THE SCHWARTZ-HILBERT VARIETY

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1. INTRODUCTION

Let θ be the collection of locally convex topological vector spaces (LCS) that are embeddable as subspaces of a power X^I, for each infinite-dimensional Banach space X. It is known that θ contains all nuclear spaces (S. A. Saxon [16]), that only Schwartz spaces are in θ (J. Diestel and R. H. Lohman [4]), and that there are Schwartz spaces not in θ (the author [1] and D. J. Randtke [15]). We shall show (Theorem 4.1) that θ coincides with the Schwartz-Hilbert variety (see below). Thus θ is strictly larger than the nuclear variety [1].

Our main tool (see the Lemma) is A. Dvoretsky's theorem [6] (see [12, p. 42]) on the existence of near-ℓ_2^n-subspaces in any Banach space. Using Dvoretsky's theorem, we show that each compact map into Hilbert space can be factored through a subspace of any infinite-dimensional Banach space (Theorem 3.3). For similar results, see C. P. Stegall and J. R. Retherford [18].

2. NOTATION AND PRELIMINARIES

X, Y, and Z are reserved for infinite-dimensional Banach spaces. We write ℓ_2 for the Hilbert space of squared-summable sequences. R, S, T, U, V are reserved for bounded linear maps. Each $T = T_\lambda$ represents a diagonal map on ℓ_2 (that is, $\lambda = (\lambda_n)$ and $T_\lambda(\alpha_n) = (\lambda_n \alpha_n)$). We note that T_λ is a positive compact map if and only if $\lambda_n \geq 0$ for all n and (λ_n) belongs to the space c_0 of null sequences. To say $S: X \to Y$ factors through Z means that there are maps $U: X \to Z$ and $V: Z \to Y$ such that $S = VU$.

A *prevariety* [2] is a collection of LCS's that is closed with respect to the formation of subspaces and arbitrary products. A *variety* [5] is a prevariety that, in addition, contains all its separated quotients. If X is a Banach space, we denote by $\rho\nu(X)$ (respectively $\nu(X)$) the smallest prevariety (variety) containing X.

It follows from Theorem 1.1 of [5, p. 209], that for each LCS E and each Banach space X, $E \in \rho\nu(X)$ if and only if E is a subspace of some power of X. A *universal generator* [5] for a variety $\mathscr{A}$ is an $E \in \mathscr{A}$ such that each $F \in \mathscr{A}$ is embeddable as a subspace of a power of E.

Let $\mathscr{S}$ be the variety of Schwartz spaces (see [8, p. 271]), let $\mathscr{H}$ be the variety $\nu(\ell_2)$, and let $\mathscr{S}\mathscr{H}$ be their intersection, the *Schwartz-Hilbert* variety. From Theorem 4.4 of [5, p. 219] and the definition of Schwartz spaces it follows that each $E \in \mathscr{S}\mathscr{H}$ has a neighborhood basis $\mathscr{U}$ such that the completion of the norm space E_U [17, p. 53] is a Hilbert space, for each $U \in \mathscr{U}$. Furthermore, for each $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$ such that the canonical map $E_V \to E_U$ [17, p. 53] is precompact. In the language of [14] and [15], each $E \in \mathscr{S}\mathscr{H}$ is a subspace of a compact projective limit of ℓ_2-spaces. Finally, let $\mathscr{N}$ be the variety of nuclear spaces (see [13]).

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3. FACTORING COMPACT MAPS INTO HILBERT SPACE

With the exception of the Lemma, the results of this section deal with compact maps into Hilbert spaces. It is shown (Theorem 3.3) that every compact map from a Banach space X into Hilbert space can be factored through a subspace of any Banach space. For convenience, we break the proof of Theorem 3.3 into propositions.

The Lemma is a consequence of Dvoretsky's theorem. A proof can be modelled on the standard construction of a basic sequence in any Banach spaces as given in [12, p. 10]. Similar results are used in [3] and [18, p. 468].

LEMMA. *Corresponding to every* $\kappa > 1$, *every Banach space* X, *and every increasing sequence* $0 = n_0 < n_1 < \cdots$ *of integers, there is a subspace* Y *of* X *with a basis* $\{e_n\}$ *such that*

$$\kappa^{-1}\left[\sum |\alpha_n|^2\right]^{1/2} \leq \left\|\sum \alpha_n e_n\right\| \leq \kappa\left[\sum |\alpha_n|^2\right]^{1/2}$$

for all sequences (α_n) *with support in* $N(k) = \{n_k + 1, n_k + 2, \cdots, n_{k+1}\}$ *for* $k = 0, 1, \cdots$.

