THE STAIRCASE REPRESENTATION OF BIQUASITRIANGULAR OPERATORS

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1. INTRODUCTION

Let $\mathscr{H}$ be a complex, separable, infinite-dimensional Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Several years ago, P. R. Halmos [9] introduced the remarkable class of *quasitriangular* operators on $\mathscr{H}$, which we shall denote by (QT). One consequence of the subsequent study of this class (see the list of references) was the spectral characterization of non-quasitriangular operators [2, Theorem 5.4]. In particular, this theorem implies that every non-quasitriangular operator on $\mathscr{H}$ has a nontrivial hyperinvariant subspace (along with its adjoint), and thus attention now naturally focuses on the class

$$(\mathrm{BQT}) = (\mathrm{QT}) \cap (\mathrm{QT})^*$$

of *biquasitriangular* operators on $\mathscr{H}$. For example, it was recently shown that (BQT) is the norm closure of the class (A) of all algebraic operators on $\mathscr{H}$ [15], and the norm-closure of the class of all nilpotent operators on $\mathscr{H}$ was also determined [3].

The purpose of this note is to present a striking matrix representation for biquasitriangular operators (which was used implicitly in [1] and [15]), and to deduce some consequences of the existence of this representation for the structure theory of biquasitriangular operators. If $T \in \mathscr{L}(\mathscr{H})$, we shall denote the spectrum of T by $\sigma(T)$, and the [left, right] Calkin spectrum of T by $[\sigma_{\ell e}(T), \sigma_{re}(T)]$ $\sigma_e(T)$. If T is a Fredholm operator, we write j(T) for the Fredholm index of T. Moreover, if $\mathscr{K}$ is a Hilbert space and T is a bounded operator mapping $\mathscr{H}$ into $\mathscr{K}$ such that $\ker(T) = \ker(T^*) = \{0\}$, we say that T is a *quasiaffinity*.

2. THE STAIRCASE MATRIX

In what follows, we shall say that an operator T in $\mathscr{L}(\mathscr{H})$ has a *staircase-matrix representation* if there exists an orthogonal decomposition of $\mathscr{H}$ of the form

$$\mathscr{H} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_n, \tag{1}$$

where the subspaces $\mathscr{H}_n$ $(1 \leq n < \infty)$ are finite-dimensional, such that the matrix of T with respect to this decomposition has the form

Received April 9, 1975.

C. Pearcy gratefully acknowledges support from the National Science Foundation.

Michigan Math. J. 22 (1975).

$$\begin{bmatrix} A_1 & B_1 & & & & & & & \\ & C_1 & & & & & & & \\ & D_1 & A_2 & B_2 & & & & & \\ & & & C_2 & & & & & \\ & & & D_2 & \ddots & & & & \\ & & & & & A_n & B_n & & \\ & & & & & & C_n & & \\ & & & & & & D_n & A_{n+1} & \\ & & & & & & & & \ddots \end{bmatrix} \tag{2}$$

where all the entries except the A_n, B_n, C_n, and D_n are understood to be 0.

THEOREM 2.1. *An operator* T *in* $\mathscr{L}(\mathscr{H})$ *is biquasitriangular if and only if for every* $\varepsilon > 0$, *there exists a compact operator* K_ε *in* $\mathscr{L}(\mathscr{H})$ *such that* $\|K_\varepsilon\| < \varepsilon$ *and such that* $T - K_\varepsilon$ *has a staircase-matrix representation.*

Proof. Suppose first that an operator T in $\mathscr{L}(\mathscr{H})$ can be written as a sum $T = S + K$, where K is compact and S has a staircase-matrix representation of the form (2) with respect to a decomposition of $\mathscr{H}$ of the form (1). To show that T is biquasitriangular, it suffices, in view of the fact that (BQT) is invariant under compact perturbations [9], to show that S is biquasitriangular. Since the finite-dimensional subspaces

$$\mathscr{H}_1, \quad \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3, \quad \cdots, \quad \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_{2n+1}, \quad \cdots$$

are all invariant under S, it follows easily from the definition [9] that $S \in (QT)$. That $S^* \in (QT)$ is just as obvious, since each of the finite-dimensional subspaces

$$\mathscr{H}_1 \oplus \mathscr{H}_2, \quad \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_4, \quad \cdots, \quad \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_{2n}, \quad \cdots$$

