EXISTENCE OF MARKOV PROCESSES ASSOCIATED WITH NONCONTRACTION SEMIGROUPS

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1. INTRODUCTION

The connection between contraction semigroups and Markov processes has been extensively studied. However, the relation between noncontraction semigroups and Markov processes remains virtually unexplored. The purpose of this article is to generalize the definition of Markov processes so that a large class of noncontraction semigroups is associated with Markov processes. Although the results of this article can be extended to the cases where the state spaces are more general, we shall consider only the case where the state space Z is the complex plane or a subset of it and is topologically equivalent to a complete separable metric space.

For a measure Q on a measurable space $(M, \mathscr{B})$, let $|Q|$, Q^r, and Q^i denote the total variation, the real part, and the imaginary part of Q, respectively. If Q is a signed measure, let Q^+ and Q^- denote the positive and negative parts of Q.

An *abstract probability space* $(M, \mathscr{B}, Q)$ is a finite measure space with $Q(M) = 1$. Q is called an *abstract probability measure*. If $\mathscr{G}$ is a sub-σ-field of $\mathscr{B}$, let $Q_{\mathscr{G}}$ denote the restriction of Q to $\mathscr{G}$. An element $G \in \mathscr{G}$ is called a *$\mathscr{G}$-null set* if $|Q_{\mathscr{G}}|(G) = 0$. Note that the previous definition of a $\mathscr{G}$-null set G requires also the condition $|Q|(G) \neq 0$ (see [5]). Here we drop this condition, because a set G with $|Q|$-measure 0 can be neglected. The set $N_{\mathscr{G}} = \{(dQ_{\mathscr{G}}/d|Q|_{\mathscr{G}}) = 0\}$ is a $\mathscr{G}$-null set that contains every other $\mathscr{G}$-null set $|Q|$-a.e. A $\mathscr{G}$-measurable r.v. (random variable) Y is said to be *$\mathscr{G}$-integrable* if $|Y|$ is $|Q_{\mathscr{G}}|$-integrable. The conditional expectation $E(X \mid \mathscr{G})$ (see [5]) of a r.v. X given a sub-σ-field $\mathscr{G}$ is the $|Q|$-a.e. uniquely defined r.v. Y, if it exists, that satisfies the conditions

(1.1) $$Y \text{ is } \mathscr{G}\text{-integrable},$$

(1.2) $$Y = 0 \text{ on } N_{\mathscr{G}},$$

(1.3) $$\int_G Y\,dQ = \int_G X\,dQ \quad \text{for each } G \in \mathscr{G}.$$

If X is bounded, then $E(X \mid \mathscr{G})$ exists if and only if the condition

(1.4) $$\int_G X\,dQ = 0$$

is satisfied for each $\mathscr{G}$-null set G (see [5, Theorem 2.1]). In the case where $|Q_{\mathscr{G}}|$ and $|Q|_{\mathscr{G}}$ are equivalent, condition (1.4) is always satisfied and hence $E(X \mid \mathscr{G})$ always exists. The following definition is a generalization of the classical definition of a Markov process.

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Definition. A family of r.v.'s $y(t)$ $(0 \leq t \leq T)$ on $(M, \mathscr{B}, Q)$ is a *Markov process* if $E\{f(y(s)) \mid \sigma(y(u),\ 0 \leq u \leq t)\}$ always exists and is $\sigma(y(t))$-measurable for each bounded, measurable function f and for $0 \leq t \leq s \leq T$.

2. EXISTENCE THEOREM

Let $B = B(Z)$ denote the Banach space, with supremum norm, of all bounded, measurable, complex-valued functions on Z. We write $f_0 = w - \lim f_n$ for $\{f_n\} \subset B$ if $\mu(f_0) = \lim \mu(f_n)$ for every finite measure μ on the Borel field $\mathscr{Z}$ of subsets of Z.

THEOREM 2.1. *Let* S_t $(0 \leq t \leq T)$ *be a semigroup of linear operators on* B *satisfying the conditions*

(2.1) $$S_t 1 = 1,$$

(2.2) $$S_t f = w - \lim S_t f_n \quad \textit{if } f = w\text{-} \lim f_n,$$

(2.3) $$\|S_t\| \leq \exp(bt) \quad \textit{for some } b \geq 0.$$

Then there exists a Markov process $x(t)$ $(0 \leq t \leq T)$ *such that*

$$E\{f(x(s+t)) \mid \sigma(x(u),\ 0 \leq u \leq s)\} = S_t f(x(s))$$

for $f \in B$ *and* $s \leq t + s \leq T$.

