \leq d(\theta) < (1-r)^{\gamma_k}\},$$

and proceed as in the proof of Theorem 4.

COROLLARY 6. *If* $B(z)$ *is a Blaschke product with* $\arg a_n$ *belonging to some set* E *of type* β *for all* $\beta < 1$, *then* $B' \in B^p$ *for all* $p < 2/3$. *If in addition* $\Sigma d_n^{\alpha} < \infty$, *then* $B' \in B^p$ *for* $p < (1+\alpha)/(1+2\alpha)$.

We remark that there exist Blaschke products with $B' \notin B^p$ for $p > 1/2$ (even with $a_n \to 1$). An example can be got from Somadasa's Example 2 [12, p. 299], together with Theorem 2.

Added in proof. Since submission of this paper, we learned that Corollary 2 had been obtained previously by C. L. Belna, in his paper *The derivative of the atomic function is in* B^p *iff* $0 < p < 2/3$, to appear in Proc. Amer. Math. Soc.

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