is invariant under S^*. To prove the other half of the theorem, suppose now that $T \in (BQT)$, and let ε be any positive number. Then, by virtue of the equivalent definitions of quasitriangularity given in [9], it follows easily that there exist increasing sequences $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ of finite-rank projections converging strongly to $1 = 1_{\mathscr{H}}$ and satisfying the further conditions

$$\begin{cases} P_n \mathscr{H} + T^* P_n \mathscr{H} \subset Q_n \mathscr{H} & (n = 1, 2, \cdots), \\ Q_n \mathscr{H} + T Q_n \mathscr{H} \subset P_{n+1} \mathscr{H} & (n = 1, 2, \cdots), \end{cases} \tag{3}$$

and

$$\begin{cases} \|(1 - P_n) T P_n\| \leq \varepsilon/2^{n+2} & (n = 1, 2, \cdots), \\ \|(1 - Q_n) T^* Q_n\| \leq \varepsilon/2^{n+2} & (n = 1, 2, \cdots). \end{cases} \tag{4}$$

It follows from (3) that

$$(1 - P_{n+1})TP_n = 0 = (1 - Q_{n+1})T^*Q_n \quad (n = 1, 2, \cdots) \tag{5}$$

and

$$(1 - P_{n+1})Q_j = 0 = (1 - Q_n)P_j \quad (1 \leq j \leq n). \tag{6}$$

Moreover, the inequalities (4) imply that if K_ε is defined by the equation

$$K_\varepsilon = \sum_{j=1}^{\infty} [(1 - P_j)TP_j + Q_j T(1 - Q_j)],$$

then K_ε is a compact operator of norm less than ε. We define $T_0 = T - K_\varepsilon$. Then, by virtue of (5) and (6), we have the equations

$$\begin{aligned}(1 - P_n)T_0P_n &= (1 - P_n)TP_n - (1 - P_n)\left[\sum_{j=1}^{\infty} (1 - P_j)TP_j + Q_jT(1 - Q_j)\right]P_n \\ &= (1 - P_n)TP_n - (1 - P_n)\left[\sum_{j=1}^{\infty} (1 - P_j)TP_j\right]P_n \\ &= (1 - P_n)TP_n - (1 - P_n)\left[\sum_{j=1}^{n} (1 - P_j)TP_j\right]P_n \\ &= (1 - P_n)TP_n - (1 - P_n)TP_n = 0 \quad (n = 1, 2, \cdots).\end{aligned} \tag{7}$$

By an analogous argument we conclude that

$$Q_n T_0(1 - Q_n) = 0 \quad (n = 1, 2, \cdots). \tag{8}$$

We define $\mathscr{H}_1 = P_1\mathscr{H}$, and for every positive integer n we set

$$\mathscr{H}_{2n} = (Q_n - P_n)\mathscr{H}, \qquad \mathscr{H}_{2n+1} = (P_{n+1} - Q_n)\mathscr{H}.$$

It follows easily from (7) and (8) that the matrix of $T_0 = T - K_\varepsilon$ with respect to the decomposition (1) has the form (2). Thus the theorem is proved.

COROLLARY 2.2. *Let* T *be any biquasitriangular operator in* $\mathscr{L}(\mathscr{H})$, *and let* ε *be any positive number. Then there exists a compact operator* K_ε *of norm less than* ε *such that the operator* $T - K_\varepsilon$ *has a staircase-matrix representation of the form* (2), *where*

(a) *for* $1 \leq n < \infty$, *each eigenvalue of* A_n [*respectively*, C_n] *has algebraic multiplicity one*,

(b) *for* $1 \leq i, j < \infty$ *and* $i \neq j$, $\sigma(A_i) \cap \sigma(A_j) = \emptyset$ *and* $\sigma(C_i) \cap \sigma(C_j) = \emptyset$,

(c) *for* $1 \leq i, j < \infty$, $\sigma(A_i) \cap \sigma(C_j) = \emptyset$.

Proof. It suffices to prove that every operator T_0 in $\mathscr{L}(\mathscr{H})$ that has a staircase-matrix representation can be perturbed by a block-diagonal compact operator K of arbitrarily small norm in such a way that (a), (b), and (c) become valid for the staircase matrix (2) of $T_0 + K$. One begins by perturbing the (1, 1)-entry of the staircase matrix for T_0; then one perturbs the (2, 2)-entry, and proceeds by induction.