PROPOSITION 3.1. *Let* X *be a Banach space, and let* $T_\lambda: \ell_2 \to \ell_2$ *be a positive compact diagonal map. Then* T_λ *factors through a subspace of* X.

Proof. Let $(\mu_n) \in c_0$ be such that $\mu_n \geq 0$ and $\mu_n^2 = \lambda_n$. Let $n_0 = 0$, and for $k \geq 1$, choose $n_k > n_{k-1}$ inductively so that $j > n_k$ implies $\mu_j \leq 2^{-k}$. Let $\kappa = 2$, and let Y be the subspace of X, with basis $\{e_n\}$, given by the Lemma. Define $U: \ell_2 \to Y$ by $U(\alpha_n) = \sum \mu_n \alpha_n e_n$, and define $V: Y \to \ell_2$ by $V\left(\sum \beta_n e_n\right) = (\mu_n \beta_n)$. We complete the proof by showing U and V are bounded linear maps, since clearly $T = VU$.

Suppose $\|(\alpha_n)\|_2 \leq 1$; then

$$\|U(\alpha_n)\| = \left\|\sum \mu_n \alpha_n e_n\right\| \leq \sum_k \left\|\sum_{N(k)} \mu_n \alpha_n e_n\right\| \leq 2\sum_k \left[\sum_{N(k)} \mu_n^2 |\alpha_n|^2\right]^{1/2}$$

$$\leq 2\left(\sum_{k\geq 1} 2^{-k}\left[\sum_{N(k)} |\alpha_n|^2\right]^{1/2} + \|T_\lambda\|\left[\sum_{N(0)} |\alpha_n|^2\right]^{1/2}\right)$$

$$\leq 2(1 + \|T_\lambda\|).$$

Now suppose $\sum \beta_n e_n \in Y$ and $\left\|\sum \beta_n e_n\right\| \leq 1$. If K is the basis constant of $\{e_n\}$ [12, p. 10], then $\left\|\sum_{N(k)} \beta_n e_n\right\| \leq K + 1$ for $k = 0, 1, \cdots$. Thus

$$\left\|V\left(\sum \beta_n e_n\right)\right\| = \|(\mu_n \beta_n)\|_2 \leq \sum_k \left[\sum_{N(k)} \mu_n^2 |\beta_n|^2\right]^{1/2}$$

$$\leq \sum_{k\geq 1} 2^{-k}\left[\sum_{N(k)} |\beta_n|^2\right]^{1/2} + \|T_\lambda\|\left[\sum_{N(0)} |\beta_n|^2\right]^{1/2}$$

$$\leq \sum_{k\geq 1} 2^{-k+1}\left\|\sum_{N(k)} \beta_n e_n\right\| + \|T_\lambda\|\, 2\left\|\sum_{N(o)} \beta_n e_n\right\| \leq (K+1)[2 + 2\cdot\|T_\lambda\|].$$

PROPOSITION 3.2. *If* $S: X \to \ell_2$ *is a compact map, then there is a positive, nonincreasing null sequence* $\lambda = (\lambda_n)$ *and a map* $R: X \to \ell_2$ *such that* $S = T_\lambda R$. *Furthermore,* R *can be chosen to be compact.*

Proof. Let B be the unit ball of X; then $K = S(B)$ is relatively compact. Let P_N be the Nth partial-sum operator on ℓ_2. For $\varepsilon > 0$ (see [12, p. 12]), there is an N such that $\|P_j x - x\| < \varepsilon$ for $x \in K$ and $j \geq N$.

Let $n_0 = 0$, and for $k \geq 1$, choose $n_k > n_{k-1}$ inductively so that $j \geq n_k$ and $x \in K$ imply $\|P_j x - x\| < 2^{-k}$. Let $\lambda_n = k^{-1}$ for $n_{k-1} < n \leq n_k$, and let $\lambda = (\lambda_n)$. For $x \in B$, define Rx to be the sequence $(\lambda_n^{-1} [Sx]_n)$, where $[Sx]_n$ is the nth coordinate of Sx. Then

$$\begin{aligned}\|Rx\|^2 &= \sum |\lambda_n^{-1} [Sx]_n|^2 \\ &\leq \sum_k (k+1)^2 \|P_{n_{k+1}}(Sx) - P_{n_k}(Sx)\|^2 \leq \sum_k (k+1)^2 2^{-k} < \infty .\end{aligned}$$

Hence the linear extension of R is bounded, and clearly $T_\lambda R = S$.