The proof of this theorem is similar to the case when $\{S_t\}$ is a contraction semigroup, except that we need to use Theorem 2.3 below, which is our main result. Let $\Omega = Z^{[0,T]}$ and $x(t, w) = w(t)$ for $w \in \Omega$ and $0 \leq t \leq T$. Let $\mathscr{F}$ denote the σ-field generated by the field $\mathscr{D}$ of all cylinder subsets of Ω (see [3, p. 108]), and let $\mathscr{F}_t = \sigma(x(u),\ 0 \leq u \leq t)$ for $0 \leq t \leq T$. It is easy to see (see [1, p. 51]) that $\{S_t\}$ is associated with a (complex) transition function $P(t, x, F)$, which is defined on $[0, T] \times Z \times \mathscr{Z}$, such that $P(t, x, F) = S_t I_F(x)$, where I_F denotes the indicator function of F. In addition to the properties satisfied by an ordinary transition function, $P(t, x, F)$ satisfies also the condition

(2.4) $$\sup_x |P|(t, x, Z) \leq \exp(bt) \quad (0 \leq t \leq T),$$

because of condition (2.3). Let P denote the usual set function on $\mathscr{D}$ defined by $P(t, x, F)$. Then P is countably additive (see [3, pp. 107-111]). Taking into account (2.4), we have the inequality $|P|(\Omega) \leq \exp(bT)$. Therefore, P can be uniquely extended to $\mathscr{F}$ so that $(\Omega, \mathscr{F}, P)$ becomes an abstract probability space.

LEMMA 2.2. *Let* $G \in \mathscr{F}_t$ *be a cylinder set, and let* $D \in \mathscr{D}$. *There exists a cylinder set* $D_t \in \mathscr{F}_t$ *such that*

$$P(GD) = \int_{GD_t} p_D \, dP,$$

where p_D *is an* $\mathscr{F}_t$*-measurable function and is bounded by* e^{bT}.

Proof. Let $G = \{(x(t_1), \cdots, x(t_n)) \in E\}$ $(0 \leq t_1 < \cdots < t_n \leq t)$, $E \in \mathscr{B}(Z^n)$, $D = \{(x(s_1), \cdots, x(s_m)) \in F\}$ $(0 \leq s_1 < \cdots < s_m)$, $F \in \mathscr{B}(Z^m)$. If $s_m < t$, then

$D \in \mathscr{F}_t$. Hence $D_t = D$ and $p_D = 1$. Assume that $s_k \leq t < s_{k+1}$ for some $k < m$. Let Proj denote the first $n + k$ coordinates' projection from Z^{n+m} on Z^{n+k}, and let $A = (\mathrm{Proj})(E \times F)$. Then the set

$$D_t = \{(x(t_1), \cdots, x(t_n), x(s_1), \cdots, x(s_k)) \in A\}$$

is a cylinder set in $\mathscr{F}_t$. For each $a \in A$, let $(E \times F)(a)$ denote the section of $E \times F$ at a. Let $s = \max(t_n, s_k)$; then it is easily seen that

$$P(GD) = \int_{GD_t} p_D dP,$$

with

$$p_D(w) = \int_{(E\times F)(a(w))} \cdots \int P(s_{k+1} - s, x(s), dy_{k+1}) \cdots P(s_m - s_{m-1}, y_{m-1}, dy_m),$$

where $a(w) = (x(t_1, w), \cdots, x(t_n, w), x(s_1, w), \cdots, x(s_k, w))$. It follows from (2.4) that p_D is bounded by $\exp(bT)$.

THEOREM 2.3. $|P_{\mathscr{F}_t}|$ *and* $|P|_{\mathscr{F}_t}$ *are equivalent.*

Proof. Let $P_t = P_{\mathscr{F}_t}$ and $|P|_t = |P|_{\mathscr{F}_t}$. It is clear that $|P_t|$ is absolutely continuous w.r.t. $|P|_t$. To see the converse, let H be an $\mathscr{F}_t$-null set. For each $\varepsilon > 0$, there exist cylinder sets $\{F_n^{jk}\}$ ($j = r, i$; $k = +, -$) of $\mathscr{F}_t$ such that

$$(2.5) \qquad \begin{cases} H \subset \bigcup_n F_n^{jk}, \\ \sum_n P_t^{jk}(F_n^{jk}) < \varepsilon\, e^{-bT}, \end{cases}$$

for each j and k. Let $\{G_m\}$ be the collection of all nonempty intersections of the form $F_{n_1}^{r+} F_{n_2}^{r-} F_{n_3}^{i+} F_{n_4}^{i-}$. The family $\{G_m\}$ consists of cylinder sets of $\mathscr{F}_t$. It follows from (2.5) that, for each j and k,

$$(2.6) \qquad \begin{cases} H \subset \bigcup_m G_m, \\ \sum_m P_t^{jk}(G_m) \leq \sum_n P_t^{jk}(F_n^{jk}) < \varepsilon\, e^{-bT}. \end{cases}$$