3. SOME CONSEQUENCES

We shall now deduce some consequences of Theorem 2.1 and Corollary 2.2. Recall that two operators A and B acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, are called *quasisimilar* if there exist bounded operators $X: \mathscr{H} \to \mathscr{K}$ and $Y: \mathscr{K} \to \mathscr{H}$ with trivial kernels and trivial cokernels such that $XA = BX$ and $AY = YB$.

THEOREM 3.1. *Let* $T \in \mathscr{L}(\mathscr{H})$. *Then the following statements are equivalent:*

(i) $T \in$ (BQT),

(ii) $T = T_0 + K$, *where* K *is compact and* T_0 *is quasisimilar to a normal operator,*

(iii) *For every* $\varepsilon > 0$, *there exists a compact operator* K_ε *such that* $\|K_\varepsilon\| < \varepsilon$ *and such that* $T - K_\varepsilon$ *is quasisimilar to a diagonable normal operator.*

Proof. That (iii) implies (ii) is obvious. Moreover, since every normal operator has the spectral splitting property (see [2] for a discussion of this concept) and quasisimilarity preserves the spectral splitting property ([6, Proposition 10.1]), the fact that (ii) implies (i) follows from the result that every operator with the spectral splitting property is quasitriangular [2, Proposition 1.3]. To complete the proof, we shall establish that (i) implies (iii). Thus, let T belong to (BQT), and let ε be a positive number. By Theorem 2.1, there exists a compact operator K_ε in $\mathscr{L}(\mathscr{H})$ such that $\|K_\varepsilon\| < \varepsilon$ and such that $T' = T - K_\varepsilon$ has a staircase-matrix representation of the form (2) relative to a decomposition of $\mathscr{H}$ as an orthogonal direct sum of finite-dimensional subspaces of the form (1). Moreover, by Corollary 2.2 we may assume that the entries A_n and C_n of the matrix (2) for T' satisfy (a), (b), and (c) in the statement of Corollary 2.2. We shall complete the proof of the theorem by showing that T' is quasisimilar to a diagonal normal operator. For each positive integer n, we define in $\mathscr{L}(\mathscr{H})$ the finite-rank projections P_n and Q_n by

$$P_n \mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_{2n-1},$$

$$Q_n \mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_{2n}.$$

From the form of the matrix (2) for T', we note that $T P_n \mathscr{H} \subset P_n \mathscr{H}$ and $T^* Q_n \mathscr{H} \subset Q_n \mathscr{H}$ for all n. We define $T_n = T \mid P_n \mathscr{H}$, and we observe that

$$\sigma(T_n) = \left[\bigcup_{1 \leq k \leq n} \sigma(A_k) \right] \cup \left[\bigcup_{1 \leq k \leq n-1} \sigma(C_k) \right]. \tag{10}$$

(Perhaps the easiest way to see this is to note that T_n is unitarily equivalent to the matrix

$$\begin{bmatrix} A_1 & 0 & B_1 & 0 & 0 & \cdots & & \\ & A_2 & D_1 & 0 & B_2 & & & \\ & & C_1 & 0 & 0 & & & \\ & & & A_3 & D_2 & & & \\ & & & & C_2 & & & \\ & \mathbf{0} & & & & \ddots & & \\ & & & & & & A_n & D_{n-1} \\ & & & & & & & C_{n-1} \end{bmatrix}$$

via a unitary permutation matrix.) Therefore T_n has $d_n = \text{rank } P_n$ distinct eigenvalues and a corresponding collection of d_n linearly independent eigenvectors (which necessarily span $P_n \mathscr{H}$). Consequently, there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of unit eigenvectors of T' such that $f_1, \cdots, f_{d_n}$ are eigenvectors of T_n and span $P_n \mathscr{H}$ ($n = 1, 2, \cdots$). Therefore, if $T' f_j = \lambda_j f_j$ $(1 \leq j < \infty)$, then

$$\{\lambda_1, \lambda_2, \cdots, \lambda_{d_n}\} = \sigma(A_1) \cup \sigma(C_1) \cup \cdots \cup \sigma(A_n),$$

and consequently all the numbers λ_j are distinct. Consider now on (ℓ_2) the diagonal normal operator N defined by

$$N(\zeta_1, \zeta_2, \cdots) = (\lambda_1 \zeta_1, \lambda_2 \zeta_2, \cdots), \qquad (\zeta_1, \zeta_2, \cdots) \in (\ell_2)$$