If R is not already compact, let $\mu_n \geq 0$ be such that $\mu_n^2 = \lambda_n$. Then $T_\lambda = T_\mu T_\mu$, $S = T_\mu(T_\mu R)$, and $T_\mu R$ is compact.

Combining Proposition 3.1 and Proposition 3.2, we obtain the following result.

THEOREM 3.3. *Let* $S: X \to \ell_2$ *be compact, and let* Y *be a Banach space; then* S *factors through a subspace of* Y.

In [7], A. Grothendieck asked: "if all operators from X to Y [Banach spaces] are nuclear, must either X or Y be finite-dimensional?" From Propositions 3.1 and 3.2, we can easily deduce that for each infinite-dimensional X and Y, there are compact nonnuclear maps from X to a subspace of Y and from a subspace of X to Y. This is essentially Theorem II.3 of C. P. Stegall and J. R. Retherford [18, p. 472] (their main tool is also Dvoretsky's theorem). For other results on Grothendieck's question, see [3], [9], and [11].

4. THE SCHWARTZ-HILBERT VARIETY

Our main result, Theorem 4.1, is somewhat surprising in that the set (4) is neither $\mathcal{N}$ nor $\mathcal{S}$. The fact that $\mathcal{S} \neq \mathcal{S}\mathcal{H} \neq \mathcal{N}$ is in [1]. Let $\mathcal{B}$ denote the class of all infinite-dimensional Banach spaces.

THEOREM 4.1. *The following classes are equal:*

(1) $\mathcal{S}\mathcal{H}$,

(2) $\bigcap_{X \in \mathcal{B}} \rho\nu(X)$,

(3) $\bigcap_{X \in \mathcal{B}} \nu(X)$.

(4) $\{E$: E *is embeddable in some power of* X, *for each* $X \in \mathcal{B}\}$.

Proof. From Section 2, we have the relations (2) = (4) $\subsetneq$ (3). The results of [4] imply that (3) is contained in $\mathcal{S}$, the variety of Schwartz spaces. Since $\ell_2 \in \mathcal{B}$, it follows that (3) $\subset$ (1). To complete the proof, it suffices to show (1) $\subset$ (2).

Let $E \in \mathscr{S}\mathscr{H}$, and let $X \in \mathscr{B}$. The class E has a neighborhood basis $\mathscr{U}$ such that

(a) $\tilde{E}_U$, the completion of E_U, is isomorphic to ℓ_2, for each $U \in \mathscr{U}$; and

(b) for each $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$ such that the canonical map $\tilde{E}_V \to \tilde{E}_U$ is compact (see Section 2).

For each such U and V in (b), Theorem 3.3 provides a subspace Z of X through which $\tilde{E}_V \to \tilde{E}_U$ factors. Hence [17, p. 53], E is a subspace of a projective limit of subspaces of X, and so $E \in \rho\nu(X)$ [17, p. 54]. Therefore $(1) \subset (2)$.

COROLLARY (Saxon). *Each nuclear space is embeddable as a subspace of some power of each infinite-dimensional Banach space.*

We explicitly mention Saxon's result because Theorem 4.1 provides a proof that does not appeal to the profound result of T. Kōmura and Y. Kōmura [10]: *the space of rapidly decreasing sequences is a universal generator for* $\mathscr{N}$. There is a trade-off, however, since we require Dvoretsky's theorem.

Our final result gives a universal generator for $\mathscr{S}\mathscr{H}$. Here $\ell_2[\mathscr{S}]$ is ℓ_2 equipped with the topology of uniform convergence on (norm) null sequences. A proof of Theorem 4.2 can be obtained from Theorem 2 of D. J. Randtke [14]. Just replace c_0 by ℓ_2 and Schwartz space by Schwartz-Hilbert space. (See also [15, Theorem 1].)

THEOREM 4.2. *$\ell_2[\mathscr{S}]$ is a universal generator for $\mathscr{S}\mathscr{H}$.*

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