For each m and $D \in \mathscr{F}_t$, Lemma 2.2 implies that

$$|P(DG_m)| \leq e^{bT} |P_t| (G_m).$$

Therefore, we obtain the relation

$$(2.7) \qquad |P| (G_m) \leq 4 \sup_{D \in \mathscr{D}} |P(DG_m)| \leq 4\, e^{bT} |P_t| (G_m).$$

Now we combine (2.6) and (2.7), and we find that

$$|P|(H) \leq \sum_m |P|(G_m) \leq 4e^{bT}\sum_m |P_t|(G_m) < 16\varepsilon .$$

Since ε is arbitrary, $|P|(H) = 0$. Thus $|P|_t$ is absolutely continuous w.r.t. $|P_t|$. This proves Theorem 2.3.

COROLLARY 2.4. $E\{f(x(s+t)) \mid \mathscr{F}_s\}$ *exists for each* $f \in B$ *and* $0 \leq s \leq s+t \leq T$.

Proof of Theorem 2.1. Let $X = f(x(s+t))$,

$$Y = S_t f(x(s)) = \int_Z P(t, x(s), dy) f(y),$$

and $\mathscr{G} = \mathscr{F}_s$. Using Theorem 2.3 and the fact that $S_t f \in B$, we see that Y satisfies conditions (1.1) and (1.2). From the construction of P on $\mathscr{D}$, it is easy to verify that X and Y satisfy condition (1.3) for each cylinder set $G \in \mathscr{G}$ and hence for each $G \in \mathscr{G}$. Therefore,

$$E\{f(x(s+t)) \mid \mathscr{F}_s\} = S_t f(x(s))$$

and $x(t)$ $(0 \leq t \leq T)$ is a Markov process. This proves Theorem 2.1.

3. EXAMPLES

(a) Let $\{S_t\}$ be the semigroup of linear operators on $B(R)$ defined by

$$S_t f(x) = \int_R g(t, y-x) f(y)\, dy ,$$

where $g(t, x) = (2\pi t)^{-1/2} \exp(tp^2/2 - x^2/2t + ipx)$ with p a real constant. Then conditions (2.1) and (2.2) are satisfied. Also, $\|S_t\| = \exp(tp^2/2)$. The semigroup $\{S_t\}$ is associated with a continuous, real-valued process (see [5]) whose transition density function is $g(t, y-x)$.

(b) Let $K(x, y)$ be defined on $Z \times Z$ so that $\sup_x \left|\int K(x, y)\, dy\right| = 0$ and $\sup_x \int |K(x, y)|\, dy < \infty$. Let K denote the integral operator on $B(Z)$ with kernel $K(x, y)$. Then the solution $u(t, x)$ of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Ku, \\ u(0) = f \end{cases}$$

has an integral representation. More precisely, $u(t, x) = E_x\{f(x(t))\}$, where $x(t)$ is equivalent to a jump process with waiting distribution $e^{-t}\,dt$ and jumping distribution $K(x, y)dy + \delta(x-y)dy$, where $\delta(x)$ is the Dirac function with unit mass at zero. For a discussion of this, see [2, p. 317]. If $Z = [-1, 1]$ and $K(x, y) = \cos \pi(x-y)$, then the transition function can be found explicitly. Indeed,

$$P(t, x, dy) = (e^t - 1)\cos \pi(x - y)\,dy + \delta(x - y)\,dy\ .$$

In this case $E_x\{\cos \pi(x(t))\} = e^t \cos \pi x$. Hence $\|e^{tK}\| \geq e^t$.

Let K be the infinitesimal generator of some semigroup of operators satisfying conditions (2.1) to (2.3). By using the Markov property, we can discuss the integral representation for solutions of the perturbed equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = Ku + Vu\,, \\ u(0) = f\,. \end{cases}$$

In particular, if $Vu = v(x)\,u(t, x)$, then (see [4, pp. 168-171])

$$u(t, x) = E_x \left\{ \exp\left(\int_0^t v(x(s))\,ds \right) f(x(t)) \right\}$$

under adequate assumptions. However, this is not the main interest of this article.

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