and also the operator $X: (\ell_2) \to \mathscr{H}$ defined by

$$X(\zeta_1, \zeta_2, \cdots) = \sum_{n=1}^{\infty} \frac{\zeta_n}{n^2} f_n .$$

An easy calculation shows that X is bounded, and another one shows that $XN = T'X$. Furthermore, it is clear that the range of X is dense in $\mathscr{H}$. We shall now show that $\ker X = \{0\}$. To this end, suppose that to the contrary there exists a nonzero vector $z = (\zeta_1, \zeta_2, \cdots)$ in (ℓ_2) such that

$$Xz = \sum_{n=1}^{\infty} \frac{\zeta_n}{n^2} f_n = 0 .$$

Choose n_0 large enough so that $\zeta_k \neq 0$ for some k satisfying $k \leq d_{n_0}$. Then, for $1 \leq j < \infty$,

$$(Q_{n_0} T' Q_{n_0})(Q_{n_0} f_j) = Q_{n_0} T' f_j = \lambda_j (Q_{n_0} f_j) .$$

Thus either $Q_{n_0} f_j = 0$, or else $Q_{n_0} f_j$ is an eigenvector for $Q_{n_0} T' Q_{n_0}$ corresponding to the eigenvalue λ_j. Since

$$\sigma(Q_{n_0} T' Q_{n_0}) = \sigma(A_1) \cup \sigma(C_1) \cup \cdots \cup \sigma(A_{n_0}) \cup \sigma(C_{n_0}) \subset \{\lambda_1, \cdots, \lambda_{d_{n_0+1}}\},$$

by an argument like the one that established (10), we see that $Q_{n_0} f_j = 0$ for all $j > d_{n_0+1}$. Furthermore, $\zeta_k Q_{n_0} f_k = \zeta_k f_k \neq 0$, and consequently the equation

$$0 = Q_{n_0} X z = \sum_{j=1}^{d_{n_0+1}} \frac{\zeta_j}{n^2} Q_{n_0} f_j$$

contradicts the fact that those $\zeta_j Q_{n_0} f_j$ $(1 \leq j \leq d_{n_0+1})$ that are nonzero are eigenvectors for $Q_{n_0} T' Q_{n_0}$ corresponding to different eigenvalues. Thus $\ker X = \{0\}$, and therefore X is a quasiaffinity satisfying the condition

$$XN = T'X. \tag{11}$$

Next we note that if the decomposition (1) of $\mathscr{H}$ is replaced by another decomposition

$$\mathscr{H} = \mathscr{K}_1 \oplus \mathscr{K}_2 \oplus \cdots, \tag{12}$$

where $\mathscr{K}_1 = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $\mathscr{K}_n = \mathscr{H}_{n+1}$ $(n \geq 2)$, then the matrix for $(T')^*$ relative to the decomposition (12) again has the form (2) for certain operators $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$. Furthermore, by virtue of the relation between the $\tilde{A}_n$ and $\tilde{C}_n$ and the A_n and C_n, it is easy to see that the sets $\sigma(\tilde{A}_n)$ and $\sigma(\tilde{C}_n)$ satisfy conditions (a), (b), and (c) of Corollary 2.2. Thus we can repeat the construction just given for T' with $(T')^*$, and it follows that there exist a normal operator N_* on (ℓ_2) and a quasiaffinity Y such that

$$(T')^* Y = Y N_*. \tag{13}$$

We can combine the equations (11) and (13) to obtain the equation

$$(N_*)^* (Y^* X) = (Y^* X) N. \tag{14}$$

Since $Y^* X$ is a quasiaffinity, it follows that N and $(N_*)^*$ are unitarily equivalent (see [14, p. 71]). If we write $U^* N U = (N_*)^*$ and take adjoints in (13), we obtain the equation

$$(UY^*) T' = N(UY^*). \tag{15}$$

But (11) and (15) imply that T' is quasisimilar to the diagonal normal operator N, and thus the proof of the theorem is complete.

4. QUASISIMILARITIES OF BIQUASITRIANGULAR OPERATORS

In this section we show by giving some examples that the property of being biquasitriangular is not preserved under quasisimilarity.

PROPOSITION 4.1. *There exists a biquasitriangular operator that is quasisimilar to a non-quasitriangular operator.*

Proof. In [14, Chapter VI, Section 4.2], a contraction T_0 was constructed that is quasisimilar to a unitary operator V and has the further property that

$$\sigma(T_0) = \sigma_{\ell e}(T_0) = \sigma_e(T_0) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}.$$

Let $T = T_0 \oplus S$, where S is a unilateral shift operator of multiplicity one. Then the spectrum of T and the left essential spectrum of T are again the closed unit disc, and it follows from the spectral characterization of quasitriangular operators [2, Theorem 5.4] that T is biquasitriangular. On the other hand, T is obviously quasisimilar to $V \oplus S$, which fails to be quasitriangular since the Fredholm index of $V \oplus S$ at the origin is -1.

The following proposition is known and extremely useful.

PROPOSITION 4.2. *Suppose that for every positive integer* n, A_n *and* B_n *are similar operators. Then* $\sum_{n=1}^{\infty} \oplus A_n$ *is quasisimilar to* $\sum_{n=1}^{\infty} \oplus B_n$.

Proof. Suppose that $S_n A_n = B_n S_n$, where for every n, S_n is an invertible operator. Then

$$\left(\sum_{n=1}^{\infty} \oplus \alpha_n S_n\right)\left(\sum_{n=1}^{\infty} \oplus A_n\right) = \left(\sum_{n=1}^{\infty} \oplus B_n\right)\left(\sum_{n=1}^{\infty} \oplus \alpha_n S_n\right)$$

and

$$\left(\sum_{n=1}^{\infty} \oplus A_n\right)\left(\sum \oplus \beta_n S_n^{-1}\right) = \left(\sum_{n=1}^{\infty} \oplus \beta_n S_n^{-1}\right)\left(\sum_{n=1}^{\infty} \oplus B_n\right),$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers chosen to make the quasiaffinities $\sum \oplus \alpha_n S_n$ and $\sum \oplus \beta_n S_n^{-1}$ bounded. The result follows.

PROPOSITION 4.3. *There exists an operator in* $\mathcal{L}(\mathcal{H})$ *of the form* $N + K$, *where* N *is normal and* K *is compact, that is quasisimilar to a non-quasitriangular operator.*

Proof. For each positive integer n, let L_n be the weighted shift operator on a Hilbert space $\mathcal{H}_n$ of dimension $2n$ whose matrix relative to some orthonormal basis for the space has the weight sequence

$$\sqrt{\frac{1}{n}},\ \sqrt{\frac{2}{n}},\ \cdots,\ \sqrt{\frac{n-1}{n}},\ 1,\ 1,\ \sqrt{\frac{n-1}{n}},\ \cdots,\ \sqrt{\frac{2}{n}},\ \sqrt{\frac{1}{n}}.$$

Then the operator $L = \sum_{n=1}^{\infty} \oplus L_n$ (popularly called the Lancaster operator or the Gabriel operator) acting on $\sum_{n=1}^{\infty} \oplus \mathcal{H}_n$ is known to be of the form $N_1 + K_1$, where N_1 is normal and K_1 is compact by virtue of the Brown-Douglas-Fillmore theorem [5]. But each L_n is obviously similar to a weighted shift G_n on $\mathcal{H}_n$ whose weight sequence is the constant sequence $\{1/n\}$. By Proposition 4.2, L is quasisimilar to the operator $G = \sum_{n=1}^{\infty} \oplus G_n$, which is clearly compact. Thus we conclude that if S denotes a unilateral shift operator in $\mathcal{L}(\mathcal{H})$ of multiplicity one, then $L \oplus S$ is quasisimilar to $G \oplus S$. It is easy to verify that $\sigma_e(L)$ is the unit disc. Hence $L \oplus S$ is also the sum of a normal operator and a compact operator, by the Brown-Douglas-Fillmore theorem [5], and $G \oplus S$ is non-quasitriangular because at points λ of the open unit disc where $G - \lambda$ is invertible, $j[(G \oplus S) - \lambda] = -1$. Thus the proof is complete.

5. OPERATORS COMMUTING WITH COMPACT OPERATORS

V. J. Lomonosov's beautiful theorem (see [11], [13]) says that every nonscalar operator in $\mathscr{L}(\mathscr{H})$ that commutes with some nonzero compact operator has a nontrivial hyperinvariant subspace. This gives rise to the question: Is it possible for a compact quasiaffinity to commute with a non-quasitriangular operator? The first piece of evidence in this direction shows that such commuting is not easy.

THEOREM 5.1. *Let* K *and* T *be nonzero operators in* $\mathscr{L}(\mathscr{H})$ *such that* K *is a compact quasiaffinity. If* T *has the property that there exists at least one scalar* λ_0 *such that* $T - \lambda_0$ *is a Fredholm operator of nonzero (necessarily finite) index, then* K *does not commute with* T.

Proof. We may suppose, without loss of generality, that $j(T - \lambda_0) > 0$. (For, if $j(T - \lambda_0) < 0$, we can apply the argument to T^* and K^*.) By the Fredholm theory, there exists a neighborhood $\mathscr{N}$ of the point λ_0 such that for $\lambda \in \mathscr{N}$, $\mathscr{M}_\lambda = \ker(T - \lambda)$ is a nonzero finite-dimensional subspace of $\mathscr{H}$. Suppose now that, contrary to the theorem, $TK = KT$. Then $(T - \lambda)K = K(T - \lambda)$ for every scalar λ, and it follows that all of the subspaces $\mathscr{M}_\lambda$ $(\lambda \in \mathscr{N})$ are invariant under K. Since $\mathscr{M}_\lambda$ is finite-dimensional, $K \mid \mathscr{M}_\lambda$ must have a nonzero eigenvalue μ_λ and an associated eigenspace $\mathscr{E}_\lambda \subset \mathscr{M}_\lambda$. Since K is compact, the collection $\{\mu_\lambda\}_{\lambda \in \mathscr{N}}$ must be at most countable, and thus there exists an uncountable subset $\mathscr{S} \subset \mathscr{N}$ such that $\mu_{\lambda_1} = \mu_{\lambda_2}$ for all λ_1, λ_2 in $\mathscr{S}$. If for each λ in $\mathscr{S}$ we choose a unit vector f_λ in $\mathscr{E}_\lambda$, then the space $\bigvee_{\lambda \in \mathscr{S}} \{f_\lambda\}$ must be infinite-dimensional (because each f_λ is an eigenvector of T corresponding to the eigenvalue λ). This contradicts the compactness of K, and the proof is complete.

The preceding theorem and the spectral characterization of non-quasitriangular operators [5, Theorem 5.4] yield the following corollary.

COROLLARY 5.2. *If* K *is a compact quasiaffinity on* $\mathscr{H}$, *and* K *commutes with a non-biquasitriangular operator* T, *then for every scalar* λ *such that* $T - \lambda$ *is a semi-Fredholm operator,* $j(T - \lambda) = \pm\infty$.

We observe that this phenomenon can actually occur.

PROPOSITION 5.3. *There exist a compact quasiaffinity* K *on* $\mathscr{H}$ *and a non-quasitriangular operator* T *on* $\mathscr{H}$ *such that* $KT = TK$.

Proof. Let V be the classical Volterra operator; that is, let

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad (f \in L_2[0, 1]).$$

Then (see [8]) V is similar to V/2. In other words, there exists an invertible operator X on $L_2[0, 1]$ such that $V/2 = XVX^{-1}$. We set

$$\mathscr{H} = L_2[0, 1] \oplus L_2[0, 1] \oplus \cdots$$

and define K and T by the matrices

$$\begin{pmatrix} V & & & \text{\Large 0} \\ & V/2 & & \\ & & V/4 & \\ \text{\Large 0} & & & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & & \text{\Large 0} \\ X & 0 & & & \\ & X & 0 & & \\ & & X & \ddots & \\ \text{\Large 0} & & & \ddots & \end{pmatrix}$$

respectively. Then it is clear that $TK = KT$, and T is not quasitriangular, since T is a semi-Fredholm operator with $j(T) = -\infty$. Since K is obviously a compact quasiaffinity, the proof is complete.

In the positive direction, we can report the following.

THEOREM 5.4. *Every biquasitriangular operator on* $\mathscr{H}$ *is the sum of a compact operator of arbitrarily small norm and an operator commuting with a compact quasiaffinity whose eigenvectors span* $\mathscr{H}$.

Proof. If T is biquasitriangular, then by virtue of Theorem 3.1, there exists a compact operator K_1 of arbitrarily small norm such that $T_1 = T - K_1$ is quasisimilar to a diagonal normal operator D. Moreover, the proof of Theorem 3.1 shows that the eigenvalues of D may all be taken to have (algebraic and geometric) multiplicity one. This implies, of course, that the commutant $\{D\}'$ of D is abelian and consists entirely of diagonal operators. Let K_2 be a compact normal diagonal operator of multiplicity one commuting with D, and let X and Y be quasiaffinities satisfying the conditions $T_1X = XD$ and $YT_1 = DY$. Then YX commutes with D, and since $\{D\}'$ is abelian, we see that $K_2YX = YXK_2$. We now define $K = XK_2Y$ and $Z = K_2YX$. Calculation shows that K commutes with T_1 and that $KX = XZ$. Since Z is the product of the two commuting diagonal operators K_2 and YX, the eigenvectors of Z span $\mathscr{H}$. Since Xf is an eigenvector for K whenever f is an eigenvector for Z, the eigenvectors of K must span $\mathscr{H}$. Since K is obviously compact, the proof is complete.

COROLLARY 5.5. *An operator* T *in* $\mathscr{L}(\mathscr{H})$ *is biquasitriangular if and only if* T *can be written as a sum* $T = K_1 + T_1$, *where* K_1 *is compact and* T_1 *commutes with a compact quasiaffinity whose eigenvectors span* $\mathscr{H}$.

Proof. Half of the corollary follows from Theorem 5.4. To prove the other half, suppose $T = T_1 + K_1$, where T_1 and K_1 are as in the statement of the corollary. Examination of the proof of the preceding theorem shows that the compact quasiaffinity K that commutes with T_1 and whose eigenvectors span $\mathscr{H}$ has the property that the eigenvectors of K^* also span $\mathscr{H}$. It follows easily that T_1 and T_1^* have the spectral splitting property, and thus they are quasitriangular along with T and T^* [2]. Thus the proof is complete.

REFERENCES

1. C. Apostol and C. Foiaş, *On the distance to the biquasitriangular operators*. Rev. Roumaine Math. Pures Appl. (to appear).
2. C. Apostol, C. Foiaş, and D. Voiculescu, *Some results on non-quasitriangular operators. IV*. Rev. Roumaine Math. Pures Appl. 18 (1973), 487-514.
3. ———, *On the norm-closure of nilpotents. II.* Rev. Roumaine Math. Pures Appl. 19 (1974), 549-557.

4. I. D. Berg, *An extension of the Weyl-von Neumann theorem to normal operators.* Trans. Amer. Math. Soc. 160 (1971), 365-371.

5. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of* C^*-*algebras.* Proc. Conf. Operator Theory, Dalhousie Univ., Halifax, 1973. Lecture Notes in Mathematics, No. 345, pp. 58-128. Springer-Verlag, New York, 1973.

6. R. G. Douglas and C. Pearcy, *A note on quasitriangular operators.* Duke Math. J. 37 (1970), 177-188.

7. ———, *Invariant subspaces of non-quasitriangular operators.* Proc. Conf. Operator Theory, Dalhousie Univ., Halifax, 1973. Lecture Notes in Mathematics, No. 345, pp. 13-57. Springer-Verlag, New York, 1973.

8. C. Foiaş and J. P. Williams, *Some remarks on the Volterra operator.* Proc. Amer. Math. Soc. 31 (1972), 177-184.

9. P. R. Halmos, *Quasitriangular operators.* Acta. Sci. Math. (Szeged) 29 (1968), 283-293.

10. ———, *Ten problems in Hilbert space.* Bull. Amer. Math. Soc. 76 (1970), 887-933.

11. V. I. Lomonosov, *Invariant subspaces for the family of operators which commute with a completely continuous operator.* Funkcional Anal. i Priolžen 7 no. 3 (1973), 55-56.

12. C. Pearcy and N. Salinas, *An invariant-subspace theorem.* Michigan Math. J. 20 (1973), 21-31.

13. C. Pearcy and A. Shields, *A survey of the Lomonosov technique in the theory of invariant subspaces.* Topics in operator theory, pp. 219-229. Math. Surveys No. 13, American Math. Soc., Providence, R.I., 1974.

14. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space.* North-Holland Publ. Co., Amsterdam; American Elsevier, New York; Akadémiai Kiadó, Budapest, 1970.

15. D. Voiculescu, *Norm-limits of algebraic operators.* Rev. Roumaine Math. Pures Appl. 19 (1974), 371-378.

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